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GENERALIZED SOLENOIDS AND C*-ALGEBRAS

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We present the continuous graph approach for some generalizations of the Cuntz-Krieger algebras. These algebras are simple, nuclear, and purely infinite, with rich K-theory. They are tied with the dynamics of a shift on an infinite path space. Interesting examples occur when the vertex spaces are unions of tori, and the shift is not necessarily expansive. We also show how the algebra of a continuous graph could be thought as a Pimsner algebra.

Introduction.

Recent papers are dealing with different generalizations of the Cuntz-Krieger algebras $\mathcal{O}_A$ (see [**Pi**], [**P1**], [**D2**], [**AR**], etc). The exact relationship between these approaches remains to be explored, but certainly there are overlaps. In [**Pi**], the author considers a Hilbert bimodule H over a C*-algebra, and creation operators on a corresponding Fock space. These operators generate the Toeplitz algebra $\mathcal{T}_H$ and, taking a quotient of this, one obtains the algebra $\mathcal{O}_H$. If the Hilbert bimodule is projective and finitely generated over an abelian, finite dimensional C*-algebra, then one recovers the algebras $\mathcal{O}_A$.

In [**P1**], the starting point is a Smale space (a compact metric space endowed with an expansive homeomorphism with canonical coordinates), on which one defines the stable and unstable equivalence relations. The associated C*-algebras have natural shift automorphisms, and the crossed products are the so called Ruelle algebras. These are strongly Morita equivalent to particular Cuntz-Krieger algebras if the Smale space is a topological Markov shift.

Our point of view is to start with a continuous oriented graph (or diagram) E, to consider the space of one-sided infinite paths (obtained by concatenation of edges in E), and to associate a groupoid (à la Renault) using the unilateral shift on this path space. The C*-algebra of this groupoid plays the role of a continuous version of the Cuntz-Krieger algebras, since these could be obtained by the same construction from a finite graph defined by a 0-1 matrix. In many cases, this groupoid algebra is simple, purely infinite, with computable K-theory. This approach offers more freedom for constructing easy, concrete examples, with prescribed K-theory. It should be mentioned that C*-algebras associated with discrete graphs were studied in [**KPRR**], [**KPR**], [**KP**]. See also the survey [**K2**].

The continuous graph approach is very similar to the point of view of polymorphisms or correspondences, introduced earlier in a measure theoretical context by Vershik and Arzumanian (see [**AR**] for a precise definition and references).

Even though our groupoid algebras could be obtained also by using the Pimsner approach, with a right choice of the Hilbert bimodule, we feel that the present point of view has certain advantages, beeing tied with the dynamics of a shift. For example, even in a case where this shift is not expansive, so the space of two-sided infinite paths has no obvious Smale space structure, we will prove that the corresponding algebra is simple and purely infinite.

In the particular case when the vertex space is a disjoint union of tori, we call the corresponding space of paths a *generalized solenoid*, and we obtain results similar to those of Brenken (see [**B**]). It is interesting to notice how these fairly complicated dynamical systems appear in a natural way from embeddings of toral algebras.

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1. Continuous graphs and dynamical systems.

Definition 1.1. By a *continuous graph* we mean a closed subset

$$E \subset V \times \{1, 2, ..., m\} \times V,$$

where V is a compact metric space. The elements of V are called *vertices*, and the elements of E are called *edges*. The set $\{1, 2, ..., m\}$ is used to label different edges between the same pair of vertices. The graph is *oriented* when for each edge $e = (v, k, w)$ we specify the origin $o(e) = v$ and the terminus $t(e) = w$.

In this paper we consider dynamical systems $(X_+, \sigma_+), (X, \sigma)$ built from a continuous oriented graph E. The space X_+ is the space of one-sided infinite paths,

$$X_+ = \{(x_i, k_i)_{i=0}^{\infty} \mid (x_i, k_i, x_{i+1}) \in E, i \geq 0\},$$

and $\sigma_+ : X_+ \to X_+$ is the unilateral shift,

$$\sigma_+(x_i, k_i)_p = (x_{p+1}, k_{p+1}).$$

The space X is the space of two-sided infinite paths, and σ is the bilateral shift. The dynamical system (X_+, σ_+) unifies in a natural way the notion of a continuous map $T : V \to V$, a (finitely-generated) semigroup or group of continuous transformations $\mathcal{S} : V \to V$ and the (unilateral) Markov shifts (when V is a finite set and E is defined by a 0-1 matrix)(see [**F**]). For example, if $T : V \to V$ is a continuous map, we can take $E = \Gamma(T)$, the graph of T (in this case $m = 1$, and we omit it). Then X_+ is homeomorphic to V, and σ_+ is conjugated to T.

