Nonabelian algebraic topology:
Part I

Ronald Brown
Department of Mathematics
School of Informatics
University of Wales, Bangor

Rafael Sivera Villanueva
Departmento de Geometria y Topologia
Universidad de Valencia(Spain)

May 31, 2004

This contains the Introductory material, planned Table of Contents, and Part I of the three Parts of this text, as well as the bibliography.

It is made available on the web in order to attract comments and suggestions. These should be sent to either of

r.brown@bangor.ac.uk

Rafael.Sivera@uv.es

It is planned to make the further parts available in the coming months.

Negotiations for publication in a series are in hand.

Production of the text is supported by a Leverhulme Emeritus Fellowship for R. Brown, 2002-2004.

Contents

Preface

Aims

Our aim for this text is to give a connected and we hope readable account of some work since about 1965 on extending to higher dimensions the theory and applications in algebraic topology of the fundamental group. This group, $\pi_1(X, x)$, for a space X with base point x, is determined by homotopy classes relative to the end points of paths , i.e. maps $f : [0, 1] \to X$ from the unit interval to X which map $0, 1$ to x. The fundamental group is one of the corner stones of basic algebraic topology, with many applications in topology, analysis, geometry, and group theory, which often, particularly in group theory, exploit nonabelian examples.

Our extension to higher dimensions of the fundamental group necessitates a parallel extension of some concepts of group theory, such as a free group. These two extensions, running side by side, allow for an account of some aspects of algebraic topology with more nonabelian features than those available from previous texts, and this explains our title. We are also able to give a new exposition of group cohomology, including nonabelian coefficients, based on analogies with homotopy theory.

Structure of the book

We divide our account into three parts, each with an Introduction.

In Part I we give some history of work on the fundamental group and groupoid, in particular explaining how the van Kampen theorem gives a method of computation of the fundamental group. We are then mainly concerned with the extension of nonabelian work to dimension 2, using the key concept, due to J.H.C. Whitehead in 1946, of *crossed module*. This is a morphism

$$\mu : M \to P$$

of groups together with an action of the group P on the right of the group M, written $(m, p) \mapsto m^p$, satisfying the two rules:

CM1) $\mu(m^p) = p^{-1}(\mu m)p$;

CM2) $m^{-1}nm = n^{\mu m}$,

for all $p \in P$, $m, n \in M$. Algebraic examples of crossed modules include normal subgroups M of P; P-modules; the inner automorphism crossed module $M \to \mathrm{A}ut\, M$; and many others. There is the beginnings of a combinatorial and also computational crossed module theory.

The standard geometric example of crossed module is the boundary morphism of the second relative homotopy group

$$\partial : \pi_2(X, X_1, x) \to \pi_1(X_1, x)$$

where X_1 is a subspace of the topological space X and $x \in X_1$. This relative homotopy group is defined in terms of certain homotopy classes of maps $I^2 \to X$. For this reason, and because they are a good model of 2-dimensional pointed homotopy theory, crossed modules are commonly seen as good candidates for 2-*dimensional groups*.

The remarkable fact is that we can calculate with these 2-dimensional structures and apply these calculations to topology using a 2-dimensional version of the van Kampen theorem for the fundamental group.

We give a substantial account of this 2-dimensional theory because the step from dimension 1 to dimension 2 involves a number of new ideas for which the reader's intuition needs to be developed. In particular, calculation with crossed modules requires some extensions of combinatorial group theory, for example to induced crossed modules. Finally in this Part, the proof of the van Kampen type theorem for crossed modules, involves a notion of *homotopy double groupoid*, based on composing squares with common edges. The intuition for this construction was the start of the theory of this book.

In Part II we extend the theory of crossed modules to *crossed complexes*, giving applications which include many basic results in homotopy theory, such as the relative Hurewicz theorem. This Part is intended as a kind of handbook of basic techniques in this border area between homology and homotopy theory.

However for the *proofs* of these results, particularly of the van Kampen type theorem and use of the tensor product and homotopy theory of crossed complexes, we have to introduce in Part III another algebraic structure, that of *cubical ω-groupoid with connection*. In principle, Part III can be read independently of the previous parts, referring back for some basic definitions.

Background

Recall that two maps $f, g : X \to Y$ of topological spaces are called *homotopic* if there is a *homotopy* $H : f \simeq f'$, by which is meant a map $H : [0,1] \times X \to Y$, such that $H(0,x) = f(x), H(1,x) = f'(x)$ for all $x \in X$. In this way we get a set $[X,Y]$ of homotopy classes of maps $X \to Y$. Spaces X, Y are *homotopy equivalent*, written $X \simeq Y$, if there are maps $f : X \to Y, g : Y \to X$ such that the composites $fg,\ gf$ are homotopic to the respective identity maps $1_Y,\ 1_X$. Then $f : X \to Y$ is called a *homotopy equivalence*. Thus a basic problem is to decide if spaces X, Y are, or are not, homotopy equivalent.

It is not surprising that the fundamental group is a homotopy invariant since it is defined in terms of homotopy classes of maps. Thus if two connected spaces are homotopy equivalent, they have isomorphic fundamental groups.

Another corner stone of algebraic topology is the theory of homology, with its abelian homology groups $H_n(X),\ n \geqslant 0$, for a topological space X. The homology groups, like the fundamental group, are homotopy invariants. A homotopy equivalence induces isomorphisms of homology groups as well of fundamental groups. However the definition of homology, and the proof of homotopy invariance, are more subtle than those of the various homotopy groups. Also the converse is false: a map may induce isomorphisms of fundamental group and homology groups, and yet not be a homotopy equivalence.

Higher homotopy groups $\pi_n(X,x)$ were defined in 1932 for $n \geqslant 2$ and they are all abelian. One definition of these is in terms of homotopy classes of maps of an n-cube I^n to X which map the boundary ∂I^n of the n-cube to the base point x, and all homotopies are constant on ∂I^n.

A further important problem in algebraic topology, with many applications, is to calculate the set $[X,Y]$ of homotopy classes of maps in terms of information on X, Y. This can be solved completely in some cases. For instance, if X is a connected CW-complex and $\pi_i(Y,y) = 0$ for $i > 1$, there is a bijection of sets

$$[X,Y] \cong [\pi_1(X,x), \pi_1(Y,y)]$$

where the right hand set is conjugacy classes of morphisms of groups. We give an analogous result when $\dim X \leqslant n$ and $\pi_i(Y,y) = 0$ for $1 < i < n$.

We describe algebraic structures in dimensions greater than 1 which develop the nonabelian character of the fundamental group: they are in some sense 'more nonabelian than groups', and they reflect better the geometrical complications of higher dimensions than the known homology and homotopy groups. We show how these methods can be applied to determine homotopy invariants of spaces, and homotopy classification of maps, in cases which include some classical results, and allow results not available by classical methods.

The development of such higher dimensional, nonabelian, methods in algebraic topology has been a programme of the first author since about 1966. Its inspiration was work of Philip Higgins in 1963 generalising the notion of presentation of groups to presentation of groupoids. This suggested a generalisation of the fundamental group on a space with base point to the fundamental *groupoid* on a *set of base points*, thus allowing a more flexible modelling of the underlying geometry of a space. This modelling also allowed more calculations, through a generalisation to groupoids of the van Kampen theorem for the fundamental group. The success of groupoids at this level suggested a programme of using groupoids in higher dimensional homotopy theory, and in particular developing a higher dimensional version of the van Kampen theorem.

Brown and Higgins found in the 1970s that *higher homotopy groupoids* could be defined, with values in what we called ω-groupoids, using maps of squares or n-cubes rather than paths. The key idea is to use not groups but groupoids, and to replace the space with base point by a *filtered space*

$$X_* : X_0 \subseteq X_1 \subseteq \cdots \subseteq X_n \subseteq \cdots \subseteq X_\infty,$$

namely a topological space X_∞ with an increasing sequence of subspaces. Then a new functor $\rho(X_*)$ was obtained using filtered maps $I^n_* \to X_*$, where I^n_r consists of all faces of the n cube of dimension $\leqslant r$. The homotopies are through filtered maps and keep the vertices of I^n fixed throughout.

An ω-groupoid is in the first instance a graded set with n groupoid structures in each dimension $n \geqslant 1$ satisfying a fairly complicated but geometrically clear set of laws. This idea yielded, after a struggle, new abstract structures underlying homotopy theory, which led to new understanding and new calculations. The use of groupoids, and of structures with algebraic operations not always defined, was essential for this work.

A pleasant surprise was that the investigation of the existence and use of higher homotopy groupoids led to links of ω-groupoids with more classical structures, particularly crossed modules and crossed complexes, on which J.H.C. Whitehead had done extensive work in the 1940s. His work gave key clues to the directions to take.

The notion of crossed complex arose from *relative homotopy theory*, in which occur groups $\pi_n(X, A, x)$, $n \geqslant 2$, and which are abelian for $n \geqslant 3$. They are defined in terms of homotopy classes of maps $I^n \to X$ which map to x the set J^{n-1} of all $(n-1)$-faces of I^n except the $(0,1)$-th face, map the remaining face to A, and all homotopies keep J^{n-1} fixed. Thus for a filtered space X_* one obtains the fundamental groupoid $\pi_1(X_1, X_0)$ and the relative homotopy groups $\pi_n(X_n, X_{n-1}, x), x \in X_0, n \geqslant 2$. The structure all these satisfy is called a crossed complex. So we obtain a functor Π from filtered spaces to crossed complexes.

The remarkable fact is that this functor Π can be calculated directly in some important cases by a Generalised van Kampen Theorem (GvKT). This theorem, like its version for the fundamental group or groupoid, is an example of a 'local-to-global' theorem. It gives a method for calculating the functor Π for some filtered spaces which are presented as a union of smaller pieces.

'Local-to-global' is a general term applied to a family of problems concerned with relating the behaviour of a large structure to the way it is built out of smaller pieces. Such problems are central in mathematics and science.

To express this idea in the cases of interest to us, we take from category theory the concept of *colimit*. This gives a general definition of a kind of gluing process, of building large structures out of smaller ones of the same type. Our aim is to build what we call 'functors' from topological data to algebraic data which 'preserve certain colimits'. This says intuitively that such functors allow a modelling by 'gluing' in algebra of the process of gluing in topology. In this way we will obtain precise and useful algebraic calculations of some homotopical information on spaces built out of smaller ones.

Many of the main aims of the book can be summarised by stating that we construct a diagram, which we call the *Main Equivalence (ME)* :

$$\text{(ME)}\qquad \begin{array}{ccc} & \text{(filtered spaces)} & \\ \swarrow^{\Pi} & & \searrow^{\rho} \\ \text{(crossed complexes)} & \underset{\gamma}{\overset{\lambda}{\rightleftarrows}} & (\omega\text{-groupoids}) \end{array}$$

such that

(A) γ, λ give an equivalence of categories;

(B) $\gamma\rho$ is naturally equivalent to Π;

(C) ρ, and hence also Π, preserves certain colimits.

The final statement we call a Generalised van Kampen Theorem (GvKT); it allows for calculations of Π, and so of certain relative homotopy groups, to get started. Corollaries of these results include:

(i) the Brouwer degree theorem (the n-sphere S^n is $(n-1)$-connected and the homotopy classes of maps of S^n to itself are classified by an integer called the *degree* of the map);

(ii) the relative Hurewicz theorem, which relates relative homotopy and homology groups;

(iii) Whitehead's theorem that $\pi_n(A \cup \{e^2_\lambda\}, A, x)$ is a free crossed $\pi_1(A, x)$-module; and

(iv) computations of the second homotopy group, and even 2-type, of the mapping cone of the map $Bf : BG \to BH$ of classifying spaces induced by a morphism $f : G \to H$ of groups.

The last two corollaries deal with constructions in crossed modules, seen as crossed complexes of length 2. These are in general nonabelian, and so these two corollaries (which we give in Part I) are not easily reachable, or not obtainable, by traditional means.

The proof of the Generalised van Kampen Theorem uses only a little knowledge of homotopy, CW-complexes, and category theory, but it is quite elaborate. The facts (A), (B), (C) are crucial. It turns out that the functor ρ is convenient for formulating and proving theorems, while the functor Π is convenient for calculation and for relating to classical constructions, such as relative homotopy groups, and chain complexes.

Our proof of the higher dimensional, local-to-global GvKT relies on methods which allow the expression of the intuitions of (i) algebraic inverse to subdivision, and (ii) of commutative cube.

