

S-duality and the giant magnon dispersion relation

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We use S-duality and planarity to propose an argument for the non-renormalization of the dispersion relation of giant magnon solutions in type IIB string theory on $AdS_5 \times S^5$. We compute the spectrum of giant magnons for (p, q) -strings from field theory at strong coupling by using the central charge properties of electrically and magnetically charged supersymmetric states in the Coulomb branch of $\mathcal{N} = 4$ super Yang-Mills. We argue that the coupling dependence of the giant magnon dispersion relation conjectured in the literature using integrability assumptions is in fact the only functional dependence compatible with S-duality.

– *Introduction.* One of the most immediate, dynamical tests of the AdS/CFT correspondence is provided by the fact that both type IIB string theory and $\mathcal{N} = 4$ super Yang-Mills are believed to be exactly invariant under $SL(2, \mathbb{Z})$ S-duality transformations. On the string theory side of the correspondence, the S-operation acts on the dilaton field by flipping its sign and interchanges, for example, fundamental strings and D1-branes [1]. In the gauge theory [2], it trades a description with gauge group G and complexified coupling constant τ for a description with group ${}^L G$ [23] and coupling constant $-1/\tau$. Albeit a rigorous, mathematical proof of these statements is still missing, a convincing body of evidence in their favor has been produced over the years.

The point of this letter will be to assume that S-duality holds as an exact symmetry in the AdS/CFT correspondence and to investigate what this implies. In particular, we will be interested in understanding what role S-duality plays in the integrability of the sigma-model for $AdS_5 \times S^5$ [3] and the corresponding integrable spin chain model in the dual $\mathcal{N} = 4$ super Yang-Mills [4].

Our main concern is to show that at strong coupling in the field theory one can describe in detail not only the exact energies of some fundamental string states, but also the energies of similar (p, q) -string states. The states we consider, the so-called *giant magnons* [5], are solitonic string solutions of the string sigma model. The general study of such solutions was undertaken in [6]. In the spin chain limit these are described by perturbative gauge theory and become magnon excitations around some ferromagnetic ground state.

Together with planarity arguments, this analysis will show that the giant magnon dispersion relation, which was originally computed under various approximations in [7], should not receive any perturbative corrections, even though they are in principle allowed by integrability [8]. Indeed, in other AdS/CFT setups [9], such renormalizations are required to interpolate between the weak and strong coupling limits [10]. From this point of view, the case of $\mathcal{N} = 4$ super Yang-Mills and its $AdS_5 \times S^5$ dual

is rather special and deserves further attention.

We will employ the approach of [11], where part of the gravitational geometry can be obtained from the distribution of eigenvalues of a certain diagonal matrix model reduction of $\mathcal{N} = 4$ super Yang-Mills on $\mathbb{R} \times S^3$. The eigenvalues localize on a 5-sphere that is identified with the S^5 of the dual geometry, where the giant magnons live. The energies of the giant magnons can be reproduced exactly by studying the off-diagonal modes of the original field theory [12] that become heavy for dynamical reasons and can be self-consistently integrated out in the ground state. We will show that the eigenvalues can be connected not only by fields carrying fundamental charges, but they can also be connected by (p, q) -dyonic excitations carrying both electric and magnetic charge and whose semiclassical energy will exactly reproduce the energies of the giant magnons for the (p, q) -strings. This will give us a spectrum of string states that is manifestly covariant under S-duality. We will moreover show that compatibility of planarity with S-duality forces the corresponding dispersion relation to be non-renormalized between weak and strong coupling.

– *Dyonic off-diagonal excitations.* It has been shown in [11] (see also [12, 13] and [14] for a review) that one can truncate $\mathcal{N} = 4$ super Yang-Mills living on $\mathbb{R} \times S^3$ to a matrix quantum mechanics of six commuting matrices given by the s -waves of the scalar fields ϕ^I . The truncation is argued to be valid for low energy computations because other modes become heavy dynamically in the most probable configurations. The dominance of commuting matrices can be shown analytically and numerically in various models [15].

In the truncation, one can diagonalize these commuting matrices simultaneously and the (bosonic) eigenvalues that one obtains turn out to localize on a 5-sphere with radius $r_0 = \sqrt{N}/2$, where N represents the rank of the gauge group or, equivalently, the number of eigenvalues. This localization is the result of a competition between an attractive quadratic potential given by the conformal coupling of the scalars to the curvature of the S^3 (a mass term) and a repulsive interaction originating from the measure Jacobian produced by the diagonalization. In this setup, the gauge dynamics can be considered to be spontaneously broken from $U(N)$ down to $U(1)^N$ to account for the charges of the various states.