Proposition 1.2. *The dynamical system (X, σ) could be obtained from (X_+, σ_+) by the usual inverse limit process by which one associates a homeomorphism to a continuous onto map.*

Proof. Indeed, let

$$\tilde{X} = \left\{ (\xi_n) \in \prod_1^\infty X_+ \mid \sigma_+(\xi_{n+1}) = \xi_n \right\}.$$

We have $\pi : \tilde{X} \to X_+, \pi(\xi_1\xi_2...) = \xi_1$, and $\tilde{\sigma}_+ : \tilde{X} \to \tilde{X}, \tilde{\sigma}_+(\xi_1\xi_2...) = (\sigma_+(\xi_1)\xi_1\xi_2...)$, such that $\sigma_+\pi = \pi\tilde{\sigma}_+$. Since

$$X_+ = \left\{ (e_n) \in \prod_1^\infty E \mid t(e_n) = o(e_{n+1}) \right\},$$

$\tilde{X} \subset \prod_1^\infty \prod_1^\infty E$ could be identified with X, the space of two-sided infinite paths, and $\tilde{\sigma}_+$ with the bilateral shift σ. □

Definition 1.3. For each continuous oriented graph E we define its dual (or opposite) graph $\hat{E}$ by

$$\hat{E} = \{(x, k, y) \mid (y, k, x) \in E\}.$$

This way we get dynamical systems $(\hat{X}_+, \hat{\sigma}_+), (\hat{X}, \hat{\sigma})$, where $\hat{X}_+$, $\hat{X}$ are constructed from $\hat{E}$, and $\hat{\sigma}_+$, $\hat{\sigma}$ are the unilateral and bilateral shift, respectively. Of course, the systems (X, σ) and $(\hat{X}, \hat{\sigma}^{-1})$ are conjugated. But (X_+, σ_+) and $(\hat{X}_+, \hat{\sigma}_+)$ could be very different.

Example 1.4. Take $V = \mathbf{T}$, the unit circle, and E the graph of the map $z \mapsto z^2$,

$$E = \{(z, z^2) \mid z \in \mathbf{T}\}.$$

Then $X_+ = \mathbf{T}, \sigma_+(z) = z^2$, and $\hat{X}_+$ is a solenoid,

$$\hat{X}_+ = \{(z_1, z_2, ...,) \mid z_n \in \mathbf{T}, z_{n+1}^2 = z_n, n \geq 1\},$$

$$\hat{\sigma}_+(z_1, z_2, ...) = (z_2, z_3, ...).$$

Note that if V has a group structure and $E \subset V \times V$ is a subgroup, then X_+ and X have also natural group structures, with componentwise multiplication.

2. The C*-algebra of a continuous graph.

In the case the two projections $o, t : E \to V, o(x,k,y) = x$ and $t(x,k,y) = y$ are onto local homeomorphisms, we can associate to the graph E a C*-algebra $C^*(E)$, using the Renault groupoid of the dynamical system (X_+, σ_+). The space X_+ is endowed with a metric defining the product topology. If δ denotes the metric on V, then one can take

$$d((x_i,k_i),(x'_i,k'_i)) = \sum_{i\geq 0} \frac{\delta(x_i,x'_i)+ \mid k_i - k'_i \mid}{2^i}$$

as a metric on X_+. Similarly, we obtain a metric on X.

The unilateral shift σ_+ is a local homeomorphism, and we consider the following locally compact r-discrete groupoid:

$$\begin{aligned}\Gamma &= \Gamma(X_+, \sigma_+)\\ &= \{(x,n,y) \in X_+ \times \mathbf{Z} \times X_+ \mid \exists k, l \geq 0, n = k-l, \sigma_+^k(x) = \sigma_+^l(y)\}.\end{aligned}$$

The range map, the source map, and the operations are given as follows:

$$r(x,n,y) = x, \quad s(x,n,y) = y,$$
$$(x,n,y)(y,p,z) = (x,n+p,z), \quad (x,n,y)^{-1} = (y,-n,x).$$

The unit space of Γ is X_+, if we identify $(x,0,x)$ with x. A basis of open sets for Γ is given by

$$Z(U,V,k,l) = \{(x, k-l, (\sigma_+^l\mid_V)^{-1} \circ \sigma_+^k(x)), x \in U\},$$

where U and V are open subsets of X_+, and k, l are such that $\sigma_+^k \mid_U$ and $\sigma_+^l \mid_V$ are homeomorphisms with the same open range.

Definition 2.1. Given a continuous oriented graph E with the maps o, t onto local homeomorphisms, we define its C*-algebra $C^*(E)$ to be $C^*(\Gamma)$, the C*-algebra of the Renault groupoid associated with the dynamical system (X_+, σ_+).