For (i), cubical ideas are essential, since there is an easy notion of subdividing a cube by hyperplanes parallel to the faces, and is is not so hard to envisage an algebraic structure on cubical sets which will model reversing this process. These ideas are clearly related to local-to-global problems. The idea of (ii) is more subtle than that of (i); it needs the full relation between the above algebraic categories to express it; and it is required in order to show that a morphism on $\rho(X_*)$ is well defined, i.e. is independent of the choices apparently made in its construction.

Thus the full structure of the diagram is necessary for our proofs, and not only that of the Generalised van Kampen Theorem (GvKT). We also use this equivalence of algebraic categories to formulate the further important properties of homotopy and of tensor product of crossed complexes, and to show how these model homotopies and tensor products of filtered spaces.

These methods do not *replace* traditional homological methods, partly because of the restrictive conditions on the colimits to which the GvKT applies. However, when it does apply, it can give direct and precise homotopical information not available by other means. Further, these methods have opened out new directions in algebraic topology and related areas.

Another theme in the book, containing results on crossed complexes and the functor Π, can be shown in the following diagram of categories and functors, which we call the Main Diagram (MD), in which Top is the category of topological spaces and continuous maps, and a notion of homotopy is assumed developed for filtered spaces and crossed complexes:

(MD)
$$\begin{array}{ccc} \text{(filtered spaces)} & \underset{\mathsf{B}}{\overset{\Pi}{\rightleftarrows}} & \text{(crossed complexes)} \\ & {}_{U}\searrow \quad \swarrow_{B} & \\ & \mathsf{Top} & \end{array}$$

satisfying the following properties:

(i) Π preserves homotopies.

(ii) There is a natural equivalence $\Pi\mathsf{B} \simeq 1$. This shows that the topology and the algebra are well related.

(iii) U is a 'forgetful' functor and $B = U\mathsf{B}$. This gives a so-called *classifying space* BC of a crossed complex C.

(iv) If X is a CW-complex with skeletal filtration X_*, and C is a crossed complex, there is a natural bijection
$$[X, BC] \cong [\Pi X_*, C],$$
where the right hand side denotes homotopy classes of morphisms in the category of crossed complexes.

This last result allows for some explicit computation of homotopy classes of maps of spaces even in cases where the fundamental groups are involved, as for example for maps of surfaces to the projective plane.

A central aspect of these homotopy classification applications is a notion of tensor product $A \otimes B$ for crossed complexes A, B and of homotopy defined as morphism $\mathcal{I} \otimes B \to C$, where $\mathcal{I}$ is the 'unit interval' groupoid or crossed complex, with two objects $0, 1$ and only one arrow $\iota : 0 \to 1$. The definition of this tensor product seems formidable. However it relies on an equivalent definition for ω-groupoids, which follows geometrically from the fact that $I^m \times I^n \cong I^{m+n}$. The transfer from ω-groupoids to crossed complexes uses the equivalence of these structures. However, for the applications of crossed complexes it is sufficient to take the definition in this context on trust, and this is what we do in Part II, with the proofs involving ω-groupoids left to Part III. The proof of the above homotopy classification result also requires results from the theory of simplicial sets, and these we have to assume in Part II.

As explained earlier, Part I is devoted to the Main Equivalence and applications in dimension 2, that is to the theory and application of crossed modules.

Prerequisites:

Large parts of this book can be read by a graduate student acquainted with general topology, the fundamental group, notions of homotopy, and some basic methods of category theory. Many of these areas, including the concept of groupoid and its uses, are covered in Brown's Topology text [30], which is in the process of being prepared for web publication.

Some aspects of category theory perhaps less familiar to a graduate student are summarised in an Appendix, particularly the notion of adjoint functor, and the preservation of colimits by a left adjoint functor. This is a basic tool of algebraic computation for those algebraic structures which are built up in several levels, since it can often show that a colimit of such a structure can be built up level by level.

Some knowledge of homology theory could be useful at a few points.

For the notion of classifying space of a crossed module or crossed complex we will need results from the theory of realisations of simplicial or cubical sets. The results needed are summarised in an Appendix.

Acknowledgements:

Obtaining these results depended on fortunate collaborations, particularly initial work on double groupoids and crossed modules with Chris Spencer in 1971-2; the long collaboration with Philip Higgins, 1974-now, resulting in twelve joint papers; and a continuing collaboration with both Chris Wensley and Tim Porter.

It is a pleasure to acknowledge also:

(i) the influence of the work of Henry Whitehead, who was Brown's supervisor until 1960, when Henry died suddenly in Princeton at the age of 55. It was then Michael Barratt who guided Brown's thesis towards the homotopy type of function spaces, and Michael's example of how to go about mathematical research is gratefully acknowledged.

(ii) the further contributions of research students at Bangor, and of other colleagues, who all contributed key ideas to the whole programme. This is discussed briefly in a Historical Account at the end of the book.

(iii) the support of the Leverhulme Trust through an Emeritus Fellowship for Brown in 2002-2004. This provided the support for: meetings of Brown and Sivera; for LaTeX work, well done by Genevieve Tan and Peilang Wu; for other travel and visitors; and also a moral impetus to complete this project.

Part I

1 and 2-dimensional results

Introduction to Part I

Part I develops in dimensions 1 and 2 that aspect of nonabelian algebraic topology related to the van Kampen Theorem (vKT).

We start by giving a Historical background, and outline the proof of the van Kampen theorem in dimension 1. It was an analysis of this proof which suggested the higher dimensional possibilities.

We then explain the functor

$$\Pi_2 : \text{(pointed pairs of spaces)} \to \text{(crossed modules)}$$

in terms of second relative homotopy groups, state a Generalised van Kampen Theprem (GvKT) for this, and give applications. These applications involve the algebra of crossed modules, and two important constructions for calculations with crossed modules, namely coproducts of crossed modules on a fixed base group (Chapter 4) and induced crossed modules (Chapter 5). The latter concept illustrates well the way in which low dimensional identifications in a space can influence higher dimensional homotopical information. Induced crossed modules also include free crossed modules, which are important in applications to defining and determining identities among relations for presentations of groups. This has a relation to the cohomology theory of groups.

Both of these chapters illustrate how some nonabelian calculations in homotopy theory may be carried out using crossed modules. They also show the advantages of having an invariant stronger than just an abelian group of even a module over a group. The latter are pale shadows of the structure of a crossed module.

Finally in this Part, Chapter 6 gives the proof of the GvKT for the functor Π_2. A major interest here is that this proof requires another structure, namely that of *double groupoid with connection,* which we abbreviate to *double groupoid*. We therefore construct a functor

$$\boldsymbol{\rho}_2 : \text{(triples of spaces)} \to \text{(double groupoids)},$$

and show that this is equivalent in a clear sense to a small generalisation of our earlier Π_2 functor, to

$$\Pi_2 : \text{(triples of spaces)} \to \text{(crossed modules of groupoids)}.$$

Here a triple of spaces is of the form (X, X_1, X_0), where $X_0 \subseteq X_1 \subseteq X$, and the pointed case is when X_0 is a singleton.

This substantial chapter develops the 2-dimensional groupoid theory which is then used in the proof of the GvKT, which gives precise situations where $\boldsymbol{\rho}_2$, and hence also Π_2, preserves colimits. The surprising fact is that in this book we are able to obtain many new nonabelian calculations in homotopy theory without any of the standard machinery of algebraic topology, such as simplicial complexes, simplicial approximation, chain complexes, or homology theory.

All this theory generalises to higher dimensions, as we show in Parts II and III, but the new ideas and basic intuitions are more easily explained in dimension 2.

Chapter 1

History

Understanding the context and historical background to the material developed here is, we believe, useful for understanding, for evaluating results, and for analysing potential developments and applications.

It is generally accepted that the notion of abstract group is a central concept of mathematics, and one which allows the successful expression of the intuitions of reversible processes. In order to obtain the higher dimensional, nonabelian, local to global results described briefly in the Preface, the concept of group has:

A) to be 'widened' to that of groupoid, which in a sense generalises the notion of group to allow a spatial component, and

B) to be 'increased in height' to higher dimensions.

Further step A) is a requirement for step B).

We would like to record that a major stimulus for this view was work of Philip Higgins in his 1963 paper [106], and this book is based largely on the resulting collaboration with Brown. Higgins writes in the Preface to [107] that "The main advantage of the transition [from groups to groupoids] is that the category of groupoids provides a good model for certain aspects of homotopy theory. In it there are algebraic models such notions as path, homotopy, deformation, covering and fibration. Most of these become vacuous when restricted to groups, although they are clearly relevant to group-theoretic problems. ... There is another side of the coin: in applications of group theory to other topics it is often the case that the natural object of study is a groupoid rather than a group, and the algebra of groupoids may provide a more concrete tool for handling concrete problems.

In fact there is a range of intuitions which abstract groups are unable to express, and for which other concepts such as groupoid, pseudogroup and inverse semigroup have turned out to be more appropriate. As Mackenzie writes in [138]:

> The concept of groupoid is one of the means by which the twentieth century reclaims the original domain of applications of the group concept. The modern, rigorous concept of group is far too restrictive for the range of geometrical applications envisaged in the work of Lie. There have thus arisen the concepts of Lie pseudogroup, of differentiable and of Lie groupoid, and of principal bundle – as well as various related infinitesimal concepts such as Lie equation, graded Lie algebra and Lie algebroid – by which mathematics seeks to acquire a precise and rigorous language in which to study the symmetry phenomena associated with geometrical transformations which are only locally defined.

A failure to accept a relaxation of the concept of group made it difficult to develop a higher dimensional theory. To see the reasons for this we need to understand the basic intuitions which a higher dimensional theory is trying to express, and to see how these intuitions were dealt with historically. This study will confirm a view that it is reasonable to examine and develop the algebra which arises in a natural way from the geometry rather than insist that the geometry has to be expressed within the current available concepts, schemata and paradigms.

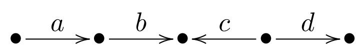
1.1 Basic intuitions

There were two simple intuitions involved. One was the notion of an

algebraic inverse to subdivision.

That is, we know how to cut things up, but do we have available an algebraic control over the way we put them together again? This is of course a general problem in mathematics, science and engineering, where we want to represent and determine the behaviour of complex objects from the way they are put together from standard pieces. Any algebra which gives new insights into questions of this form, and yields new computations, clearly has arguments in its favour.

We explain this a bit more in a very simple situation. We often translate geometry into algebra. For example, a figure as follows:

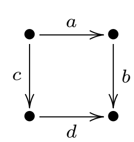


is easily translated into

$$abc^{-1}d.$$

Again, given a diagram as follows:

(1.1.1)



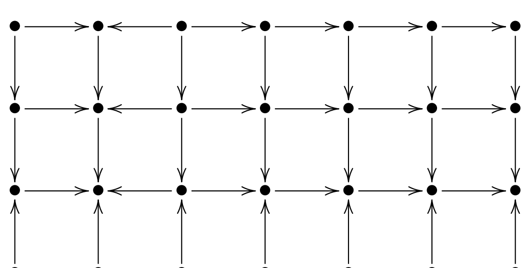
it is easy to write

$$ab = cd, \qquad \text{or} \qquad a = cdb^{-1}.$$

All this is part of the standard repertoire of mathematics. The formulae given make excellent sense as part of say the theory of groups. We also know how to calculate with such formulae.

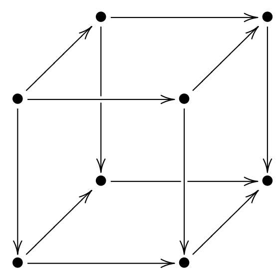
The problem comes when we try to express similar ideas in one dimension higher. How can one write down algebraically the following picture, where each small square is supposed labelled?

(1.1.2)



Again, how can one write down algebraically the formulae corresponding to the above commutative square (1.1.1) but now for the cube:

(1.1.3)



What does it mean for the *faces* of the cube to commute, or for the top face to be the composition, in some sense, of the other faces?

It is interesting that the step from a linear statement to a 2-dimensional statement should need a lot of apparatus; it took a long time to find an appropriate formulation. As we shall see later, the 2-dimensional composition (1.1.2) requires double groupoids or double categories, while the second (1.1.3) requires double groupoids with *thin structure*, or with *connections*.

Thus the step from dimension 1 to dimension 2 is the critical one, and for this reason most of Part I of this book is devoted 2-dimensional case. Further reasons are that the theory is less technical than it becomes in higher dimensions, and that the new features of the 2-dimensional theory need to be well understood before passing to higher dimensions. It is also intriguing that so much can be done once one has the mathematics to express the intuitions, and that the mathematical structures then control the ways the calculations have to go.

