HIGHER DIMENSIONAL ALGEBROIDS AND CROSSED COMPLEXES

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DECLARATION

The work of this thesis has been carried out by the candidate and contains the results of his own investigations . The work has not already been accepted in substance for any degree , and is not being concurrently submitted in candidature for any degree . All sources of information have been acknowledged in the text .

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SUMMARY

The equivalence between the category of crossed modules (over groups) and the category of special double groupoids with connections and with one vertix was proved by R.Brown and C.B.Spencer . More generally , C.B.Spencer and Y.L.Wong have shown that there exists an equivalence between the category of 2-categories and the category of double categories with connections .

R.Brown and P.J.Higgins have generalised the first result : they proved that there exists an equivalence between the category of ω-groupoids and the category of crossed complexes (over groupoids) .

In this thesis we develop a parallel theory in a more algebraic context , with expectation of applications in non-abelian homological and homotopical algebra . We prove an equivalence between the category of crossed modules (over algebroids) and the category of special double algebroids with connections . Moreover we prove a similar result for the 3-dimensional case , that is , we prove that there exists an equivalence between the category $(\underline{Crs})^3$ of 3-truncated crossed complexes and the category $(\underline{\omega\text{-}Alg})^3$ of 3-tuple algebroids . Also we end this work by giving a conjecture for the higher dimensional case . In particular , we have

<u>Theorem</u>: The functors γ , λ form an adjoint equivalence

$$\gamma : \underline{DA}^! \longleftrightarrow \underline{C} : \lambda$$

where $\underline{DA}^!$ is the category of special double algebroids with connections and $\underline{C}$ is the category of crossed modules .

Theorem: The functors γ , λ form an adjoint equivalence

$$\gamma : (\underline{\omega\text{-Alg}})^n \longleftrightarrow (\underline{\text{Crs}})^n : \lambda$$

for n = 3 , 4 .

Finally we give a conjecture whose validity would be sufficient for the general equivalence of categories of ω-algebroids and crossed complexes .

In chapter VI we explain some results which have been obtained in the case of groupoids and higher dimensional groupoids , and suggest the possibility of obtaining similar results in the case of algebroids and higher dimensional algebroids .

CONTENTS

INTRODUCTION

A. AIMS AND BACKGROUND:

1. Overall aim:

There are many useful analogies between the theory of groups and the theory of algebras which are exploited for example in homological algebra . Some interesting generalisations of groups are groupoids , crossed modules , crossed complexes , double groupoids and ω-groupoids , dating respectively from 1926 [Brandt - 1] , 1946 [J.H.C.Whitehead - 1,2] , 1949 [Blakers - 1] , 1965 [Ehresmann - 1] and 1977 [Brown-Higgins - 8] . Corresponding to groupoids as generalisations of groups , we have algebroids as generalisations of algebras , a theory due to B.Mitchell (1972) . There are also notions of crossed modules of algebras . But a theory of double and n-tuple algebroids does not seem to be available , and it is our aim to investigate this idea .

In order to see the motivation for this investigation and the kind of result to be expected , we first recall some facts on the group case .

2. Crossed modules , crossed complexes in groups and ω-groupoids:

First , a group homomorphism $\partial : M \dashrightarrow P$ is said to be a crossed P-module (in groups) if there is given an action of P on M , $(p,m) \dashrightarrow {}^{P}m$ which satisfies the following axioms :
(i) $\partial({}^{P}m) = p\,(\partial m)\,p^{-1}$ (ii) ${}^{\partial m}m' = m\,m'\,m^{-1}$ for $m,m' \in M$ and $p \in P$. Standard examples of crossed modules are :

1) the inclusion N ---→ P of a normal subgroup N of the group P , with the action of P on N given by conjugation ;

2) the zero morphism 0 : M ---→ P in which M is a P-module in the usual sense ;

3) the boundary map $\partial : \pi_2(X,Y,x_0) \longrightarrow \pi_1(Y,x_0)$ from the second relative homotopy group to the fundamental group with the standaed action of $\pi_1(Y,x_0)$ on $\pi_2(X,Y,x_0)$.

As this last example suggests , crossed modules can be used to model certain homotopy types . In fact from the stand point of homotopy theory , crossed modules should be viewed as "2-dimensional groups" . It is reasonable to ask then , what are the n-dimensional groups (or crossed modules) ?