To understand the structure of $C^*(E)$, consider the homomorphism $c : \Gamma \to \mathbf{Z}$, $c(x,n,y) = n$, and let's denote by R^∞ the subgroupoid $c^{-1}(0)$. If we denote by B the C^*-algebra of the equivalence relation R^∞, the local homeomorphism σ_+ induces a *-endomorphism α of B by the formula

$$\alpha(f)(x,y) = \frac{1}{\sqrt{p(\sigma_+(x))p(\sigma_+(y))}} f(\sigma_+(x), \sigma_+(y)), f \in C_c(R^\infty),$$

where for $x \in X_+$, $p(x)$ is the number of paths z such that $\sigma_+(z) = x$. Moreover, assuming that σ_+ is not one-to-one, α is induced by a non unitary isometry v, in the sense that $\alpha(f) = vfv^*$, where

$$v(x,n,y) = \begin{cases} (p(\sigma_+(x)))^{-1/2}, & \text{if } n = 1 \text{ and } y = \sigma_+(x)\\ 0, & \text{otherwise.}\end{cases}$$

Indeed, $v^*v = 1$, and

$$vv^*(x,n,y) = \begin{cases} p(\sigma_+(x))^{-1} & \text{if } \sigma_+(x) = \sigma_+(y) \\ 0, & \text{otherwise.} \end{cases}$$

Thus, α is a proper corner endomorphism of B, and $C^*(E)$ is isomorphic to the crossed product $B \times_\alpha \mathbf{N}$ (see [**R1**]).

In order to compute the K-theory of $C^*(E)$, we can use the exact sequence

$$\begin{array}{ccccc} K_0(C^*(R^\infty)) & \xrightarrow{\text{id}-\alpha_0} & K_0(C^*(R^\infty)) & \xrightarrow{i_0} & K_0(C^*(E)) \\ \partial_1 \uparrow & & & & \downarrow \partial_0 \\ K_1(C^*(E)) & \xleftarrow{i_1} & K_1(C^*(R^\infty)) & \xleftarrow{\text{id}-\alpha_1} & K_1(C^*(R^\infty)) \end{array}$$

where $i : C^*(R^\infty) \to C^*(E)$ is the inclusion map.

If on E we consider the equivalence relation R defined by t: two edges (x,k,y) and (x',k',y') are equivalent iff $y = y'$, then the C*-algebra $C^*(R)$ is a continuous trace algebra with spectrum V, and there is a canonical embedding

$$\Phi : C(V) \to C^*(R),$$

$$\Phi(f)((x,k,y),(x',k',y)) = \begin{cases} f(x), & \text{if } x = x' \text{ and } k = k' \\ 0, & \text{otherwise.} \end{cases}$$

Using the same method as in the Main Result of [**D2**], we get:

Theorem 2.2. *If Φ_0 and Φ_1 are the maps induced on K-theory by the embedding $\Phi : C(V) \to C^*(R)$, and if the K-theory groups $K^0(V)$ and $K^1(V)$ are free and finitely generated, then*

$$K_0(C^*(E)) = \ker(\text{id} - \Phi_1) \oplus K^0(V)/(\text{id} - \Phi_0)K^0(V),$$
$$K_1(C^*(E)) = \ker(\text{id} - \Phi_0) \oplus K^1(V)/(\text{id} - \Phi_1)K^1(V).$$

Using this theorem, we can get interesting examples of simple purely infinite C*-algebras with prescribed K-theory groups. In particular, in the next example, we construct C*-algebras A_n with $K_0(A_n) = 0$ and $K_1(A_n) = \mathbf{Z}_n$.

Example 2.3. Let $V = V_1 \cup V_2$, where $V_i,\ i = 1,2$ are copies of the unit circle, and

$$E = \{(v,w) \in V_1 \times V_1 \mid v = w^2\} \cup \{(v,w) \in V_1 \times V_2 \mid v = w\} \cup$$
$$\{(v,w) \in V_2 \times V_1 \mid v = w\} \cup$$
$$\{(v,k,w) \in V_2 \times \{1,2,...,n+2\} \times V_2 \mid w = v^n\}.$$

Then

$$\Phi : C(V_1) \oplus C(V_2) \longrightarrow C(V_1) \otimes \mathbf{M}_2 \oplus C(V_2) \otimes \mathbf{M}_{n(n+2)+1},$$

$$\Phi(f \oplus g) = \begin{pmatrix} \sigma_2 f & 0 \\ 0 & \sigma_1 g \end{pmatrix} \oplus \begin{pmatrix} \sigma_1 f & 0 & 0 & 0 \\ 0 & \hat{\sigma}_n g & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \hat{\sigma}_n g \end{pmatrix}.$$

Here $\sigma_k f(z) = f(z^k)$, and $\hat{\sigma}_k$ is the k-times around embedding (the homomorphism compatible with the covering $z \to z^k$). There are $n+2$ copies of $\hat{\sigma}_n g$ in the definition of Φ. Note that

$$\Phi_0 = \begin{pmatrix} 1 & 1 \\ 1 & n(n+2) \end{pmatrix}, \quad \Phi_1 = \begin{pmatrix} 2 & 1 \\ 1 & n+2 \end{pmatrix}.$$