1.2 The fundamental group and homology

The above questions on 2-dimensional compositions did not arise out of the void but from a historical context which we now explain.

The intuition for a *Nonabelian Algebraic Topology* was seen early on in algebraic topology, after the ideas of homology and of the fundamental group $\pi_1(X, x)$ of a space X at a point x of X were developed.

The motivation for Poincaré's definition of the fundamental group in his 1895 paper [156] seems to be from the notion of monodromy, that is the change in the value of a meromorphic function of many complex variables as it is analytically continued along a loop avoiding the singularities. This change in value depends only on the homotopy class of the loop, and this consideration led to the notion of the group $\pi_1(X, x)$ of homotopy classes of loops at x, where the group structure arises from composition of loops. Poincaré called this group the *fundamental group*, and this fundamental group $\pi_1(X, x)$, with its relation to covering spaces, surface theory, and the later combinatorial group theory, came to play an increasing rôle in the geometry, complex analysis and algebra of the next hundred years.

It also seems possible that an additional motivation arose from dynamics, in the classification of orbits in a phase space.

The utility of groups in homotopy theory is increased by the relations between the fundamental group considered as a functor from based topological spaces to groups

$$\pi_1 : \mathsf{Top}_* \to \mathsf{Groups}$$

and another functor called the *classifying space*

$$B : \mathsf{Groups} \to \mathsf{Top}_*,$$

which is the composite of the *geometric realisation* and the *nerve functor* N from groups to simplicial sets.

We shall review the properties of B in Section 2.4. Now let us note that B and π_1 are inverses in some sense. To be more precise, BG is a based space that has all homotopy groups trivial except the fundamental group, which itself is isomorphic to G. Moreover, if X is a connected based CW-complex and G is a group, then there is a natural bijection

$$[X, BG]_* \cong Hom(\pi_1 X, G),$$

where the square brackets denote pointed homotopy classes of maps.

It follows that there is a map

$$X \to B\pi_1 X$$

inducing an isomorphism of fundamental groups. It is in this sense that groups are said to model homotopy 1-types, and a computation of a group G is also regarded as a computation of the 1-type of the classifying space BG.

The fundamental group of a space may be calculated in many cases using the Seifert-van Kampen theorem (see Section 1.5), and in others using fibrations of spaces. The main result on the latter, for those familiar with fibrations, is that if $1 \to K \to E \to G \to 1$ is a short exact sequence of groups, then the induced sequence $BK \to BE \to BG$ is a fibration sequence of spaces. Conversely, if $F \xrightarrow{i} X \xrightarrow{p} Y$ is a fibration sequence of spaces, and $x \in F$ then there is an induced exact sequence of groups and based sets

$$\cdots \longrightarrow \pi_1(F, x) \xrightarrow{i_*} \pi_1(X, x) \xrightarrow{p_*} \pi_1(Y, px) \to \pi_0(F) \to \pi_0(X) \to \pi_0(Y).$$

This result gives some information on $\pi_1(X, x)$ if the other groups are known and even more if the various spaces are connected. We shall go back to this sequence in Section 2.6.

Much earlier than the definition of the fundamental group there had been higher dimensional topological information obtained in terms of Betti numbers and torsion coefficients. These were together formulated into the idea of abelian homology groups $H_n(X)$ of a space X defined for all $n \geqslant 0$, and which gave very useful topological information on the space. They measured the presence of ‘holes’ in X of various dimensions and various types. The origins of homology theory lie in integration, the theorems of Green and Stokes, and complex variable theory.

The notion of boundary and of a cycle as having zero boundary is crucial in the methods and results of this theory, but was always difficult to express precisely until Poincaré brought in simplicial decompositions, and the notion of a ‘chain’ as a formal sum of oriented simplices. It seems that the earlier writers thought of a cycle as in some sense a ‘composition’ of the pieces of which it was made, but this was, and still is, difficult to express precisely. Dieudonné in [75] suggests that the key intuitions can be expressed in terms of cobordism. In any case the notion of ‘formal sum’ fitted well with integration, where it was required to integrate over a formal sum of domains of integration, with the correct orientation for these.

It was also found that if X is connected then the group $H_1(X)$ is the fundamental group $\pi_1(X, x)$ made abelian:

$$H_1(X) = \pi_1(X, x)^{\mathrm{ab}}.$$

It was thus clear that the nonabelian fundamental group gave much more information than the first homology group. However, the homology groups were defined in all dimensions. So there was pressure to find a generalisation to all dimensions of the fundamental group.

1.3 The search for higher dimensional versions of the fundamental group

According to [75], Dehn had some ideas on this search in the 1920's, as would not be surprising. The first published attack on this question is the work of Čech, using the idea of classes of maps of spheres instead of maps of circles. He submitted his paper on higher homotopy groups $\pi_n(X, x)$ to the International Congress of Mathematicians at Zurich in 1932. The story is that Alexandroff and Hopf quickly proved that these groups were commutative for $n \geqslant 2$, and so on these grounds persuaded Čech to withdraw his paper. All that appeared in the Proceedings of the Congress was a brief paragraph [65].

The main algebraic reason for this commutativity was the following result, in which the two compositions $\circ_1, \circ_2$ are thought of as compositions of 2-spheres in two directions.

Theorem 1.3.1 *Let S be a set with two monoid structures $\circ_1,\ \circ_2$ each of which is a morphism for the other. Then the two monoid structures coincide and are Abelian.*

Proof The condition that the structure $\circ_1$ is a morphism for $\circ_2$ is that the function

$$\circ_1 : (S, \circ_2) \times (S, \circ_2) \to (S, \circ_2)$$

is a morphism of monoids, where $(S, \circ_2)$ denotes S with the monoid structure $\circ_2$. This condition is equivalent to the statement that for all $x, y, z, w \in S$

$$(x \circ_2 y) \circ_1 (z \circ_2 w) = (x \circ_1 z) \circ_2 (y \circ_1 w).$$

This can be interpreted as saying that the diagram

$$\begin{bmatrix} x & y \\ z & w \end{bmatrix} \qquad \begin{array}{l} \ulcorner\!\!\to 2 \\ \downarrow \\ 1 \end{array}$$

has only one composition. Here the arrows indicate that we are using matrix conventions in which the first coordinate gives the rows, and the second coordinate gives the columns. This law is commonly called the *interchange law.*

We now use some special cases of the interchange law. Let $e_1,\ e_2$ denote the identities for the structures $\circ_1,\ \circ_2$. Consider the matrix

$$\begin{bmatrix} e_1 & e_2 \\ e_2 & e_1 \end{bmatrix}$$

This yields easily that $e_1 = e_2$. We write then e for e_1.

Now we consider the matrix composition

$$\begin{bmatrix} x & e \\ e & w \end{bmatrix}$$

Interpreting this in two ways yields

$$x \circ_1 w = x \circ_2 w.$$

So we write $\circ$ for $\circ_1$.

Finally we consider the matrix composition

$$\begin{bmatrix} e & y \\ z & e \end{bmatrix}$$

and find easily that $y \circ z = z \circ y$. This completes the proof.

Incidentally, it will also be found that associativity comes for free. We leave this to the reader. □

This result seemed to kill any possibility of "nonabelian algebraic topology", or of any generalisations to higher dimensions of the fundamental group. In 1935, Hurewicz published the first of his celebrated notes on higher homotopy groups, and the latter are often referred to as the Hurewicz homotopy groups. The abelian higher homotopy groups came to be accepted, a considerable amount of work in homotopy theory has moved as far as possible from group theory and the nonabelian fundamental group, and the original concern about the abelian nature of the higher homotopy groups came to be seen as a quirk of history, an unwillingness to accept a basic fact of life. Indeed, Alexandroff in his Obituary Notice for Čech referred to the unfortunate lack of appreciation of Čech's work on higher homotopy groups, resulting from too much attention to the disadvantage of their abelian nature [5].

However important nonabelian work using the notion of crossed module was done in dimension 2 by J.H.C. Whitehead in 1941, 1946 and 1949 – these crossed modules are a central theme of this book. Brown remembers Henry Whitehead remarking in 1958 that early workers in homotopy theory were fascinated by the action of the fundamental group on higher homotopy groups. Again, many were dissatisfied with the fact that the composition in higher homotopy groups was independent of the direction. Deeper reasons for this independence are contained in the theory of iterated loop spaces (see the book by Adams [2] or the boopks and survey articles by May [143, 144, 145, 146].

1.4 The origin of the concept of abstract groupoid

A groupoid is defined formally as a small category in which every arrow is invertible. For more detail and a survey see [27, 174].

There are two important, related and relevant differences between groupoids and groups. One is that groupoids have a partial multiplication, and the other is that the condition for two elements to be composable is a geometric one (namely the end point of one is the starting point of the other). This partial multiplication allows for groupoids to be thought of as "groups with many identities". The other is that the geometry underlying groupoids is that of directed graphs, whereas the geometry underlying groups is that of based sets, i.e. sets with a chosen base point. It is clear that graphs are more interesting than sets, and can reflect more geometry. Hence people find in practice that groupoids can reflect more geometry than can groups alone. It seems that the objects of a groupoid allow the addition of a spatial component to group theory.

An argument usually made for groups is that they give the mathematics of reversible processes, and hence have a strong connection with symmetry. This argument applies even more strongly for groupoids. For groups, the processes all start and return to the same position. This is like considering only journeys which start at and end at the same place. However to analyse a reversible process, such as a journey, we must describe the intermediate steps, the stopping places. This requires groupoids, since in this setting the processes described are allowed to start at one point and finish at another. This clearly allows a more flexible and powerful analysis, and confirms a basic intuition that, in dimension 1, groupoids are more convenient than groups for writing down an 'algebraic inverse to subdivision'.

The definition of groupoid arose from Brandt's attempts to extend to quaternary forms Gauss' work on a composition law of binary quadratic forms, which has a strong place in *Disquitiones Arithmeticae.* It is of interest here that Bourbaki [18], p.153, cites this composition law as an influential early example of a composition law which arose not from numbers, even taken in a broad sense, but from distant analogues[1]. Brandt found that each quaternary quadratic form had a left unit and a right unit, and that two forms were composable if and only if the left unit of one was the right unit of the other. This led to his 1926 paper on groupoids [19]. (A modern account of this work on composition of forms is given by Kneser *et al.* [124].)

Groupoids were then used in the theory of orders of algebras. Curiously, groupoids did not form an example in Eilenberg and Mac Lane's basic 1945 paper on category theory [81]. Groupoids appear in Reidemeister's 1932 book on topology [162], as the edge path groupoid, and for handling isomorphisms of a family of structures. The fundamental groupoid of a space was well known by the 1950's, and Crowell and Fox write in [72]:

> A few [definitions], like that of a group or of a topological space, have a fundamental importance to the whole of mathematics that can hardly be exaggerated. Others are more in the nature of convenient, and often highly specialised, labels which serve principally to pigeonhole ideas. As far as this book is concerned, the notions of category and groupoid belong to the latter class. It is an interesting curiosity that they provide a convenient systematisation of the ideas involved in developing the fundamental group.

The fundamental groupoid $\pi_1(X, A)$ on a set A of base points is introduced and used in [30]. Its successes suggest the value of an aesthetic approach to mathematics, namely that the concept which feels right and gives the good exposition is likely to be the most powerful one. In this viewpoint, much good mathematics enables difficult things to bec ome easy, and an important part of the development of good mathematics is finding: (i) the appropriate underlying structures, (ii) the appropriate language to describe these structures, and (iii) means of calculating with these structures.

There is no benefit today in arithmetic in Roman numerals. There is also no benefit today in insisting that the group concept is more fundamental than that of groupoid; one uses each at the appropriate place. It is as well to distinguish the sociology of the use of a mathematical concept from the scientific consideration of its relevance to the progress of mathematics.

It should also be said that the development of new concepts and language is a different activity from the successful employment of a range of known techniques to solve already formulated problems.

The notion that groupoids give a more flexible tool than groups in some situations is only beginning to be widely appreciated. One of the most significant of the books which use the notion seriously is Connes book "Noncommutative geometry", published in 1994 [69]. He states that Heisenberg discovered quantum mechanics by considering the *groupoid of transitions* for the hydrogen spectrum, rather than the usually considered group of symmetry of an individual state. This fits with the previously expounded philosophy. The main examples of groupoids in his book are equivalence relations and holonomy groupoids of foliations.