J.H.C.Whitehead gave a partial answer to this by introducing what he called a "homotopy system" , but which are now called crossed complexes . These gadgets consist of a sequence of groups

$$\cdots \xrightarrow{\partial_n} C_n \xrightarrow{\partial_{n-1}} C_{n-1} \xrightarrow{\partial_{n-2}} \cdots \xrightarrow{\partial_3} C_3 \xrightarrow{\partial_2} C_2 \xrightarrow{\partial_1} C_1 \rightrightarrows C_0$$

where C_0 is a single point and satisfy the axioms ;

i) ∂_1 is a crossed module ;

ii) C_n is abelian for n ⟩ 3 ;

iii) $\partial^2 = 0$;

iv) C_1 acts on C_n , n ⟩ 2 and $\partial_1 C_2$ acts trivially on C_n for n ⟩ 3 .

The standard example of a crossed complex is obtained from a pointed filtered space (c.f. [Br-3]) .

Work in homotopy theory has developed the well known notion of "groupoids" , which are categories in which every arrow is invertible .

Since a crossed module has been considered as a "higher dimensional group" , the question aries : what is a higher dimensional groupoid ? Ehresmann [Eh-1] has defined the notion of double groupoid . R-Brown and C.B.Spencer have proved that there exists an equivalence between the category of crossed modules (over groups) and the category of double groupoids with special connections and one vertex . But the general case has been defined in [B-Hi-8] ; namely they have defined ω-groupoids and crossed complexes (over groupoids) by using the cubical set notion . Moreover they have proved in [B-Hi-2] there exist an equivalence between the category of ω-groupoids and the category of crossed complexes (over groupoids) .

The above discussion of the development of group theory in this direction is summarised in the diagram

Groups ⟶ Groupoids ⟶ Double groupoids ⟶ ω-groupoids .

There are in fact a remarkable collection of equationally defined categories of (many-sorted) algebras which are non-trivially equivalent to ω-groupoids . These are summarised in the following diagram :

```
                          [J-1]                           [B-Hi-3]
poly-T-Complexes <----> cubical T-complexes ------> ω-groupoids
       ^                                             ↗    ^
       |                                                  |
 [J-1] |                             [B-Hi-2]            | [B-Hi-4]
       |                                                  |
       v                                       ↙          v
Simplicial T-  <------> crossed complexes <------> ∞-groupoids
 complexes      [As-1]                      [B-Hi-4]
```

in which the arrows denote explicit functors which are equivalence of categories . The symbols in square brackets give references to the proofs .

3. Crossed modules and Crossed complexes over algebras:

The work of [Ge-1] essentially involves the notion of crossed modules in associative and commutative algebras under a different name . Also the work of [K-L-1] in algebraic K-theory has introduced crossed modules of Lie algebras . The definition of crossed modules in associative algebras is given on page (9,10) .

The notion of crossed modules of algebras has been generalised to crossed complexes over algebras [Po-3] , namely ;

Let R be a commutative ring and let K be an R-algebra . A crossed complex of R-algebras is a sequence of R-algebras

$$\cdots \longrightarrow C_n \xrightarrow{\partial_n} C_{n-1} \longrightarrow \cdots \xrightarrow{\partial_2} C_2 \xrightarrow{\partial_1} K$$

in which

i) ∂_1 is a crossed K-module ,

ii) C_i for $i \geq 1$ is a K-module on which $\partial_1 C_1$ operates trivially

iii) for $i \geq 1$, $\partial_{i+1} \partial_i = 0$.

Now one can ask , what are the higher dimensional algebras ? In this thesis we shall give a partial answer to this question and we will give some extra conjectures .

4. Algebraic geometry:

The idea of this work arose from the consideration of bringing crossed module ideas into commutative algebra and algebraic geometry ; namely an ideal in a polynomial ring corresponds to an affine algebraic variety . Crossed modules in commutative algebras are generalisations of ideals . One would like to know the geometric analogue of a crossed module , but nothing seems to be known on this question .

The original motivation for this thesis was to see if it would be easier to find analogues of "double commutative algebroids" in algebraic geometry , assuming such were equivalent to crossed modules . This lead to the problem of finding analogues for algebroids of the work of Brown - Higgins on ω-groupoids , and this problem has since occupied our full attention .