It follows that

$$\ker(\mathrm{id} - \Phi_0) = 0, \quad \mathbf{Z}^2/(\mathrm{id} - \Phi_0)\mathbf{Z}^2 = 0,$$
$$\ker(\mathrm{id} - \Phi_1) = 0, \quad \mathbf{Z}^2/(\mathrm{id} - \Phi_1)\mathbf{Z}^2 = \mathbf{Z}_n,$$

therefore the corresponding C*-algebra $C^*(E)$ has $K_0 = 0,\ K_1 = \mathbf{Z}_n$. One can check that every orbit with respect to the equivalence relation R^∞ is dense, therefore $C^*(R^\infty)$ and $C^*(E)$ are simple. The latter algebra is purely infinite because it appears as a crossed product of an inductive limit of circle algebras by an endomorphism that does not preserve any trace (see Theorem 2.1 in [**R2**]).

Definition 2.4. Recall that $\sigma_+ : X_+ \to X_+$ is (positive) expansive if there is a constant $c > 0$ such that $x \neq y$ implies $d(\sigma_+^n(x), \sigma_+^n(y)) \geq c$ for some integer $n \geq 0$. An element $x \in X_+$ is *eventually periodic* if there are two integers $p \neq q$ with $\sigma_+^p(x) = \sigma_+^q(x)$.

In [**De**], Proposition 4.2, it is proved that if σ_+ is expansive and the eventually periodic points form a dense set with empty interior, then $C^*(\Gamma)$, and therefore $C^*(E)$, is nuclear, purely infinite, and belongs to the bootstrap class $\mathcal{N}$.

Note that in the above hypotheses, the groupoid $\Gamma = \Gamma(X_+, \sigma_+)$ is essentially free, i.e. the set of points in the unit space with trivial isotropy is dense.

We will see in the last section that even for non-expansive σ_+, the C*-algebra $C^*(E)$ could be purely infinite. Of course, it can not be finite as long as the endomorphism α is induced by a non unitary isometry v. If σ_+ is minimal (i.e. each orbit with respect to the equivalence relation R^∞ is dense), then this C*-algebra is also simple.

Remark 2.5. When σ_+ is expansive, there are other C*-algebras associated with the continuous graph E. According to [**AR**], in this case, the space X of two-sided infinite paths has a Smale space structure, and one may consider the stable equivalence relation:

$$R_s = \{(x, y) \in X \times X \ \mid d(\sigma^n(x), \sigma^n(y)) \to 0 \text{ as } n \to +\infty\}.$$

Then $C^*(R_s)$ is strongly Morita equivalent to $C^*(R^\infty)$, and its Ruelle algebra $C^*(R_s)\times \mathbf{Z}$ is strongly Morita equivalent to $C^*(E)$ (see [**AR**], Theorem 4.5).

Another C*-algebra which could be associated with the continuous graph E is the crossed product $C(X)\times_\sigma \mathbf{Z}$.

3. The connection with the Pimsner algebras $\mathcal{O}_H$.

In this paragraph, we recall the Pimsner construction from [**Pi**], and we show how the C*-algebra of a continuous graph could be thought as $\mathcal{O}_H$, for a particular Hilbert bimodule H. To a pair (H, A), where H is a (right) Hilbert module over a C*-algebra A, and A acts to the left on H via a $*$-homomorphism $\varphi : A \to L(H)$, Pimsner constructs a C*-algebra $\mathcal{O}_H$, which generalizes both the crossed products by $\mathbf{Z}$ and the Cuntz-Krieger algebras. The algebra $\mathcal{O}_H$ is a quotient of the generalized Toeplitz algebra $\mathcal{T}_H$, generated by the creation operators $T_\xi,\ \xi \in H$ on the Fock space $\mathcal{H}_+ = \bigoplus_{n=0}^{\infty} H^{\otimes n}$. Here $H^{\otimes 0} = A$, and for $n \geq 1$, $H^{\otimes n}$ denotes the n-th tensor power of H, balanced via the map φ. By definition, $T_\xi a = \xi a$, for $a \in A$, and $T_\xi(\xi_1 \otimes ... \otimes \xi_n) = \xi \otimes \xi_1 \otimes ... \otimes \xi_n$, for $\xi_1 \otimes ... \otimes \xi_n \in H^{\otimes n}$.