On the other hand, in books on category theory the role of groupoids is often fundamental (see for example Mac Lane and Moerdijk [136]). In foliation theory, which is a part of differential topology and geometry, the notion of holonomy groupoid is widely used. For surveys of the use of groupoids, see [29, 107, 138, 174].

1.5 The van Kampen Theorem

A change of prospects came about in a roundabout way, in the mid 1960s. R. Brown was writing the book [30] and became dissatisfied with the standard treatments of the van Kampen Theorem, which is a basic tool for

[1] C'est vers cette même époque que, pour le premier fois en Algèbre, la notion de loi de composition s'étend, dans deux directions différents, à des élements qui ne présentent plus avec les ⟨⟨nombres⟩⟩ (au sens le plus large donné jusque-là à ce mot) que des analogies lointaines. La première de ces extensions est due à C.F.Gauss, à l'occasion de ses recherches arithmétiques sur les formes quadratiques ...

computing the fundamental group of a space X given as the union of two connected open subsets U_1, U_2 with connected intersection U_{12}. For those familiar with the concepts, the result is that the natural morphism

$$\pi_1(U_1, x) *_{\pi_1(U_{12},x)} \pi_1(U_2, x) \to \pi_1(X, x) \tag{1.5.1}$$

induced by inclusions is an isomorphism. The group on the left hand side of the above arrow is the free product with amalgamation; it is the construction for groups corresponding to $U_1 \cup U_2$ for spaces, as we shall see later in discussing pushouts. This version of the theorem was given by Crowell [71], based on lectures by R.H. Fox. One important consequence is that the fundamental group shared the same possibilities and difficulties of computation as general abstract groups.

The problem was with the connectivity assumption on U_{12}, since this prevented the use of the theorem for deducing the result that the fundamental group of the circle S^1 is isomorphic to the group $\mathbb{Z}$ of integers. (See Section 1.7 where $\pi_1(S^1)$ is calculated.) If S^1 is the union of two connected open sets, then their intersection cannot be connected. So the fundamental group of the circle is usually determined by the method of covering spaces. Of course this method is basic stuff anyway, and needs to be explained, but having to make this detour, however attractive, is unaesthetic.

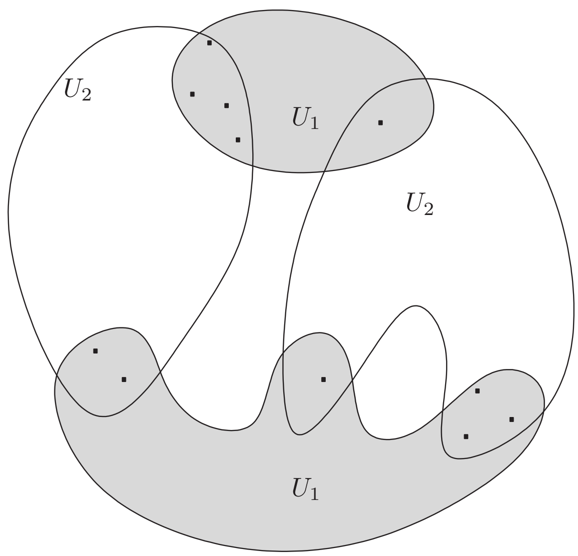


Figure 1.1: Example of spaces in a van Kampen type situation

It was found that a uniform method could be given using nonabelian cohomology [22], but a full exposition of this became turgid. Then Brown came across a paper by Philip Higgins entitled 'Presentations of groupoids with applications to groups' [106], which among other things defined free products with amalgamation of groupoids. We will explain something about groupoids a bit later. It seemed reasonable to insert an exercise in the book on an analogous result to (1.5.1) for the fundamental groupoid $\pi_1(X)$, namely that the natural morphism of groupoids

$$\pi_1(U_1) *_{\pi_1(U_{12})} \pi_1(U_2) \to \pi_1(X) \tag{1.5.2}$$

is an isomorphism. It then seemed desirable to write out a solution to the exercise, and lo and behold! the solution was much better than all the turgid stuff on nonabelian cohomology. Further work yielded the idea that it was sensible to generalise from the fundamental group $\pi_1(X, x)$ on a base point x to the fundamental groupoid $\pi_1(X, A)$ on a set A chosen according to geometric reasons. In particular if U_{12} is not connected it is not clear from which component of U_{12} a base point should be chosen. So one hedges one's bets, and chooses a set of base points, one in each component of U_{12}. One finds that the natural morphism

$$\pi_1(U_1, A) *_{\pi_1(U_{12},A)} \pi_1(U_2, A) \to \pi_1(X, A) \tag{1.5.3}$$

is also an isomorphism and that the proof of this result using groupoids is simpler than the original proof of (1.5.1) for groups. One also obtains a new range of calculations. For example, U_1, U_2, U_{12} may have respectively 27, 63, and 283 components, and yet X could be connected - a description of the fundamental group of this situation in terms of groups alone is not so easy.

In view of these results the book [30] was redirected to give a full account of groupoids and the van Kampen Theorem. A conversation with G.W.Mackey in 1967 informed Brown of Mackey's work on ergodic groupoids. It seemed that if the idea of groupoid arose in two separate fields, then there was more in this than met the eye. As background to Mackey's methods of relating group actions to groupoids the book was strengthened with a account of covering spaces in terms of groupoids, following the initial lead of Higgins and of Gabriel and Zisman [91].

Later Grothendieck was to write (1985):

> "The idea of making systematic use of groupoids (notably fundamental groupoids of spaces, based on a given set of base points), however evident as it may look today, is to be seen as a significant conceptual advance, which has spread into the most manifold areas of mathematics. . . . In my own work in algebraic geometry, I have made extensive use of groupoids - the first one being the theory of the passage to quotient by a "pre-equivalence relation" (which may be viewed as being no more, no less than a groupoid in the category one is working in, the category of schemes say), which at once led me to the notion (nowadays quite popular) of the nerve of a category. The last time has been in my work on the Teichmüller tower, where working with a "Teichmüller groupoid" (rather than a "Teichmüller group") is a must, and part of the very crux of the matter . . . "

1.6 Proof of the van Kampen theorem for the fundamental groupoid

In this section we sketch a proof that the morphism induced by inclusions

$$\eta : \pi_1(U_1, A) *_{\pi_1(U_{12},A)} \pi_1(U_2, A) \to \pi_1(X, A) \tag{1.6.1}$$

is an isomorphism when U_1, U_2 are open subsets of $X = U_1 \cup U_2$ and A meets each path component of U_1, U_2 and $U_{12} = U_1 \cap U_2$.

What one would expect is that the proof would construct directly an inverse to η. Alternatively, the proof would verify in turn that η is surjective and injective.

The proof we give might at first seem roundabout, but in fact it follows an important procedure, that of *verifying a universal property*. One advantage of this procedure is that we do not need to show that the free product with amalgamation of groupoids exists in general, nor do we need to give a construction of it at this stage. Instead we define the free product with amalgamation by its universal property, which enables us to go directly to an efficient proof of the van Kampen Theorem. It also turns out that the universal property is convenient for many explicit calculations.

We use the notion of pushout in the category of groupoids. It is a special case of the pushout in categories that we study in the Appendix. Let us recall the definition in this case. We say that the groupoid G and the two morphisms of groupoids $G_1 \xrightarrow{j_1} G$ and $G_2 \xrightarrow{j_2} G$ are the *pushout* of the two morphisms of groupoids $G \xrightarrow{i_1} G_1$ and $G \xrightarrow{i_2} G_2$ if they satisfy

PO1) the diagram

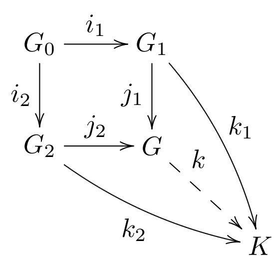
(1.6.2)
$$\begin{array}{ccc} G_0 & \xrightarrow{i_1} & G_1 \\ {\scriptstyle i_2}\downarrow & & \downarrow{\scriptstyle j_1} \\ G_2 & \xrightarrow[j_2]{} & G \end{array}$$

is a commutative square, i.e. $j_1 i_1 = j_2 i_2$,

PO2) it is universal with respect to this type of diagram, i.e. for any groupoid K and morphisms of groupoids $G_1 \xrightarrow{k_1} K$ and $G_2 \xrightarrow{k_2} K$ such that the following diagram is commutative

(1.6.3)
$$\begin{array}{ccc} G_0 & \xrightarrow{i_1} & G_1 \\ {\scriptstyle i_2}\downarrow & & \downarrow{\scriptstyle k_1} \\ G_2 & \xrightarrow[k_2]{} & K \end{array}$$

then there is a unique morphism of groupoids $k : G \to K$ such that $kj_1 = k_1, kj_2 = k_2$. The two diagrams are often combined into one as follows:

(1.6.4)


We think of a pushout square as given by a standard input, the pair (i_1, i_2), and a standard output, the pair (j_1, j_2). The properties of this standard output are defined by reference to *all other* commutative squares with the same (i_1, i_2). At first sight this might seem strange, and logically invalid. However a pushout square is somewhat like a computer program: given the data of another commutative square of the right type, then the output will be a morphism (k in the above diagram) with certain defined properties.

It is a basic feature of universal properties that the standard output, in this case the pair (j_1, j_2) making the diagram commute, is determined up to isomorphism by the standard input (i_1, i_2). The further details will be given in the Appendix.

Thus in our case, we have

Theorem 1.6.1 *If U_1, U_2 are open subsets of X, $X = U_1 \cup U_2$, and A is a subset of $U_{12} = U_1 \cap U_2$ meeting each path component of U_1, U_2, U_{12} (and therefore of X), the following diagram of morphisms induced by inclusion*

$$\begin{array}{ccc} \pi_1(U_{12}, A) & \xrightarrow{i_1} & \pi_1(U_1, A) \\ {\scriptstyle i_2}\downarrow & & \downarrow{\scriptstyle j_i} \\ \pi_1(U_2, A) & \xrightarrow[j_2]{} & \pi_1(X, A) \end{array}$$

is a pushout of groupoids.

Proof So we suppose given a commutative diagram of morphisms of groupoids

$$\begin{array}{ccc} \pi_1(U_{12},A) & \xrightarrow{i_1} & \pi_1(U_1,A) \\ {\scriptstyle i_2}\downarrow & & \downarrow{\scriptstyle k_1} \\ \pi_1(U_2,A) & \xrightarrow[k_2]{} & K \end{array} \tag{1.6.5}$$

We have to prove that there is a unique morphism $k : \pi_1(X, A) \to K$ such that $kj_1 = k_1, kj_2 = k_2$.

Let us take an element $[\alpha] \in \pi_1(X, A)$ with $\alpha : (I, \partial I) \to (X, A)$. By the Lebesgue covering lemma ([30, 3.6.4]) there is a subdivision

$$0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = 1$$

of I into intervals by equidistant points such that α maps each $[t_i, t_{i+1}]$ into U_1 or U_2 (possibly in both). Choose one of these for each i and call it U^i and α_i the restriction of α. This subdivision determines a decomposition

$$\alpha = \alpha_0\alpha_1 \ldots \alpha_{n-1}.$$

Of course the point $\alpha(t_i)$ need not lie in A, but it lies in $U^i \cap U^{i-1}$ and this intersection can only be U_1, U_2 or U_{12}. By the connectivity conditions, for each $i = 0, 1, \cdots, n-1$, we may choose a path c_i in $U^i \cap U^{i-1}$ joining $\alpha(t_i)$ to A. Moreover, if $\alpha(t_i)$ already lies in A (which is the case when $i = 0$ and when $i = n$), we choose c_i to be the constant path at $\alpha(t_i)$.