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It is possible to study the off-diagonal excitations of this matrix model in a perturbative way, by treating them as free harmonic oscillators whose mass depends on the diagonal modes [12]. A computation shows that the frequency of these modes scales as the distance between pairs of eigenvalues. More precisely, the off-diagonal element $\delta\phi_{ij}^I$ (with $i \neq j$) has a mass squared

$$m_{ij}^2 = 1 + \frac{\lambda}{4\pi^2} |\hat{x}_i - \hat{x}_j|^2, \quad (1)$$

where the constant term comes from the curvature of the S^3 , $\lambda \equiv g_{YM}^2 N = g_{YM}^2 (2r_0^2)$ is the 't Hooft coupling, and $\hat{x}_i \equiv \vec{x}_i/|\vec{x}_i|$ is the unit-normalized 6-vector formed by the i -th eigenvalues of the scalars ϕ^I . The off-diagonal modes can then be interpreted as fundamental string bits connecting different eigenvalues on the S^5 . The off-diagonal fields carry $U(1)$ gauge charges and the Gauss constraint forces them to be assembled into closed polygons, thereby providing a combinatorial picture of closed strings as being formed by gluing several open string bits into a loop. Notice that at strong coupling the mass of these modes is very large and they can be integrated out of the low-energy effective theory. In principle, the integration procedure might generate additional interactions between the eigenvalues that are not included by just performing the truncation to diagonal configurations, so in general one could expect that the radius of the sphere can be renormalized by these interactions to a new value $r \neq r_0$. This changes λ in the expression (1) above to a more general function of the coupling constants λ and $\tau \equiv \theta/2\pi + i4\pi/g_{YM}^2$

$$m_{ij}^2 = 1 + \frac{h(\lambda, \tau)}{4\pi^2} |\hat{x}_i - \hat{x}_j|^2, \quad (2)$$

where $h(\lambda, \tau)$ is to be determined. This renormalization would account for the expected renormalization of the giant magnon dispersion relation.

In this letter we want to include magnetic charges in this picture and consider the off-diagonal modes as bits of (p, q) -strings, and not just of fundamental strings. In the planar limit and at strong coupling the off-diagonal modes are not very sensitive to the compactness of the sphere S^3 , as their Compton wave-length $l_C \sim 1/\lambda$ is much shorter than the curvature radius of the space, so that the 3-sphere can be essentially replaced by flat space at the Compton wave-length scale of the charged particles. This is true so long as they can be considered to be heavy relative to the size of the sphere, which requires a strong coupling limit in the 't Hooft coupling. In this limit, notice that configurations of constant commuting matrices on the sphere become configurations on the moduli space of flat directions of the $\mathcal{N} = 4$ field theory on flat space. The masses of these electric objects are not only calculable, but they are protected by supersymmetry, since the fundamental fields transform in short representations of supersymmetry on flat space. The $\mathcal{N} = 2$ central charge that these states carry has to

be identified with their electric charge [16], and the orientation and size is determined by the expectation values of the scalar fields. Duality (together with holomorphy) permits us not only to calculate the central charges and masses of the fundamental charges, but also of magnetic and dyonic charges. This was instrumental in the solution of $\mathcal{N} = 2$ super Yang-Mills [17]. For the case of $\mathcal{N} = 4$ super Yang-Mills, this was calculated by Sen [18].

It is immediate to compute the mass \tilde{m}_{ij} of these (p, q) charged objects. This is obtained from (2) by taking into account the expression for the mass of the (p, q) -dyons

$$T_{(p,q)} = T_{(1,0)} |p - q\tau|, \quad T_{(1,0)} = \frac{\sqrt{\lambda}}{2\pi}. \quad (3)$$

Here we have written the (p, q) -dyon mass in terms of λ and τ . One finds that \tilde{m}_{ij}^2 reads

$$\tilde{m}_{ij}^2 = 1 + \frac{h(\lambda, \tau) |p - q\tau|^2}{4\pi^2} |\hat{x}_i - \hat{x}_j|^2. \quad (4)$$

As already mentioned, the constant factor arises from the curvature coupling of the scalar fields to the background metric of the S^3 , which should be the same for all BPS (Bogomolny-Prasad-Sommerfeld) protected scalar particles if we enforce S-duality. In general this constant gets replaced by $(\ell + 1)^2$, where ℓ is the orbital angular momentum quantum number on the S^3 [13]. This term reproduces the bound state dispersion relation for giant magnons [19]. This is also the momentum squared operator on the sphere, which is also part of the dispersion relation in flat space because of Lorentz symmetry.