To give another description of the algebra $\mathcal{O}_H$, Pimsner considers a new pair $(H_\infty, \mathcal{F}_H)$, where $\mathcal{F}_H$ is the C*-algebra generated by all the compact operators $K(H^{\otimes n}),\ n \geq 0$ in $\varinjlim L(H^{\otimes n})$, and $H_\infty = H \otimes \mathcal{F}_H$. The advantage is that H_∞ becomes an $\mathcal{F}_H - \mathcal{F}_H$ bimodule, such that the adjoint H_∞^* is also an $\mathcal{F}_H - \mathcal{F}_H$ bimodule. The C*-algebra $\mathcal{O}_H$ is represented on the two-sided Fock space

$$\mathcal{H}_\infty = \bigoplus_{n\in\mathbf{Z}} H_\infty^{\otimes n},$$

where for $n < 0, H_\infty^{\otimes n}$ means $(H_\infty^*)^{\otimes -n}$. In fact, it is isomorphic to the C*-algebra generated by the multiplication operators $M_\xi \in L(\mathcal{H}_\infty)$, where $\xi \in H_\infty$, and $M_\xi \omega = \xi \otimes \omega$.

Given a continuous graph E such that the origin and terminus maps $E \to V$ are onto local homeomorphisms, let $A = C(V)$, and let $H = C(E)$ (as a vector space), with the structure of Hilbert A-module given by

$$(\xi f)(e) = \xi(e) f(t(e)),\ \xi \in H, f \in A, e \in E,$$

$$\langle \xi, \eta\rangle(v) := \sum_{t(e)=v} \overline{\xi(e)}\eta(e), v \in V,\ \xi, \eta \in H.$$

In other words, the inner product is given by $\langle \xi, \eta \rangle = P(\bar{\xi}\eta)$, where $P : C(E) \to C(V)$ is the conditional expectation

$$(P\xi)(v) = \sum_{t(e)=v} \xi(e).$$

The left module structure is given by

$$\varphi : A \to L(H),\ (\varphi(f)\xi)(e) = f(o(e))\xi(e)\ f \in A, \xi \in H.$$

Note that indeed $\varphi(f)$ is in $L(H)$, having the adjoint $\varphi(\bar{f}), f \in A$.

To prove that $\mathcal{O}_H$ with this choice of A, H and φ is isomorphic to $C^*(E)$, let's identify the C*-algebra $\mathcal{F}_H$ in this case.

Note that $H \otimes_\varphi H$ is a quotient of $C(E) \otimes C(E)$, where we identify $\xi f \otimes \eta$ with $\xi \otimes \varphi(f)\eta$ for any $\xi, \eta \in H$ and any $f \in A$. Therefore $H \otimes_\varphi H$ could be identified as a vector space with the continuous functions on the set

$$\{(e_1, e_2) \in E \times E \mid\ t(e_1) = o(e_2)\}.$$

This set will be denoted by X_2, and is precisely the set of paths of length 2. In a similar way, $H^{\otimes n}$ is identified (as a vector space) with $C(X_n)$, where X_n is the set of paths of length n. The Hilbert A-module structure on $H^{\otimes n}$ for $n \geq 2$ is given by

$$(\xi f)(x) = \xi(x) f(t_n(x)), x \in X_n$$

where $t_n : X_n \to V, t_n(e_1 e_2 ... e_n) = t(e_n)$, and by

$$\langle \xi, \eta \rangle_n = P_n(\bar{\xi}\eta).$$

Here P_n is the conditional expectation

$$P_n : C(X_n) \to C(V), P_n(\xi)(v) = \sum_{t_n(x)=v} \xi(x).$$

Proposition 3.1. *The C*-algebra $K(H)$ is isomorphic with $C^*(R)$, where*

$$R = \{(e_1, e_2) \in E \times E \mid\ t(e_1) = t(e_2)\}$$

is the equivalence relation associated with the map t. The map $\varphi : A \to L(H)$ could be identified with the embedding $\Phi : C(V) \to C^*(R)$, defined before Theorem 2.2. Moreover, $K(H^{\otimes n}) \simeq C^*(R_n)$, where

$$R_n = \{(x, y) \in X_n \times X_n \mid\ t_n(x) = t_n(y)\}$$

is the equivalence relation associated with t_n.

Proof. Taking into account the fact that o and t are local homeomorphisms, we have $L(H) = K(H)$, since H is algebraically finitely generated.

Now $K(H) = H \otimes H^*$, the tensor product balanced over A, where H^* is the adjoint of H. Since $\xi f \otimes \eta^* = \xi \otimes f\eta^*$, it follows that, as a set, $K(H) = C(R)$. The multiplication of compact operators turns out to be the convolution product on $C(R)$, therefore, as C*-algebras, $K(H) = C^*(R)$. □

Corollary 3.2. *We have* $\mathcal{F}_H = \varinjlim C^*(R_n)$. *Therefore,* $\mathcal{F}_H$ *is isomorphic to the algebra* $C^*(R^\infty)$.