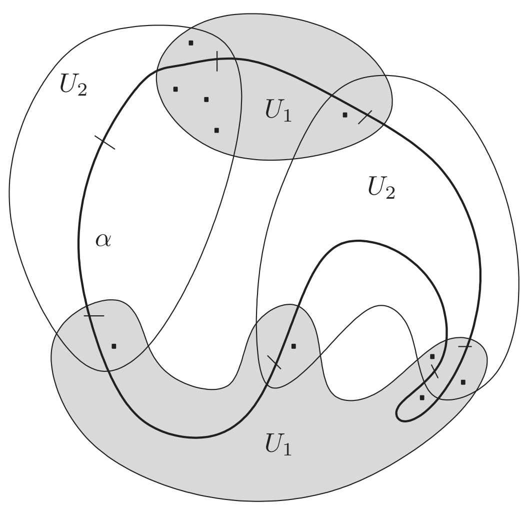


Figure 1.2: A path α in a van Kampen type situation

For each $0 \leqslant i < n$ we have the path $\beta_i = c_i^{-1}\alpha_i c_{i+1}$ in U^i joining points of A. It is clear that

$$[\alpha] = [\beta_0][\beta_1] \cdots [\beta_{n-1}] \in \pi_1(X, A).$$

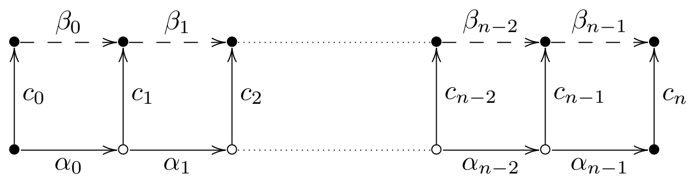
Notice that β_i also represents a class in $\pi_1(U^i, A)$. Let us call $\psi_i = k_i([\beta_i])$. If the homomorphism of groups that commutes the external square k exists, the value of $k([\alpha])$ is determined, because

$$k([\alpha]) = k([\beta_0][\beta_1] \cdots [\beta_{n-1}]) = k([\beta_0])k([\beta_1]) \cdots k([\beta_{n-1}]) = \psi_0\psi_1 \cdots \psi_{n-1}.$$

This proves uniqueness of k. We have also proved that $\pi_1(X, A)$ is generated as a groupoid by the images of $\pi_1(U_1, A), \pi_1(U_2, A)$.

We have yet to prove that the element $k([\alpha])$ is independent of all the choices made. Before going into that, notice that the construction we have just made can be interpreted diagrammatically as follows. The starting situation looks like the bottom side of the diagram

(1.6.6)



where the solid circles denote points which definitely lie in A.

The way of getting β_i may be seen as composing with a retraction from above like the one in the Fig 1.3.

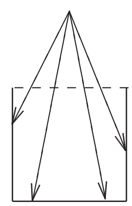
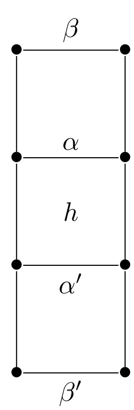


Figure 1.3: Retraction from above-centre

If necessary, this retraction also provides a homotopy $\alpha \simeq \beta_0\beta_1 \cdots \beta_{n-1}$ rel end points. This is the first of lots of filling arguments where we have defined a map in a subset of the boundary of a cube and fill the whole cube by appropriate retractions. This is studied in all generality in Chapter 16, using 'expansions' and 'collapses'.

We shall use another filling argument in I^3 to prove independence of choices. Suppose that we have a homotopy rel end points $h : \alpha \simeq \alpha'$ of two maps $(I, \partial I) \to (X, A)$. We can perform the construction in (1.6.6) for each of α, α', and then glue the three homotopies together.

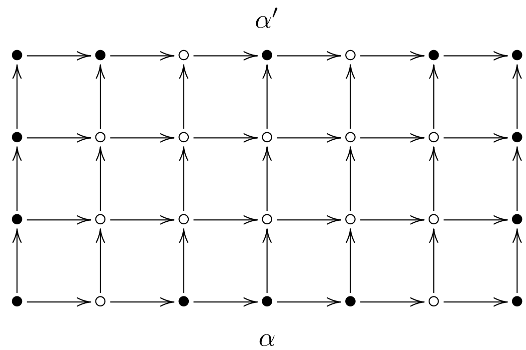
(1.6.7)



So, replacing βs by αs, we can assume the maps α, α' have subdivisions $\alpha = [\alpha_i]$, $\alpha' = [\alpha'_j]$ such that each α_i, α'_j has end points in A and lies in one of U_1, U_2. Since h is a map $I^2 \to X$, we may again by the Lebesgue covering lemma make a subdivision $h = [h_{lm}]$ such that each h_{lm} lies in one of U_1, U_2. Also by further subdivision as necessary, we may assume this subdivision of h refines on $I \times \partial I$ the given subdivisions of α, α'.

The problem is that none of the vertices of this subdivision are necessarily mapped into A, except those on $\partial I \times I$ (since the homotopy is rel vertices and α, α' both map ∂I to A) and those on $I \times \partial I$ determined by the subdivisions of α, α'. So the situation looks like the following:

(1.6.8)



We want to deform h to $h' : \alpha \simeq \alpha'$, a new homotopy rel end points between the same maps, having the same subdivision as does h, and such that any subsquare mapped by h into U_i, $i = 1, 2$ remains so in h', and any vertex already in A is not moved. This is done inductively by filling arguments in the cube I^3.

Let us imagine the 3-dimensional cube I^3 as $I^2 \times I$ where I^2 has the subdivision we are working with in h. Define the bottom map to be h. We have to fill I^3 so that in the top face we get a similar diagram but with all the vertices solid, i.e. in A, and each subsquare in the top face lies in the same U_i as the corresponding in the bottom one.

We start by defining the map on all 'vertical' edges, i.e. on $\{v\} \times I$ for all vertices in the partition of I^2. If the image of a vertex lies in U_{12} but not in A, then we choose a path in U_{12} joining it to a point of A. We work similarly for the case of vertices with images in $U_1 \setminus U_{12}$, $U_2 \setminus U_{12}$. Let us call e_{lm} the path we have chosen between the vertex $h(s_l, t_m)$ and A. (These e_{lm} are constant if $h(s_l, t_m)$ lies already in A.) This gives us the map on the vertical edges of I^3.

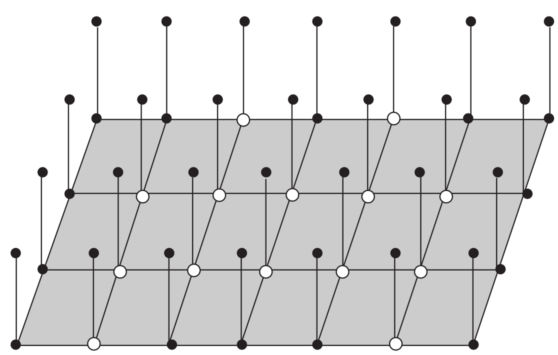


Figure 1.4: Extending to the edges

From now on, we restrict our construction to the part of I^3 over the square $S_{lm} = [s_l, s_{l+1}] \times [t_m, t_{m+1}]$. Let us call $\sigma_{lm} = h|_{[s_l,s_{l+1}]\times\{t_m\}}$ and $\tau_{lm} = h|_{\{s_l\}\times[t_m,t_{m+1}]}$. Then, using the retraction of Figure 1.3 on each lateral face, we can fill all the faces of a 3-cube except the top one. Now, using the retraction from a point on a line perpendicular to the centre of the top face, as in the following Figure 1.5

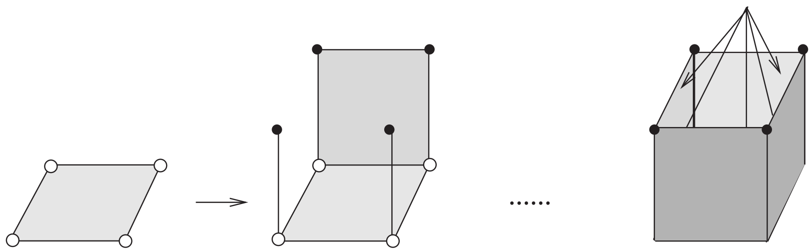
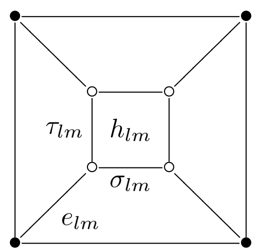


Figure 1.5: Extending to the lateral faces

we get at the top face a map that looks like

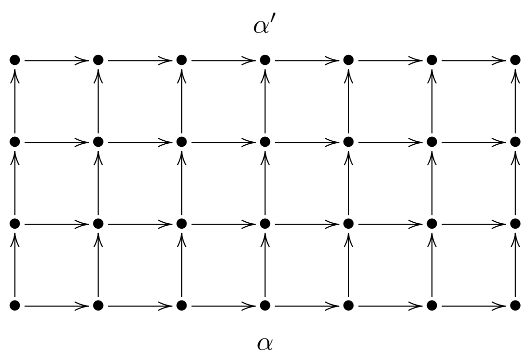
(1.6.9)



Thus, in particular, it is a map into U_i sending all vertices in A.

If we do the above construction in each square of the subdivision, we get a top face of the cube that is an homotopy rel end points between two paths in the same classes as α and α', and subdivided in such a way that each subsquare goes into some U_i sending all vertices into A. Each of these squares produces a commutative square of paths in one of $\pi_1(U_i, A)$, $i = 1, 2$. Thus the diagram can be pictured as

(1.6.10)



Applying k_i to each subsquare we get a commutative square l_i in K. Since $k_1 i_1 = k_2 i_2$, we get that the l_i compose in K to give a square l in K.

Now comes the vital point. Since **the composite of commutative squares in a groupoid produces a commutative square**, the external square l is commutative.

But because of the way we constructed it, two sides of this composite commutative square l in K are identities. Therefore the opposite sides of l are equal. This shows that our element $k([\alpha])$ is independent of the choices made, and so proves that k is well defined as a function on arrows of the fundamental groupoid $\pi_1(X, A)$.

The proof that k is a morphism is now quite simple, while uniqueness has already been shown. So we have shown that the diagram in the statement of the theorem is a pushout of groupoids.

This completes the sketch proof. □

In the case of commutative squares, there is another way of expressing the above argument on the composition of commutative squares being a commutative square, namely by working on formulae for each individual square as in the expression $a = cdb^{-1}$ for (1.1.1). Putting together two such squares as in

(1.6.11)
$$\begin{array}{ccccc} \bullet & \xrightarrow{a} & \bullet & \xrightarrow{e} & \bullet \\ {\scriptstyle c}\downarrow & & {\scriptstyle b}\downarrow & & \downarrow{\scriptstyle f} \\ \bullet & \xrightarrow[d]{} & \bullet & \xrightarrow[g]{} & \bullet \end{array}$$

allows cancellation of the middle term

$$ae = (cdb^{-1})(bgf^{-1}) = cdgf^{-1}$$

which if $c = 1, f = 1$ reduces to $ae = dg$. This argument extends to longer gluings of commutative squares, and hence extends, by induction, and in the other direction, to a subdivision of a square.

We would like to extend the above argument to the faces of a cube, and then to an n-dimensional cube.

For a cube, the expression of one of the faces in terms of the others can be done (see the Homotopy Addition Lemma 6.7.7) and then can be used to prove a 2-dimensional van Kampen theorem. That is done in Section 6.8.

It is much more difficult to follow this route in the general case and a more roundabout method is developed in Chapter 16. The algebra to carry out this argument in dimension n is given in Chapter 15. It is interesting that such a complicated and subtle algebra seems to be needed to make it all work.

Remark 1.6.2 One of the nice things about proving the theorem by verifying the universal property is that the proof uses some calculations in a general groupoid K, and groupoids have, in some sense, the minimal set of properties needed for the result. This avoids a calculation in $\pi_1(X, A)$, and somehow makes the calculations as easy as possible. The same characteristics hold in some other verifications of universal properties, for example in the computation of the fundamental group of an orbit space in [30]. We will see a similar situation later for double groupoids.

1.7 The fundamental group of the circle

In order to interpret the last theorem, one has to set up the basic algebra of computational groupoid theory. In particular, one needs to be able to deal with presentations of groupoids. This is done to a good extent in [107, 30]. Here we can give only the indications of the theory.

The theory of groupoids may be thought of as an algebraic analogue of the theory of groups, but based on directed graphs rather than on sets. For some discussion of the philosophy of this, see [31].

We refer to the Appendix for the construction of a free groupoid over a directed graph.

Let us get over with some basic definitions in groupoid theory. A groupoid G is called *connected* if $G(a, b)$ is non empty for all $a, b \in Ob(G)$. The maximal connected subgroupoids of G are called the *(connected) components* of G.

If a is an object of the groupoid G, then the set $G(a,a)$ inherits a group structure from the composition on G, and this is called the *object group* of G at a and is written also $G(a)$. The groupoid G is called *simply connected* if all its object groups are trivial. If it is connected and simply connected, it is called 1-*connected*, or an *indiscrete groupoid*.

A standard example of an indiscrete groupoid is the groupoid $I(S)$ on a set S. This has object set S and arrow set $S \times S$, with $s, t : S \times S \to S$ being the first and second projections. The composition on $I(S)$ is given by

$$(a,b)(b,c) = (a,c), \quad \text{for all } a,b,c \in S.$$

A directed graph X is called *connected* if the free groupoid $F(X)$ on X is connected, and is called a *forest* if every object group $F(X)(a)$ of $F(X), a \in Ob(X)$, is trivial. A connected forest is called a *tree*. If X is a tree, then the groupoid $F(X)$ is indiscrete.