The regime where these calculations are valid is equivalent to a decoupling limit where the sphere S^3 becomes of infinite radius and the off-diagonal modes become BPS protected states effectively living in flat space. This requires first taking large N , at strong 't Hooft coupling and then taking τ to be finite.

– *The giant magnon dispersion relation.* We have now all the ingredients to analyze the dispersion relation of the giant magnon of type IIB strings on $AdS_5 \times S^5$. We consider the solution corresponding to a (p, q) -string. Such string has the same classical sigma model as a fundamental string (described by a Nambu-Goto action)

$$S_{(p,q)} = \frac{\sqrt{\lambda}}{2\pi} |p - q\tau| \int d^2\sigma \sqrt{-\det g_{\alpha\beta}}, \quad (5)$$

modulo a different overall factor given by the different tension of the two objects (3). Being defined by the same sigma-model, both a fundamental string and a (p, q) -string will admit the same classical giant magnon solution, with just a different dependence on τ . One can then immediately generalize the result of [5] and write down the strong coupling dispersion relation for giant magnons with both electric and magnetic charge

$$(E - J)_{(p,q)} = \frac{\sqrt{\lambda} |p - q\tau|}{\pi} \left| \sin \frac{k}{2} \right|, \quad (6)$$

where k represents the world-sheet momentum of the giant magnon.

This can be justified at large (infinite) N and finite coupling because long classical D-strings that do not self-intersect can not break in flat space. Remember that when N is large the $AdS_5 \times S^5$ is very large in string units and can be replaced by flat space at scales much larger than the string scale.

Now we shall compute the dispersion relation of giant magnon solutions corresponding to (p, q) -strings. To this end, we would need to consider a 2-impurity state with large momentum, which is given at weak coupling by some BMN-like operator [20]

$$|k, J\rangle \sim \sum_{\ell=0}^J e^{2\pi i k \ell / J} \text{Tr}(Z^\ell [X, Z] Z^{J-\ell} [Y, Z]). \quad (7)$$

At strong coupling, the Z 's are described by diagonal modes, whereas X and Y are described by off-diagonal modes [12]. Here, we include the possibility that the off-diagonal degrees of freedom might be described by magnetic charges as well and we write

$$|k, J\rangle \sim \sum_{\ell=0}^J e^{2\pi i k \ell / J} \sum_{i,j=1}^N z_i^\ell (M^\dagger)_j^i z_j^{J-\ell-2} (\widetilde{M}^\dagger)_i^j |0\rangle. \quad (8)$$

By analogy, this can be thought of as a magnetic trace operator. The z 's represent the (collective) coordinates of eigenvalues, while the M^\dagger and \widetilde{M}^\dagger represent the Fock space raising operators for the corresponding states with given electric and magnetic charges. This is allowed so long as one can argue that the electric/magnetic impurities are well separated from each other and that the system does not back-react substantially in their presence. We assume this is consistent at this stage. One can argue this is allowed by noticing that the z variables are also excitations of the eigenvalue degrees of freedom, so the charged objects are in a sea of photon superpartners that can keep them apart from each other. One can ignore the bound state problem if there are sufficiently many such photons. Also, the force between eigenvalues due to the presence of the charged particles is of order one, while the force due to the collective repulsion of eigenvalues scales like a power of N . In the expression (8) the sum over eigenvalue pairs ensures that the discrete gauge symmetry of permutation of eigenvalues is implemented. The vacuum also contains the information of the wave function of the collective coordinate degrees of freedom.

Following [12], the sum is done over z 's on the sphere S^5 and it is interpreted as an element of the Fock space of the off-diagonal modes. The sum has a very sharp maximum norm in the amplitude for a fixed angle between the eigenvalues lying on a diameter of the S^5 . We can compute the energy of the state (8) and obtain the functional form of the dispersion relation valid for arbitrary

coupling

$$(E - J)_{(p,q)} = \sqrt{1 + \frac{h(y, \tilde{\tau})}{\pi^2} \sin^2 \frac{k}{2}}. \quad (9)$$

Here $h(y, \tilde{\tau})$, with $y \equiv 1/\lambda$ and $\tilde{\tau} \equiv |p - q\tau|$, is an unknown function that we wish to determine. We are using the variable y rather than λ to stress the fact that we are expanding around $\lambda \rightarrow \infty$. An important point is that, although we don't know *a priori* the function $h(y, \tilde{\tau})$, the emergent geometry approach we are using guarantees that we will have the square root behavior for any value of the coupling constant, as explained in [12].