Proof. Note that for $n \geq 1$, the inclusion $\phi_n : \mathbf{C}^*(R_n) \to \mathbf{C}^*(R_{n+1})$,

$$(\phi_n)(f)(x_1...x_{n+1}, y_1...y_{n+1}) = \begin{cases} f(x_1...x_n, y_1...y_n) & \text{if } \ x_{n+1} = y_{n+1} \\ 0, & \text{otherwise} \end{cases}$$

is just the map $K(H^{\otimes n}) \to K(H^{\otimes n+1}),\ \ T \mapsto T \otimes I$. Here R_1=R. □

In order to establish an isomorphism between $\mathbf{C}^*(\Gamma)$ and $\mathcal{O}_H$, we show that they appear as the $\mathbf{C}^*$-algebras associated to isomorphic Fell bundles over the group $\mathbf{Z}$. This point of view was suggested by Abadie, Eilers and Exel in [**AEE**]. The definition of a Fell bundle and of the associated C^*-algebra is taken from [**K1**].

To the pair $(H_\infty, \mathcal{F}_H)$, we can associate the Fell bundle $\mathcal{B}$, where $\mathcal{B}_n := H_\infty^{\otimes n}, n \in \mathbf{Z}$. The multiplication is given by the tensor product, identifying $H_\infty^* \otimes H_\infty$ with $\mathcal{F}_H$ and $H_\infty \otimes H_\infty^*$ with the ideal $\mathcal{F}_H^1$ of $\mathcal{F}_H$, generated by $K(H^{\otimes n})$ with $n \geq 1$. But $\mathcal{F}_H^1$ is equal to $\mathcal{F}_H$ in our case. The involution is obvious. Then

$$L^2(\mathcal{B}) = \mathcal{H}_\infty = \bigoplus_{n \in \mathbf{Z}} H_\infty^{\otimes n}.$$

Since $\mathcal{H}_\infty$ is generated by $\mathcal{F}_H$ and H_∞, it follows that the C^*-algebra generated by the operators M_ξ is isomorphic to $C^*(\mathcal{B})$. Hence, $\mathcal{O}_H \simeq C^*(\mathcal{B})$.

For the groupoid Γ and $l \in \mathbf{Z}$, take

$$\Gamma_l := \{(x, k, y) \in \Gamma \mid k = l\} = \{(x, y) \in X \times X \mid x_n = y_{n+l} \text{ for large } n\},$$

and $\mathcal{D}_l = \overline{C_c(\Gamma_{-l})}$ (closure in $C^*(\Gamma)$). This way, we obtain a $\mathbf{Z}$-grading on $C^*(\Gamma)$, and it is easy to see that this C*-algebra could be recovered as $C^*(\mathcal{D})$. But

$$\mathcal{D}_0 = C^*(R^\infty) \simeq \mathcal{F}_H = \mathcal{B}_0,$$

and

$$\mathcal{D}_1 = \overline{C_c(\Gamma_{-1})} \simeq H \otimes_A \mathcal{F}_H = H_\infty = \mathcal{B}_1.$$

We get

Proposition 3.3. *With the above choice of* A, H *and* φ, *the* C^**-algebras* $C^*(E)$ *and* $\mathcal{O}_H$ *are isomorphic.*

4. Generalized solenoids.

A solenoid is a compact connected abelian group of finite dimension. For example, if $\mathbf{T}$ is the unit circle,

$$\mathbf{T}(m) = \{z \in \mathbf{T}^{\mathbf{Z}} \mid \ z_k^m = z_{k+1},\ k \in \mathbf{Z}\}$$

is such a group, for any integer $m > 1$. The bilateral shift σ on $\mathbf{T}(m)$, $\sigma(z)_p = z_{p+1}$ is a homeomorphism, and in many respects it is an analogue of the Bernoulli shift. In [B], Brenken considered the dynamical system (G_0, σ) for G_0 the connected component of the identity of the group

$$G = \{z \in (\mathbf{T}^d)^{\mathbf{Z}} \ \mid \ Fz_k = Mz_{k+1},\ k \in \mathbf{Z}\},$$

where M, F are surjective endomorphisms of the d-torus, given by matrices $M, F \in \mathbf{M}_d(\mathbf{Z})$ with nonzero determinant. (Note that the case $d = 1, M = 1, F = m$ corresponds to the above example $\mathbf{T}(m)$.) The space G_0 has a natural local product structure, being a principal bundle over $\mathbf{T}^d$ with fiber the Cantor set. Moreover, it has a Smale space structure, and the author identifies the C^*-algebras associated with the stable and unstable equivalence relations.