Let G be a connected groupoid and a_0 be an object of G. For each $a \in Ob(G)$ choose an arrow $\tau a : a_0 \to a$, with $\tau a_0 = 1_{a_0}$. Then an isomorphism

$$\phi : G \to G(a_0) \times I(Ob(G))$$

is given by $g \mapsto ((\tau a)g(\tau b)^{-1}, (a,b))$ when $g \in G(a,b)$ and $a,b \in Ob(G)$. The composition of ϕ with the projection yields a morphism $\rho : G \to G(a_0)$ which we call a *deformation retraction*, since it is the identity on $G(a_0)$ and is in fact homotopic to the identity morphism of G, though we do not elaborate on this fact here.

It is also standard [30, 8.1.5] that a connected groupoid G is isomorphic to the free product groupoid $G(a_0) * T$ where $a_0 \in Ob(G)$ and T is any wide, tree subgroupoid of G. The importance of this is as follows.

Suppose that X is a graph which generates the connected groupoid G. Then X is connected. Choose a maximal tree T in X. Then T determines for each a_0 in $Ob(G)$ a retraction $\rho_T : G \to G(a_0)$ and the isomorphisms

$$G \cong G(a_0) * I(Ob(G)) \cong G(a_0) * F(T)$$

show that a morphism $G \to K$ from G to a groupoid K is completely determined by a morphism of groupoids $G(a_0) \to K$ and a graph morphism $T \to K$ which agree on the object a_0.

We shall use later the following proposition, which is a special case of [30, 6.7.3]:

Proposition 1.7.1 *Let G, H be groupoids with the same set of objects, and let $\phi : G \to H$ be a morphism of groupoids which is the identity on objects. Suppose that G is connected and $a_0 \in Ob(G)$. Choose a retraction $\rho : G \to G(a_0)$. Then there is a retraction $\sigma : H \to H(a_0)$ such that the following diagram, where ϕ' is the restriction of ϕ:*

(1.7.1)
$$\begin{array}{ccc} G & \xrightarrow{\rho} & G(a_0) \\ {\scriptstyle\phi}\downarrow & & \downarrow{\scriptstyle\phi'} \\ H & \xrightarrow[\sigma]{} & H(a_0) \end{array}$$

is commutative and is a pushout of groupoids.

This result can be combined with Theorem 1.6.1 to determine the fundamental group of the circle S^1.

Corollary 1.7.2 *The fundamental group of the circle S^1 is a free group on one generator.*

Proof We represent S^1 as the union of two semicircles E^1_+, E^1_- with intersection $\{-1, 1\}$. Then both fundamental groupoids $\pi_1(E^1_+, \{-1, 1\})$ and $\pi_1(E^1_-, \{-1, 1\})$ are easily seen to be isomorphic to the connected groupoid $\mathcal{I}$ with object set $\{-1, 1\}$ and trivial object groups. In fact this groupoid is the free groupoid on one generator $\iota : -1 \to 1$.

Also, $\pi_1(\{-1, 1\}, \{-1, 1\})$ is the discrete groupoid on these objects $\{-1, 1\}$. By an application of Theorem 1.6.1 we get a pushout of groupoids

$$\begin{array}{ccc} \{-1,1\} & \longrightarrow & \mathcal{I} \\ \downarrow & & \downarrow \\ \mathcal{I} & \longrightarrow & \pi_1(S^1, \{-1,1\}) \end{array}$$

From the previous Proposition, we have a pushout of groupoids

$$\begin{array}{ccc} \mathcal{I} & \longrightarrow & \{1\} \\ \downarrow & & \downarrow \\ \pi_1(S^1, \{-1,1\}) & \longrightarrow & \pi_1(S^1, 1) \end{array}$$

Gluing them, we get a pushout of groupoids

$$\begin{array}{ccc} \{-1,1\} & \longrightarrow & \{1\} \\ \downarrow & & \downarrow \\ \mathcal{I} & \longrightarrow & \pi_1(S^1, 1) \end{array}$$

and the result follows by an easy universal argument. □

Note that S^1 may be regarded as a pushout in the category of topological spaces

$$\begin{array}{ccc} \{-1,1\} & \longrightarrow & \{1\} \\ \downarrow & & \downarrow \\ [-1,1] & \longrightarrow & S^1 \end{array}$$

The correspondence between these last two diagrams was for R.Brown a major incentive to exploring the use of groupoids. Here we have a successful algebraic model of a space, but of a different type from that previously considered. An aspect of its success is that groupoids have structure in two dimensions, namely 0 and 1, and this is useful for modelling the way spaces are built up using identifications in dimensions 0 and 1.

Another interesting aspect is that the groupoid $\mathcal{I}$ is finite, and it is easy to explore all its properties. By contrast, the integers form an infinite set, and discussion of its properties usually requires induction.

The problem was to find analogous methods in higher dimensions.

1.8 Higher order groupoids

The successes of the use of groupoids in 1-dimensional homotopy theory and the successes in group theory as exposed in the books [30, 107] suggested the potential interest in the use of groupoids in higher dimensional homotopy theory. In particular, it seemed possible that a higher dimensional van Kampen theorem could be proved if the 'right' higher homotopy groupoids could be constructed, with properties analogous to those which enabled the proof of this theorem in dimension 1.

Experiments by Brown to obtain such a construction in the years 1965-74 proved abortive. However in 1971 Chris Spencer came to Bangor as a Science Research Council Research Assistant, and in this and a subsequent period considerable progress was made on the discovering the algebra of double groupoids. It was in this time that the relation with crossed modules was made, so linking the notion of double groupoids with more classical ideas.

Crossed modules had been defined by J.H.C. Whitehead in 1946 [177] in order to express the properties of the properties of the boundary map

$$\partial : \pi_2(X, X_1, x) \to \pi_1(X_1, x)$$

of the second relative homotopy group, a group which is in general nonabelian. He gave the first nontrivial determination of this group in showing that when X is formed from X_1 by attaching 2-cells, then $\pi_2(X, X_1, x)$ is isomorphic to the free crossed $\pi_1(X_1, x)$-module on the characteristic maps of the 2-cells.

This result was a crucial clue to Brown and Higgins in 1974. On the one hand it showed that a universal property, namely freeness, did exist in 2-dimensional homotopy theory. Also, if our proposed theory was to be any good, it should have this theorem as a corollary. However, Whitehead's theorem was about *relative homotopy groups*, which suggested that we should look at a relative theory, i.e. a space X with a subspace X_1. With the experience obtained by then, we quickly found a satisfactory, even simple, construction of a relative homotopy double groupoid $\rho_2(X, X_1, x)$ and a proof of a van Kampen theorem, as envisaged.

The equivalence between these sorts of double groupoids and crossed modules proved earlier by Brown and Spencer, then gave the required van Kampen type theorem for the second homotopy crossed module, and so new calculations of second relative homotopy groups.

So we have a pattern of proof:

A) construct a homotopically defined multiple groupoid,

B) prove it is equivalent to a more familiar homotopical construction, and

C) prove a van Kampen theorem in the multiple groupoid context.

The three combined give new nonabelian, higher dimensional, local-to-global results. This pattern has been followed in the corresponding result for crossed complexes, which is dealt with in our Part II, and results for the cat^n-groups of Loday [128]. However, we do not discuss the latter in this book.

Crossed modules had occurred earlier in other places. In the mid 1960s the great school of Grothendieck in Paris had considered sets with two structures, that of group and of groupoid, and had proved these were equivalent to crossed modules. However this result was not published, and so was known only to a restricted group of people.

It is now clear that once one moves to higher version of groupoids, the presence of crossed modules is inevitable, and is an important part of the theory and applications. This is why Part I is devoted entirely to the crossed modules and double groupoid area.

Chapter 2

Homotopy theory and crossed modules

In this chapter we explain how crossed modules over groups arose in topology, and give some of the later developments. The beginning lies in the first half of the last century.

The topologist Henry Whitehead was steeped in the combinatorial group theory of the 1930's, and much of his work can be seen as trying to extend the methods of group theory to higher dimensions, keeping the interplay with geometry and topology. These attempts led to greatly significant work, such as the theory of simple homotopy types. His ideas on crossed modules have taken longer to come into wide use, but they can be regarded as equally significant.

One of his starting points was the van Kampen Theorem for the fundamental group. This tells us in particular how the fundamental group is affected by the attaching of a 2-cell, or of a family of 2-cells, to a space. Namely, if $X = A \cup \{e_i^2\}_{i \in I}$, where the 2-cell e_i^2 is attached by a map which for convenience we suppose is $f_i : (S^1, 1) \to (A, x)$, then each f_i determines an element ϕ_i in $\pi_1(A, x)$, and a consequence of the van Kampen Theorem for the fundamental group is that the group $\pi_1(X, x)$ is obtained from the group $\pi_1(A, x)$ by adding the relations ϕ_i, $i \in I$.

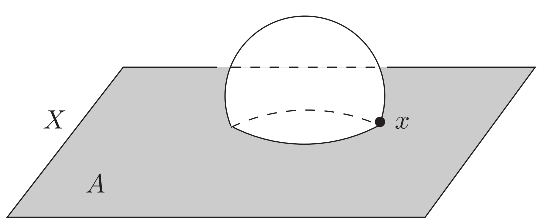


Figure 2.1: Picture of an attached 2-cell

The next problem was clearly to determine the effect on the higher homotopy groups of adding cells to a space. So his 1941 paper [176] was entitled 'On adding relations to homotopy groups'. If we could solve this in general then we would in particular be able to calculate all homotopy groups of spheres. Work over the last 60 years has shown the enormous difficulty of this task.

In this paper he gave important results in higher dimensions, but he was able to obtain information on second homotopy groups of $X = A \cup \{e_i^2\}_{i \in I}$. His results were clarified by him in two subsequent papers using the notion of *crossed module* and then *free crossed module*. This formulation became the key for Brown and Higgins to higher order van Kampen Theorems, as we shall see later. His basic method of proof uses what is now called transversality, and has become the foundation of a technique called 'pictures' [110]. His algebraic methods have also been recently exploited rather differently and in a more algorithmic way in [59] to compute second homotopy modules.

We begin the chapter by giving a definition of the fundamental crossed module

$$\Pi_2(X, A, x) = (\partial : \pi_2(X, A, x) \to \pi_1(A, x))$$

of a pointed pair of spaces and explaining some of Whitehead's work. Then we state two central results:
- the 2-dimensional van Kampen theorem, in Section 2.3;
- the notion of classifying space of a crossed module, in Section 2.4.

It is these two combined which give many of the important homotopical applications of crossed modules (including Whitehead's results). However the construction of the classifying space, and the proof of its properties, needs the methods of crossed complexes of Part II. We give applications of the 2-dimensional van Kampen theorem in Chapters 4 and 5 and prove it in Chapter 6. This sets the scene for the corresponding higher dimensional results of Part II.

Section 2.5 shows that crossed modules are equivalent to another algebraic structure, that of cat^1-groups. This is used in Section 2.6 to obtain the cat^1-group of a fibration, which yields an alternative way of obtaining the fundamental crossed module.

Section 2.7 shows that crossed modules are also equivalent to 'categories internal to groups', or, equivalently, to groupoids internal to groups. This is important philosophically, because groupoids are a generalisation of equivalence relations, and equivalence relations give an expression of the idea of quotienting, a fundamental process in mathematics and science, because it is concerned with classification. We can think of groupoids as giving ways of saying not only that two objects *are equivalent*, but also *how they are equivalent*: the arrows between two objects give different 'equivalences' between them., which can sometimes be regarded as 'proofs' that the objects are equivalent.

Moving now to the case of groups, to obtain a quotient of a group P we need not just an equivalence relation, but this equivalence relation needs to be a *congruence*, i.e. not just a subset but also a subgroup of $P \times P$. An elementary result in group theory is that a congruence on a group P is determined completely by a normal subgroup of P. The corresponding result for groupoids is that a groupoid with a group structure is equivalent to a crossed module $M \to P$ where P is the group of objects of the groupoid.

This family of equivalent structures – crossed modules, cat^1-groups, group objects in groupoids – gives added power to each of these structures. In fact in Chapter 6 we will use crucially another related structure, that of *double groupoids with connection*. This is equivalent to an important generalisation of a crossed module, that of *crossed module of groupoids*, which copes with the varied base points of second relative homotopy groups.