There are still many problems in relating this work to algebraic geometry , but we believe this will eventually be possible .

B. STRUCTURE AND MAIN RESULTS:

In chapter I we give an example to show how algebras are appropriately generalised to algebroids and we show that the category of R-algebroids is a monoidal closed category . We give the definition of a crossed module over an associative algebra and introduce the definition of a crossed module over an algebroid . Also we deduce some properties of crossed modules similar to the well known properties of crossed modules over groups .

In chapter II , we define an algebroid in one higher dimension . In fact we introduce the notion of a double algebroid by using double categories ; namely a double R-algebroid D is four related R-algebroids

$$(D,D_1,\partial_1^i,\varepsilon_1,+_1,*_1,\cdot\ )\ ,\ (D,D_2,\partial_2^i,\varepsilon_2,+_2,*_2,\cdot_2)$$

$$(D_1,D_0,\delta_1^i,\varepsilon,+,*,.)\quad ,\quad (D_2,D_0,\delta_2^i,\varepsilon,+,*,.)$$

where i = 0,1 and these algebroids satisfies the following axioms :

i) $\varepsilon_2^i \, \partial_2^j = \varepsilon_1^j \, \partial_1^i$ $\qquad$ $i,j \in \{0,1\}$

ii) $\partial_2^i(\alpha +_1 \beta) = \partial_2^i\alpha + \partial_2^i\beta \quad , \quad \partial_1^i(\alpha +_2 \beta) = \partial_1^i\alpha + \partial_1^i\beta$

$\partial_2^i(\alpha *_1 \beta) = \partial_2^i\alpha * \partial_2^i\beta \quad , \quad \partial_1^i(\alpha *_2 \beta) = \partial_1^i\alpha * \partial_1^i\beta$

for $i = 0,1$, $\alpha,\beta \in D$ and both sides are defined .

iii) $r \cdot_1 (\alpha +_2 \beta) = (r \cdot_1 \alpha) +_2 (r \cdot_1 \beta)$,

$r \cdot_2 (\alpha +_1 \beta) = (r \cdot_2 \alpha) +_1 (r \cdot_2 \beta)$

$r \cdot_1 (\alpha *_2 \beta) = (r \cdot_1 \alpha) *_2 (r \cdot_1 \beta)$

$r \cdot_2 (\alpha *_1 \beta) = (r \cdot_2 \alpha) *_1 (r \cdot_2 \beta)$

$r \cdot_1 (s \cdot_2 \alpha) = s \cdot_2 (r \cdot_1 \alpha)$

for $\alpha,\beta \in D$, $r,s \in R$ and both sides are defined .

iv) $(\alpha +_1 \beta) +_2 (\gamma +_1 \delta) = (\alpha +_2 \gamma) +_1 (\beta +_2 \delta)$

$(\alpha *_1 \beta) *_2 (\gamma *_1 \delta) = (\alpha *_2 \gamma) *_1 (\beta *_2 \delta)$

$(\alpha +_1 \beta) *_2 (\gamma +_1 \delta) = (\alpha *_2 \gamma) +_1 (\beta *_2 \delta)$

$(\alpha +_2 \beta) *_1 (\gamma +_2 \delta) = (\alpha *_1 \gamma) +_2 (\beta *_1 \delta)$

for $\alpha,\beta,\gamma,\delta \in D$ and both sides are defined .

v) $\varepsilon_1(a + a_1) = \varepsilon_1 a +_2 \varepsilon_1 a_1$, $\varepsilon_2(b + b_1) = \varepsilon_2 b +_1 \varepsilon_2 b_1$

for $a,a_1 \in D_1$, $b,b_1 \in D_2$ and the additions are defined .

Thus we get a category of double R-algebroids <u>DA</u> .

We can ask now what is the relation between the category of crossed modules (over algebroids) and the category of double R-algebroids . At this stage we prove the following ;

<u>Proposition</u>: If D is a double R-algebroid , then we have two crossed modules associated with D . That is , there exist two functors from the category of double R-algebroids to the category of crossed modules (over algebroids) .

In the end of this chapter we give some examples on this notion .