Definition 4.1. By a *generalized solenoid* we mean the space X of two-sided infinite paths with edges in the graph E described bellow. Let $V = \mathbf{T}_1^d \sqcup ... \sqcup \mathbf{T}_N^d$ be the disjoint union of N copies of the d-dimensional torus $\mathbf{T}^d$, and let $L = (l(i,j))_{i,j}$ be an $N \times N$ matrix with positive integer entries (the "incidence" matrix of the graph). We require that in the matrix L each row and each column has at least a nonzero entry. For each pair (i, j) with $l(i, j) \geq 1$, consider a family of closed, connected subgroups $G_1^{ij}, G_2^{ij}, ..., G_{l(i,j)}^{ij}$ of $\mathbf{T}_i^d \times \mathbf{T}_j^d$, not necessarily distinct, such that all the projections on $\mathbf{T}_i^d$ and $\mathbf{T}_j^d$ are surjective. For the pairs (i, j) with $l(i, j) = 0$, this family is empty by definition. We take E to be the disjoint union of all the groups G_k^{ij}, $1 \leq i, j \leq N, 1 \leq k \leq l(i, j)$, with obvious origin and terminus maps.

It is known (see [KS]) that there are families of $d \times d$ nonsingular matrices with integer entries,

$$\mathcal{A}_{ij} = \{A_1^{ij}, ..., A_{l(i,j)}^{ij}\},\ \ \mathcal{B}_{ij} = \{B_1^{ij}, ..., B_{l(i,j)}^{ij}\}.$$

such that

$$G_l^{ij} = \{(z, w) \in \mathbf{T}_i^d \times \mathbf{T}_j^d \mid A_l^{ij} z = B_l^{ij} w\}.$$

The matrices A_k^{ij}, B_k^{ij} are not necessarily distinct. Note that a generalized solenoid X has no longer a group structure, and the dynamical system (X, σ) is an analogue of the Matkov shift.

Example 4.2. Let $d = 2, N = 2$,

$$A_1^{11} = \begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix}, B_1^{11} = \begin{pmatrix} -1 & 0 \\ 2 & 3 \end{pmatrix}, A_1^{12} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, B_1^{12} = \begin{pmatrix} 3 & 0 \\ 1 & 1 \end{pmatrix},$$

$$A_2^{12} = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}, B_2^{12} = \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix},$$

$$A_1^{22} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, B_1^{22} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, A_2^{22} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, B_2^{22} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The space of edges is

$$\{((z,w),1,(t,u)) \in \mathbf{T}_1^2 \times \{1\} \times \mathbf{T}_1^2 \mid \ z^3w = t^{-1},\ zw = t^2u^3\} \cup$$

$$\{((z,w),1,(t,u)) \in \mathbf{T}_1^2 \times \{1\} \times \mathbf{T}_2^2 \mid \ z = t^3, w^2 = tu\} \cup$$

$$\{((z,w),2,(t,u)) \in \mathbf{T}_1^2 \times \{2\} \times \mathbf{T}_2^2 \mid \ z^2w = t^{-1}u, w = t\} \cup$$

$$\{((z,w),1,(t,u)) \in \mathbf{T}_2^2 \times \{1\} \times \mathbf{T}_2^2 \mid \ z = t^2, w = u\} \cup$$

$$\{((z,w),2,(t,u)) \in \mathbf{T}_2^2 \times \{2\} \times \mathbf{T}_2^2 \mid \ z = t, w = u\}.$$

The corresponding embedding $C(V) \to C^*(R)$ of toral algebras is

$$\Phi : C(\mathbf{T}^2) \oplus C(\mathbf{T}^2) \to C(\mathbf{T}^2) \otimes \mathbf{M}_3 \oplus C(\mathbf{T}^2) \otimes \mathbf{M}_7,$$

$$\Phi(f \oplus g) = \Phi_{11}(f) \oplus \begin{pmatrix} \Phi_{12}(f) & 0 \\ 0 & \Phi_{22}(g) \end{pmatrix},$$

where

$$\Phi_{11}(f) = \hat{\sigma}_{B_1^{11}} \circ \sigma_{A_1^{11}}(f), \Phi_{12}(f) = \begin{pmatrix} \hat{\sigma}_{B_1^{12}} \circ \sigma_{A_1^{12}}(f) & 0 \\ 0 & \hat{\sigma}_{B_2^{12}} \circ \sigma_{A_2^{12}}(f) \end{pmatrix},$$

and

$$\Phi_{22}(g) = \begin{pmatrix} \hat{\sigma}_{B_1^{22}} \circ \sigma_{A_1^{22}}(g) & 0 \\ 0 & \hat{\sigma}_{B_2^{22}} \circ \sigma_{A_2^{22}}(g) \end{pmatrix}.$$

Here $\sigma_A : C(\mathbf{T}^2) \to C(\mathbf{T}^2)$ denotes the $*$-homomorphism induced by the covering A, defined by $(\sigma_A f)(z) = f(Az)$, and $\hat{\sigma}_A : C(\mathbf{T}^2) \to C(\mathbf{T}^2) \otimes \mathbf{M}_{|\det A|}$ is the homomorphism compatible with A, in the sense that $\hat{\sigma}_A \circ \sigma_A(f) = f \otimes I_{|\det A|}$.