2.1 Homotopy groups and relative homotopy groups

Recall that two maps $f, g : X \to Y$ between two topological spaces are said to be homotopic if f can be continuously deformed to g. Formally, they are *homotopic*, and this is denoted by $f \simeq g$, if there is a map

$$F : X \times I \to Y$$

such that $F_0(x) = F(x, 0) = f(x)$ and $F_1(x) = F(x, 1) = g(x)$. The map F is called a *homotopy* from f to g.

This definition gives an equivalence relation among the set of maps from X to Y. The quotient set is denoted $[X, Y]$ and the equivalence class of a map f is denoted by $[f]$.

Sometimes we are interested in considering only deformations that keep some subset fixed. If $A \subseteq X$, we say that two maps as above are *homotopic relative to* A, and denote this by $f \simeq g \operatorname{rel} A$, if there is a homotopy F from f to g satisfying $F(a,t) = f(a)$ for all $a \in A$, $t \in I$. This definition gives another equivalence relation among the set of maps from X to Y. The quotient set is denoted $[X,Y]_A$ and the equivalence class of a map f is again denoted by $[f]$.

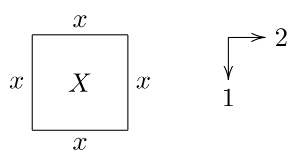
A particular case of this definition is when we study maps sending a fixed subset A of X to a given point $y \in Y$. Then the quotient set corresponding to maps from X to Y sending all A to y with respect to homotopy rel A, is written as $[(X,A),(Y,y)]$ or, when $A = \{x\}$, as $[X,Y]_*$.

To define the *homotopy groups of a space*, we consider homotopy classes of maps from particular spaces. Namely if $x \in X$, the n-th homotopy group of X based at x is defined as

$$\pi_n(X,x) = [(I^n, \partial I^n), (X,x)]$$

where ∂I^n is the boundary of I^n. The elements of $\pi_n(X,x)$ are classes of maps that can be pictured for $n=2$ as in the following diagram:

(2.1.1)

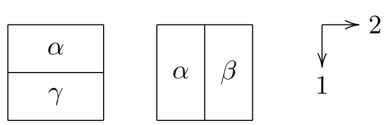


where we use throughout all the book a matrix like convention for directions. One of the reasons for this will become clear in Chapter 6.

In the case $n = 1$ we obtain the fundamental group $\pi_1(X,x)$. For all $n \geqslant 1$ there initially seem to be n group structures on this set induced by the composition of representatives given for $1 \leqslant i \leqslant n$ by

$$(f +_i g)(t_1, t_2, \ldots, t_n) = \begin{cases} f(t_1, t_2, \ldots, 2t_i, \ldots, t_n) & \text{if } 0 \leqslant t_i \leqslant 1/2, \\ g(t_1, t_2, \ldots, 2t_i - 1, \ldots, t_n) & \text{if } 1/2 \leqslant t_i \leqslant 1. \end{cases}$$

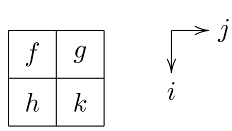
Remark 2.1.1 For the case $n = 2$ the following diagrams picture the two compositions.



$\alpha +_1 \gamma$ $\qquad$ $\alpha +_2 \beta$

Theorem 2.1.2 *If $n \geqslant 2$, then any of the multiplications $+_i$, $i = 1, \cdots, n$ on $\pi_n(X,x)$ induce the same group structure, and all these group structures are abelian.*

Proof By Theorem 1.3.1, we need only to verify the interchange law for the compositions $+_i, +_j, 1 \leqslant i < j \leqslant n$. It is easily seen that if $f, g, h, k : (I^n, \partial I^n) \to (X,x)$ are representatives of elements of $\pi_n(X,x)$, then the two compositions obtained by evaluating the following matrix in two ways



in fact coincide. The verification consists in checking the formula for such a multiple composition. □

We shall need later that π_n is functorial in the sense that to any map $\phi : X \to Y$ there is associated a homomorphism of groups

$$\phi_* : \pi_n(X, x) \to \pi_n(Y, \phi(x))$$

defined by $\phi_*[f] = [\phi f]$, and which satisfies the usual functorial properties $(\phi\psi)_* = \phi_*\psi_*, 1_* = 1$.

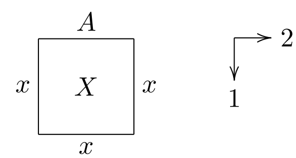
Now we may repeat everything for maps of triples and homotopies among them. By a *based pair of spaces* (X, A, x) is meant a space X, a subspace A of X and a base point $x \in A$. The n^{th} *relative homotopy group* $\pi_n(X, A, x)$ of the based pair (X, A, x) is defined as the homotopy classes of maps of triples

$$\pi_n(X, A, x) = [(I^n, \partial I^n, J^{n-1}), (X, A, x)]$$

where $J^{n-1} = \{1\} \times I^{n-1} \cup I \times \partial I^{n-1}$. That is we consider maps $\alpha : I^n \to X$ such that $\alpha(\partial I^n) \subseteq A$ and $\alpha(J^{n-1}) = \{x\}$ and homotopies through maps of this kind.

The picture we shall have in mind as representing elements of $\pi_n(X, A, x)$ is

(2.1.2)



As before, a multiplication on $\pi_n(X, A, x)$ is defined by the compositions $+_i$ in any of the last $(n-1)$ directions. It is not difficult to check that any of these multiplications gives a group structure and analogously to Theorem 2.1.2 these all agree and are abelian if $n \geqslant 3$. Also, for any maps of based pairs $\phi : (X, A, x) \to (Y, B, y)$, there is a homomorphism of groups

$$\phi_* : \pi_n(X, A, x) \to \pi_n(Y, B, y)$$

as before.

The homotopy groups defined above fit nicely in an exact sequence called the *homotopy exact sequence of the pair* as follows:

(2.1.3)

$$\begin{array}{l} \cdots \to \pi_n(X, x) \xrightarrow{j_*} \pi_n(X, A, x) \xrightarrow{\partial_n} \pi_{n-1}(A, x) \xrightarrow{i_*} \pi_{n-1}(X, x) \to \cdots \\ \xrightarrow{j_*} \pi_2(X, A, x) \xrightarrow{\partial_2} \pi_1(A, x) \xrightarrow{i_*} \pi_1(X, x) \xrightarrow{j_*} \\ \xrightarrow{j_*} \pi_1(X, A, x) \xrightarrow{\partial_1} \pi_0(A, x) \xrightarrow{i_*} \pi_0(X, x) \end{array}$$

where i_* and j_* are the homomorphisms defined by the inclusions, and ∂ is given by restriction, i.e. for any $[\alpha] \in \pi_n(X, A, x)$ represented by a map $\alpha : (I^n, \partial I^n, J^{n-1}) \to (X, A, x)$, we define $\partial[\alpha] = [\alpha']$ where α' is the restriction of α to the face $\{0\} \times I^{n-1}$, which we identify with I^{n-1}.

This exact sequence is of abelian groups and homomorphisms until $\pi_2(X, x)$, of groups and homomorphisms until $\pi_1(X, x)$, and of based sets for the last three terms. The amount of exactness for the last terms is the same as for the exact sequence of a fibration of groupoids [25, 27].

The final interesting piece of structure is the existence of a $\pi_1(A, x)$-*action* on all the terms of the above exact sequence which are groups. Let us define this action. For any $[\alpha] \in \pi_n(X, A, x)$ and any $[\omega] \in \pi_1(A, x)$, we define the map

$$F = F(\alpha, \omega) : I^n \times \{0\} \cup J^{n-1} \times I \to X$$

given by α on $I^n \times \{0\}$ and by ω on $\{t\} \times I$, for any $t \in J^{n-1}$. Then we have defined F on the subset of I^{n+1} indicated in Figure 2.2

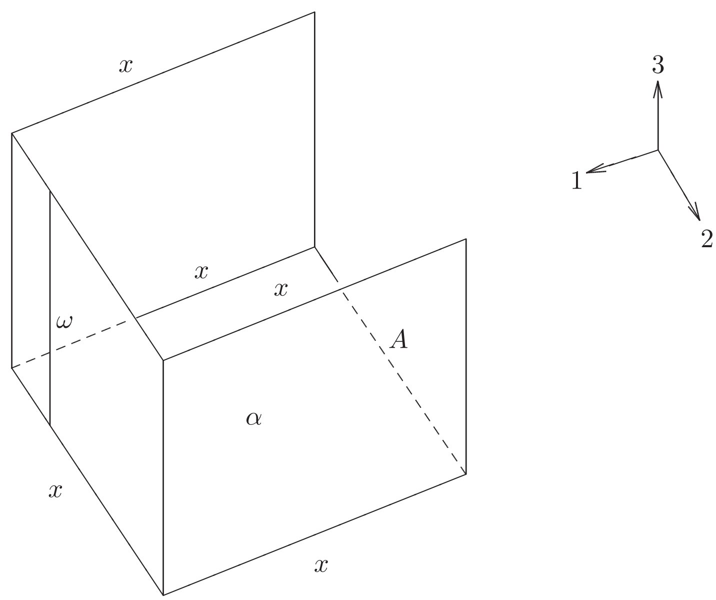


Figure 2.2: Action of $\pi_1(A, x)$

Now, we compose with the retraction

$$r : I^{n+1} \to I^n \times \{0\} \cup J^{n-1} \times I$$

given by projecting from a point $P = (0, \frac{1}{2}, \frac{1}{2}, \cdots, \frac{1}{2}, 2)$ and indicated in Figure 2.3, getting a map $Fr : I^{n+1} \to X$ extending F.

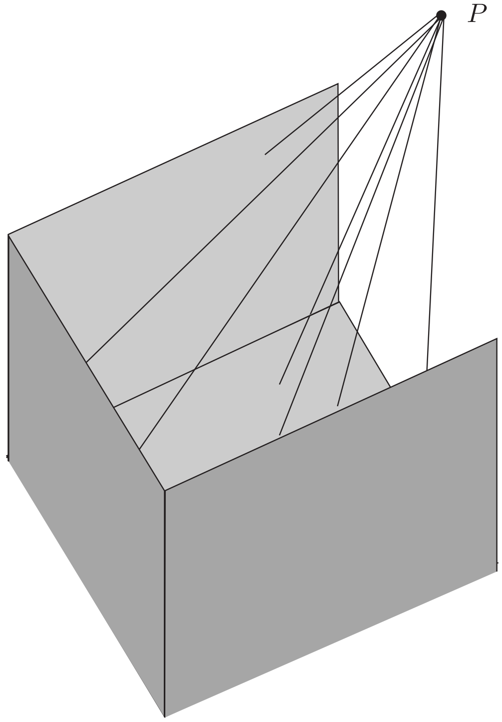


Figure 2.3: Retraction from above-lateral

The "restriction" map

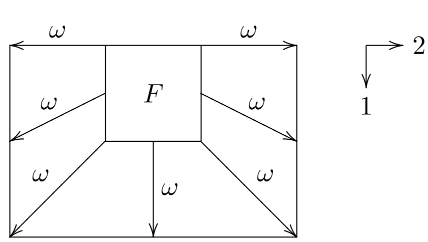
$$I^n \cong I^n \times \{1\} \hookrightarrow I^{n+1} \xrightarrow{Fr} X$$

represents an element $[\alpha]^{[\omega]} \in \pi_n(X, A, x)$.

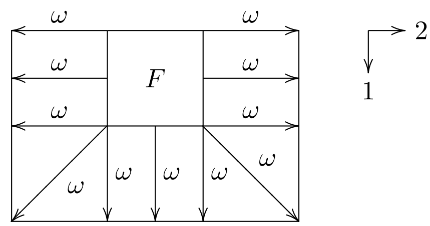
We leave the reader to develop proofs that the action is an action of a group on a group, that is that various axioms are satisfied. However all this will follow in a more algebraic fashion using the theory given in chapter 16.

Notice that in this definition we use another of the filling arguments that we have started using in the proof of Theorem 1.6.1 in Section 1.6. Arguments of the same kind prove that the assignment just defined is independent of the several choices involved (α, ω and the extension of F), and it defines an action.

Remark 2.1.3 Notice that when $n = 2$ the map representing $[\alpha]^{[\omega]}$ could be drawn



or, equivalently we could have chosen the one described as follows



In a similar way, we may define an action of $\pi_1(X,x)$ on $\pi_n(X,x)$. In our case, this gives an action of $\pi_1(A,x)$ on both $\pi_n(A,x)$ and $\pi_n(X,x)$. Moreover, all maps in the homotopy exact sequence are maps of $\pi_1(A,x)$-groups.