In chapter III , we define the notion of a special double R-algebroid (this is a double R-algebroid with $D_1 = D_2$) and we define a "thin" structure on D which is a morphism $\theta : \square D_1 \dashrightarrow D$ (where $\square D_1$ is a double R-algebroid with commutating squares) . An element $\theta(a \overset{c}{\underset{b}{}} d)$ is called <u>thin</u> , where $a,b,c,d \in D_1$. Also we define a <u>connection</u> on D to be a pair of functions $\Gamma,\Gamma':D_2 \dashrightarrow D$ which satisfy

i) $\Gamma'a *_2 \Gamma a = \varepsilon_1 a$, $\Gamma'a *_1 \Gamma a = \varepsilon_2 a$

ii) $\Gamma'(ab) = (\Gamma'a *_1 \varepsilon_1 a) *_2 (\varepsilon_2 a *_1 \Gamma'b)$

$\Gamma(ab) = (\Gamma a *_1 \varepsilon_2 b) *_2 (\varepsilon_1 b *_1 \Gamma b)$

iii) $\Gamma'(a + a_1) *_2 (\alpha +_1 \beta) *_2 \Gamma(d+d_1) = (\Gamma'a *_2 \alpha *_2 \Gamma d) +_2 (\Gamma'a_1 *_2 \beta *_2 \Gamma d_1)$

where $\alpha,\beta \in D$ with boundaries $(a \overset{c}{\underset{b}{}} d)$, $(a_1 \overset{c}{\underset{b}{}} d_1)$ respectively

iv) $\Gamma'ra *_2 (r \cdot_1 \alpha) *_2 \Gamma rd = r \cdot_2 (\Gamma'a *_2 \alpha *_2 \Gamma d) = \Gamma'a *_2 (r \cdot_2 \alpha) *_2 \Gamma d$.

<u>Theorem 1</u> : Let D be a special double R-algebroid with connection Γ' , Γ . Then there is a morphism of special double R-algebroids $\theta : \square D_1 \dashrightarrow D$ which is the identity on D_1 and $\Gamma a = \theta(a \overset{a}{\underset{1}{}} 1)$, $\Gamma'b = \theta(1 \overset{1}{\underset{b}{}} b)$, where $a,b \in D_1$.

In fact , the reason for defining these two structures on D is that ; the axioms for connection do not involve the addition or scalar multiplication , where as the axioms for a thin structure do ; also it is easier to deal with connection than with thin structure .

Also we define a morphism between two special double R-algebroids with connection . Thus we have a category DA! of special double R-algebroids with connection . Then we get a functor from the category of special double R-algebroids with connection to the category of crossed modules .

Now , to get a functor from the category of crossed modules to the category of special double R-algebroids with connection we introduce the notion of a "folding operation: Φ which has the effect of "folding" all edges $\alpha \in D$ onto the edge $\partial_1^0\alpha$. We prove ;

Proposition: There exists a functor from the category of crossed modules to the category of special double R-algebroids with connections .

In fact , we prove

Theorem 2: These functors are equivalent .

Finally we introduce the notion of a reflection on a special double R-algebroid which gives an equivalence between the two algebroid structures .

In chapter IV , we define an ω-algebroid (without connections) by using the cubical complex idea namely ;

An ω-algebroid (without connections) $\underline{A} = \{A_n; \partial_i^\alpha , \varepsilon_i\}$ is a cubical complex and for $n \geqslant 1$, A_n has n algebroid structures over A_{n-1} of the form $(A_n, +_i, *_i, \cdot_i, \partial_i^0, \partial_i^1, \varepsilon_i)$ related appropriately to each other and to ∂_i^0 , ∂_i^1 , ε_i . Thus we can define finite dimensional versions of the above definition . Therefore we get ;

Algebras $\to$ Algebroids $\to$ Double algebroids $\to$ ω-Algebroids .