Remark 4.3. Given a generalized solenoid X, let's denote by K the space of two-sided infinite paths in the discrete graph with N vertices, where from the vertex i to the vertex j there are $l(i,j)$ edges. On the Cantor set K consider the Markov shift τ. Note that there is a natural map $\rho : X \to K$, $\rho((x_n, k_n)_{n\in\mathbf{Z}}) = (k_n)_{n\in\mathbf{Z}}$. Moreover, $\rho\sigma = \tau\rho$. Therefore, (X, σ) is in fact an extension of a Markov shift (K, τ). In the above example, the incidence matrix L is

$$L = \begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}.$$

Note that the fiber of ρ has a group structure, therefore X could be seen as a group bundle over the Cantor set K. The groups are in fact solenoids if they are connected.

The space X is also fibered over $V = \mathbf{T}^d_1 \sqcup ... \sqcup \mathbf{T}^d_N$ by the map $\pi : X \to V, \pi((x_n, k_n)_{n\in\mathbf{Z}}) = x_0$. The fibers of π are totally disconnected, since the set $\{x_n \in V \mid \pi((x_n, k_n)_{n\in\mathbf{Z}}) = x_0\}$ is finite for each fixed $x_0 \in V$ and $n \in \mathbf{Z}$.

The following example arose in a discussion with Jack Spielberg.

Example 4.4. Let $V = \mathbf{T}$, the unit circle, and

$$E = \{(z, 1, z^2) \mid z \in V\} \cup \{(z^3, 2, z) \mid z \in V\}.$$

Then

$$X = \{(z_n, k_n) \in (V \times \{1, 2\})^{\mathbf{Z}} \mid k_n = 1 \Rightarrow \\ z_{n+1} = z_n^2,\ k_n = 2 \Rightarrow\ z_{n+1}^3 = z_n\}.$$

We will show that $\sigma : X \to X,\ \sigma(z_n, k_n)_p = (z_{p+1}, k_{p+1})$ is not expansive, therefore the space (X, σ) has not a Smale space structure.

It suffices to show that for any $\varepsilon > 0$, we can find two distinct sequences (z_n, k_n) and (w_n, k_n) such that $\delta(z_n, w_n) \geq \varepsilon$ for all $n \in \mathbf{Z}$. Fix $z_0, w_0 \in V$. The idea is that, taking in a certain order squares, cubes, square roots and cubic roots, the corresponding vertices remain close together. We can choose two sequences of integers $(a_n)_{n\geq 1}, (b_n)_{n\geq 1}$ such that

$$\lim_{n\to\infty} \frac{2^{a_1+...+a_n}}{3^{b_1+...+b_n}} = 1.$$

Consider the symmetric sequence $(k_n)_{n\in\mathbf{Z}}$ described as

$$...\underbrace{2...2}_{b_2}\underbrace{1...1}_{a_2}\underbrace{2...2}_{b_1}\underbrace{1...1}_{a_1}\underbrace{\bar{1}...1}_{a_1}\underbrace{2...2}_{b_1}\underbrace{1...1}_{a_2}\underbrace{2...2}_{b_2}...,$$

where the bar indicates k_0. Given $\varepsilon > 0$, we can choose z_0 and w_0 sufficiently close together (but distinct), and z_n and w_n in a consistent way (when we take square or cubic roots) such that $\delta(z_n, w_n) \geq \varepsilon$. It follows that

$$d(\sigma^p(z_n, k_n), \sigma^p(w_n, k_n)) \geq \varepsilon\ \ \forall p \in \mathbf{Z},$$

and the shift is not expansive.

Nevertheless, the orbits with respect to R^∞ are dense in X_+, and there is no shift invariant trace, therefore the C*-algebra $C^*(E)$ is simple and purely infinite.

Note that in this example, the dynamical system (X_+, σ_+) is an extension of the Bernoulli shift $(\{1,2\}^{\mathbf{N}}, \tau)$. The fibers of the map $\rho : X_+ \to \{1,2\}^{\mathbf{N}}$ are circles over the sequences which contain only a finite numbers of 2's, and solenoids over the sequences containing infinitely many 2's.

It is interesting to notice that $C^*(E)$ and $C^*(\hat{E})$ are both simple, purely infinite, with K-theory

$$K_0(C^*(E)) = K_1(C^*(\hat{E})) = \mathbf{Z}_2, \quad K_1(C^*(E)) = K_0(C^*(\hat{E})) = \mathbf{Z}_3.$$

In [**P1**] it is proved that the Ruelle algebra associated to the graph of the map $z \mapsto z^p$ on the unit circle is isomorphic to the one obtained from the dual graph. Whether this is true for more general graphs is an open question.

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