The scope of this letter is to demonstrate that the only functional dependence compatible with the S-duality of $\mathcal{N} = 4$ super Yang-Mills is

$$h(y, \tilde{\tau}) = \frac{1}{y} |p - q\tau|^2. \quad (10)$$

This means that S-duality protects this quantity from being renormalized, so that the (generalization to (p, q) -strings of the) one-loop result of [7, 8] is in fact exact to all orders in perturbation theory.

The first step toward proving (10) makes use of the emergent geometry approach described in the previous section. As explained in [12], the dependence on the coupling constant appearing under the square root of the giant magnon dispersion relation (9) is fixed by the geometry of the eigenvalue distribution. In particular, it depends on the mass of the off-diagonal modes and on the radius of the 5-sphere. In the case of the (p, q) -strings, the mass of the off-diagonal modes (4) fixes

$$h(y, \tilde{\tau}) = f(y, \tau) |p - q\tau|^2, \quad (11)$$

with $f(y, \tau)$ unknown.

We use at this point S-duality. We apply a transformation

$$\tau \rightarrow -\frac{1}{\tau}, \quad y \rightarrow \frac{y}{|\tau|^2}, \quad (12)$$

to $h(y, \tilde{\tau})$ in (11) (which is the function corresponding to a (p, q) -string) and set the result equal to the $h(y, \tilde{\tau})$ for a $(q, -p)$ -string. S-duality maps in fact $p \rightarrow q$ and $q \rightarrow -p$. Redefining for convenience $g(u, v) \equiv u f(u, v)$ we find that $g(y, \tau)$ has to satisfy

$$g\left(\frac{y}{|\tau|^2}, -\frac{1}{\tau}\right) = g(y, \tau). \quad (13)$$

This is a modular equation whose only solution in the limit $N \rightarrow \infty$ and $\lambda \rightarrow \infty$, we claim, is a constant. To prove this we first need to invoke planarity. This is justified from weak coupling to strong coupling by arguing that integrability interpolates from the weak coupling spin chain to strong coupling by summing planar diagrams only. Beyond that, there can be $1/N$ corrections, but these are ignored at large N .

In this limit the function $g(y, \tau)$ cannot depend on τ separately, so that (13) becomes

$$g\left(\frac{y}{|\tau|^2}\right) = g(y). \quad (14)$$

Taking derivatives with respect to y of this equation around $y = 0$, we see that all these derivatives have to be zero and therefore g is a constant. All of this assumes that the $y \rightarrow 0$ limit is smooth (*i.e.* $y = 0$ is not an essential singularity), as one would expect to be the case in the planar limit. Another way to see this is by recalling that the θ angle cannot arise in perturbation theory, as it is a non-perturbative effect given by instantons. Our formula for the dispersion relation (9) is inherently perturbative, thus we expect that g cannot depend on variations of θ around $y = 0$ and then it has to be constant.

This is valid in an expansion around $y = 0$. We can extrapolate this result to weak coupling, where we can compute the constant value of g . This is equal to $g = 1$ [7, 8], thus proving (10).

– *Discussion.* We have provided an argument for the non-renormalization of the dispersion relation of giant magnon solutions of type IIB strings in $AdS_5 \times S^5$. Our proof is not based on diagrammatic techniques, but rather on bootstrapping S-duality, planarity, and a certain ma-

trix model arising in the strong coupling limit of $\mathcal{N} = 4$ super Yang-Mills on $\mathbb{R} \times S^3$. An important point to stress is that this computation was possible because around $N \rightarrow \infty$ and $\lambda \rightarrow \infty$ (the regime we have focused on in this letter) the regions of validity of the “electric” description and of its “magnetic” dual do in fact overlap, thus making possible to apply S-duality. We have also identified the complete spectrum of (p, q) -giant magnons in the process, obtaining a collections of states that can be mapped into each other consistently under S-duality.

Recently there has been much interest in the integrability of the AdS_4/CFT_3 correspondence for M2-branes proposed in [9] (see [21]). According to the logic followed in this letter, one should not expect a similar non-renormalization theorem to hold in that context. One has in fact no S-duality in type IIA string theory (nor in M-theory) to protect the coupling dependence of the dispersion relation of giant magnons in $AdS_4 \times CP^3$, and one finds in fact that this quantity depends on an interpolating function $h(\lambda)$, known only in the weak and strong coupling limits [10, 22].

– *Acknowledgments.* This work was supported in part by the U.S. Department of Energy under grant DE-FG02-91ER40618 and NSF grant PHY05-51166.

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- [23] ${}^L G$ represents the Langlands dual group. In this letter we are interested in the case $G = U(N)$, which is self-dual.