Also we define a crossed complex $\underline{M}$ (over algebroid) to consist of a sequence of morphisms of R-algebroids over M_0

$$\underline{M}: \quad \ldots \to M_n \xrightarrow{\delta} M_{n-1} \xrightarrow{\delta} \ldots\ldots \xrightarrow{\delta} M_2 \xrightarrow{\delta} M_1$$

satisfying the relations ;

i) each $\delta : M_n \to M_{n-1}$, $n \geq 2$ is the identity on M_0 .

ii) M_1 operates on the right and on the left on each M_n $(n \geq 2)$, by actions $(a,m) \to {}^a m$, $(m,b) \to m^b$, whenever $m \in M_n(x,y)$, $a \in M_1(w,x)$, $b \in M_1(y,z)$.

iii) If $m \in M_n(x,y)$, $m' \in M_2(y,z)$, $m'' \in M_2(w,x)$, then

$$m^{\delta m'} = \begin{cases} 0_{xz} & \text{if } n \geq 3 \\ mm' & \text{if } n=2 \end{cases}$$

$${}^{\delta m''}m = \begin{cases} 0_{wy} & \text{if } n \geq 3 \\ m''m & \text{if } n=2 \end{cases}$$

Finally we prove that ;

<u>Theorem 3</u>: There exists a functor γ from the category of ω-algebroids (without connections) to the category of crossed complexes (over algebroids) .

In chapter V , §1 we define an ω-algebroid with connections and the morphisms between them and also we give the definition of a finite dimensional versions of an ω-algebroid .

In § 2 we introduce the notion of "folding operation" Φ , which has a similar effect to the folding operation in the two dimensional case . Also we give the relations between this operation and the axioms of 3 and 4 - tuple algebroids , that is , Proposition: Let $a \in A_n$. Then Φa belongs to the associated crossed complex $\gamma\underline{A}$.

Proposition: i) If $a,b \in A_n$ with $\partial_j^\alpha a = \partial_j^\alpha b$, for $\alpha = 0,1$, then $\Phi(a +_j b) = \Phi a +_n \Phi b$.

ii) For $n = 3,4$, if $a,b \in A_n$ with $\partial_j^1 a = \partial_j^0 b$, then

$$\Phi(a *_j b) = {}^{u_j a}(\Phi b) +_n (\Phi a)^{v_j b} .$$

where $u_j a = \partial_1^0 \ldots \partial_{j-1}^0 \partial_{j+1}^0 \ldots \partial_n^0 a$ and $v_j b = \partial_1^1 \ldots \partial_{j-1}^1 \partial_{j+1}^1 \ldots \partial_n^1 b$.

iii) If $a \in A_n$ and $r \in R$, then

$$\Phi(r \cdot_j a) = r \cdot_n \Phi a .$$

Proposition: 1) For $n = 3$, let $a \in A_2$. Then $\Phi\varepsilon_i a = \Phi\Gamma_j a = \Phi\Gamma'_j a = 0$ in dimension 3 for $1 \leqslant i \leqslant 3$ and $1 \leqslant j \leqslant 2$.

2) For $n = 4$, let $a \in A_3$. Then $\Phi\varepsilon_i a = \Phi\Gamma_j a = \Phi\Gamma'_j a = 0$ in dimension 4 for $1 \leqslant i$ n$\leqslant 4$ and $1 \leqslant j \leqslant 3$.

Also we define a thin structure on $\underline{A}$ as follows ; let $a \in A_n$, then a is called thin if and only if $\Phi a = 0$.

In \$3 we construct the coskeleton in terms of "shells" for an n-tuple algebroid and we define ∂_i^α , ε_i, Γ_i , Γ'_i and the operations on $\square A_n$ to prove the following ;

Proposition: If $(A_n,\ldots,A_0)$ is an n-tuple algebroid , then $(\square A_n, A_n,\ldots,A_0)$ is an (n+1)-tuple algebroid .

Proposition: Let $\underline{A}$ be an ω-algebroid and let $\underline{M} = \gamma\underline{A}$ be its associated crossed complex . Let $a \in \square A_{n-1}$ and $\xi \in M_n(u,v)$ where $u = \beta_0 a$, $v = \beta_1 a$. Then a neccessary and sufficient condition for the existence of $b \in A_n$ such that $\underline{\partial} b = a$ and $\Phi b = \xi$ is that $\delta\xi = \delta\Phi\underline{\partial} a$. Further if b exist , it is unique .

In $4 we construct a functor λ from the category of 3-truncated crossed complexes to the category of 3-tuple algebroids by using the folding operation . Also we prove that ;

<u>Theorem 4</u>: The functors γ , λ form an adjoint equivalence

$$\gamma : (\underline{\omega\text{-Alg}})^3 \longleftrightarrow (\underline{\mathrm{Crs}})^3 : \lambda \ .$$