All the above constructions can be repeated for *based r-ads* $X_* = (X; X_1, X_2, \dots, X_r, x)$, where all X_i are subspaces of X. Homotopy groups $\pi_n X_*$ are defined for $n \geqslant r+1$ and are abelian for $n \geqslant r+2$. There are various long exact sequences relating the homotopy groups of $(r+1)$-ads and r-ads. An account of these is in [116]. The homotopy groups of an $(r+1)$-ad are also a special case of the homotopy groups of an r-cube of spaces [128, 53, 93]. All these groups are important for discussing the failure of excision for relative homotopy groups, to which we have referred earlier, and whose analysis in some cases using nonabelian methods will be an important feature of this book.

2.2 Whitehead's work on crossed modules

We start with the basic definition of crossed module. All crossed modules in this chapter are going to be over a group, so we shall not insist in this point.

Definition 2.2.1 A *crossed module (over a group)* $\mathcal{M} = (\mu : M \to P)$ is a morphism of groups $\mu : M \to P$ called the boundary of $\mathcal{M}$ together with an action $(m,p) \mapsto m^p$ of P on M satisfying the two axioms

CM1) $\mu(m^p) = p^{-1}\mu(m)p$

CM2) $n^{-1}mn = m^{\mu n}$

for all $m, n \in M$, $p \in P$.

When we wish to emphasise the codomain P, we call $\mathcal{M}$ a *crossed P-module.*

Basic algebraic examples of crossed modules are:

- A *conjugation crossed module* is an inclusion of a normal subgroup $N \lhd G$, with action given by conjugation. In particular, for any group P the identity map $\mathrm{Id}_P : P \to P$ is a crossed module with the action of P on itself by conjugation. T. Porter has remarked that the concept of crossed module can be seen as an 'externalisation' of the concept of normal subgroup. That is, an inclusion is replaced by a homomorphism with special properties. This process occurs in other algebraic situations.
- if M is a group, its *automorphism crossed module* has the form $(\chi : M \to \mathrm{Aut}(M))$ where χm is the inner automorphism mapping n to $m^{-1}nm$. If A satisfies $\mathrm{Inn}(M) \leqslant A \leqslant \mathrm{Aut}(M)$ and $\chi(M) \subseteq A$, we also call the automorphism crossed module to $(\chi : M \to A)$.
- A *P-module crossed module* has zero boundary and M is a P-module.

- A *central extension* crossed module $(\mu : M \to P)$ has surjective boundary with kernel contained in the centre of M and $p \in P$ acts on $m \in M$ by conjugation with any element of $\mu^{-1}p$.
- Any homomorphism $(\mu : M \to P)$, with M abelian and $\operatorname{Im} \mu$ in the centre of P, provides a crossed module with P acting trivially on M.

The category $\mathsf{XMod/Groups}$ of crossed modules has as objects all crossed modules over groups. Morphisms in $\mathsf{XMod/Groups}$ from $\mathcal{M}$ to $\mathcal{N}$ are pair of group homomorphisms (g, f) forming commutative diagrams with the two boundaries,

$$\begin{array}{ccc} M & \xrightarrow{g} & N \\ {\scriptstyle\mu}\downarrow & & \downarrow{\scriptstyle\nu} \\ P & \xrightarrow[f]{} & Q \end{array}$$

and preserving the action in the sense that for all $m \in M, p \in P$ we have $g(m^p) = (gm)^{fp}$. If P is a group, then the category XMod/P of *crossed P-modules* is the subcategory of $\mathsf{XMod/Groups}$ whose objects are the crossed P-modules and whose morphisms are the group homomorphisms $g : M \to N$ such that g preserves the action (i.e. $g(m^p) = (gm)^p$, for all $m \in M,\ p \in P$), and $\nu g = \mu$.

Here are some elementary general properties of crossed modules which we will often use.

Proposition 2.2.2 *For any crossed module $\mu : M \to P$, μM is a normal subgroup of P, i.e. $\mu M \lhd P$.*

Proof This is immediate from CM1). □

The *centraliser* $C(S)$ of a subset S of a group M is the set of elements of M which commute with all elements of S. In particular, $C(M)$ is written ZM and called the *centre* of M and is abelian. Any subset of ZM is called *central* in M.

The *commutator* of elements m, n of a group M is the element $[m, n] = m^{-1}n^{-1}mn$. The *commutator subgroup* $[M, M]$ of M, is the normal subgroup of M generated by all commutators. We write M^{ab} for the abelian group $M/[M, M]$, the *abelianisation* of M.

Proposition 2.2.3 *Let $\mu : M \to P$ be a crossed module, and let $C = \mathit{Coker}\, \mu$. Then*

(i) *$\mathit{Ker}\, \mu$ is central in M.*

(ii) *$\mu(M)$ acts trivially on ZM.*

(iii) *ZM and $\mathit{Ker}\, \mu$ inherit an action of C to become C-modules.*

(iv) *P acts on M^{ab} and $\mu(M)$ acts trivially on M^{ab} which inherits an action of C to become a C-module.*

Proof Axiom CM2) shows that if $m, n \in M$ and $\mu n = 1$ then $mn = nm$. This proves (i). On the other hand, and by CM2) and CM1), $mn = nm$ implies $m^{\mu n} = m$, and this proves (ii). Then (iii) follows using these and Proposition 2.2.2, which implies $C = P/\mu(M)$.

Since $[m, n]^p = [m^p, n^p]$ for $m, n \in M,\ p \in P$, we have $[M, M]$ is P-invariant, so that P acts on M^{ab}. However in this action $\mu(M)$ acts trivially since if $m, n \in M$ then

$$m^{\mu n} = n^{-1}mn = m \bmod [M, M].$$

□

Thus for any crossed module ($\mu : M \to P$) with $C = \text{Coker } \mu$, $\pi = \text{Ker } \mu$ we have an exact sequence of C-modules

$$\pi \longrightarrow M^{\text{ab}} \longrightarrow (\mu M)^{\text{ab}} \longrightarrow 1.$$

The first map is not injective in general. To see this, consider the crossed module $\chi : M \to \mathsf{Aut}(M)$ associated to a group M. Then $\pi = \text{Ker } \chi = ZM$, the centre of M. There are groups M for which

$$1 \neq ZM \subseteq [M, M],$$

for example the quaternion group, the dihedral groups and many others. For all these the composite map $\pi \to ZM \to M^{\text{ab}}$ is trivial and so not injective. These examples give point to the following useful result.

Proposition 2.2.4 *If there is a section $s : \mu M \to M$ of μ which is a group homomorphism (but not necessarily a P-map) then M is isomorphic as a group to $\pi \times \mu M$. Further $[M, M] \cap \pi = 1$, and the map $\pi \to M^{\text{ab}}$ is injective.*

Proof Because s is a section (i.e. μs is the identity on μM) we have that $M = (\pi)(\text{Im } s)$ and $\pi \cap (\text{Im } s) = \{1\}$. Because the action of Im s on π is trivial, we have an internal product decomposition $M = (\pi) \times (\text{Im } s)$. Furthermore, by Proposition 2.2.3 we know that π is abelian so $[M, M] = [\text{Im } s, \text{Im } s]$.

So, $[M, M] \cap \pi = \{1\}$ and $\pi \to M^{\text{ab}}$ is injective. □

An important example where the section s exists is when $\mu(M)$ is a free group. The well known Schreier Theorem of combinatorial group theory, that a subgroup of a free group is itself free (see for example [131] or [118] and also [107] for a groupoid proof) assures us that this is the case when M itself is free.

The major geometric example of a crossed module is the following, where the basic definitions were given in the last Section. Let (X, A, x) be a based pair of spaces, that is X is a topological space and $x \in A \subseteq X$. Whitehead showed that the boundary map

(2.2.1) $$\partial : \pi_2(X, A, x) \to \pi_1(A, x),$$

from the second relative homotopy group of (X, A, x) to the fundamental group $\pi_1(A, x)$, together with the standard action of $\pi_1(A, x)$ on $\pi_2(X, A, x)$, has the structure of crossed module. This result and its proof will be seen in various lights in this book. Because of this example it is convenient and sensible to regard crossed modules $\mu : M \to P$ as 2-dimensional versions of groups, with P, M being respectively the 1- and 2-dimensional parts. This analogy also will be pursued in more detail later. At this stage we only note that the full description of the 2-dimensional part requires specification of its 1-dimensional foundation and of the way the two parts fit together: that is, we need the whole structure of crossed module.

Now we see that we have a functor from based pairs of topological spaces to crossed modules

(2.2.2) $$\Pi_2 : \mathsf{Top}^2_* \to \mathsf{XMod/Groups}$$

which sends the based pair (X, A, x) to the crossed module given in (2.2.1) above. (Later we shall formulate a groupoid version of this functor, allowing the base point to vary, but it is best to get familiar with this special case at first.)

The work of Whitehead on crossed modules over the years 1941-1949 contained in [176, 177, 179] and mentioned in the Introduction to this Chapter can be summarised as follows.

He started trying to obtain information on how the higher homotopy groups of a space are affected by adding cells. For the fundamental group, the answer is a direct consequence of the van Kampen Theorem:

> adding a 2-cell corresponds to adding a relation to the fundamental group, adding an n-cell for $n \geqslant 3$ does not change the fundamental group.

So the next question is:

> how is the second homotopy group affected by adding 2-cells?, i.e. if $X = A \cup \{e_i^2\}$, what is the relation between $\pi_2(A)$ and $\pi_2(X)$?

In the first paper ([176]), he formulated a geometric proof of a theorem in this direction. In the second paper ([177]) he gave the definition of crossed module and showed that the second relative homotopy group $\pi_2(X, A, x)$ of a pair of spaces could be regarded as a crossed module over the fundamental group $\pi_1(A, x)$. In the third paper ([179]) he introduced the notion of free crossed module and showed that his previous work could be reformulated as showing that the second relative homotopy group $\pi_2(X, A, x)$ was isomorphic to the *free crossed module* on a set of generators corresponding to the 2-cells. This concept of free crossed module will be studied in detail in Section 3.4.

He was not in fact able to obtain any detailed computations as a result of this result, but it was fundamental to his work on the classification of homotopy 2-types, and, together with the concept of chain complex with operators that we shall develop in the second part, on a range of realisation problems [176, 177].

The proof he gave was difficult to read, since it was spread over three papers, with some notation changes, and that is why a repackaged version of the proof by Brown was accepted for publication [25]. The main ideas of the proof included knot theory, and also transversality, techniques of which became fashionable only in the 1960s (see also [110]). A number of other proofs have been given, including one we give in this book (see Corollary 5.4.8) in which the result is seen as a special case of a 2-dimensional van Kampen type theorem.

The way this work was developed by Whitehead seems a very good example of what Grothendieck has called 'struggling to bring new concepts out of the dark' through the search for the underlying structural features of a geometric situation.

Whitehead's work on free crossed modules parallelled independent work by Reidemeister and his student Renee Peiffer at about the same time on the closely related notion of identities among relations [162, 154], which we deal with in Section 3.1. Whitehead also acknowledged in [176] that some of his results on second homotopy groups were also obtainable from work of Reidemeister on chain complexes with operators, now recognised as given by the complex of cellular chains of the universal cover of the space, and which has been extensively used for example in simple homotopy theory [68].

2.3 The 2-dimensional van Kampen Theorem

Whitehead's theorem on free crossed modules referred to in the last section demonstrated that a particular universal property was available for homotopy theory in dimension 2. This suggested that there was scope for some broader kind of universal property at this level.

It also gave a clue to a reasonable approach. Such a universal property, in order to be broader, would clearly have to include Whitehead's theorem. Now this theorem is about the fundamental crossed module of a particular pair of spaces. So the broader principle should be about the fundamental crossed modules of *pairs* of spaces. The simplest property would seem to be, in analogy to the van Kampen Theorem, that the functor

$$\Pi_2 : \mathsf{Top}^2_* \to \mathsf{XMod/Groups}$$

described in (2.2.2) preserves certain pushouts. This led to the formulation of the next theorem. Also there had been a long period of experimentation by Brown and Spencer on the relations between crossed modules and double groupoids [61, 60], and by Higgins on calculation with crossed modules, so that the proof of the theorem, and the deduction of interesting calculations, came fairly quickly in 1974.

The next two theorems correspond to Theorem C of this Brown and Higgins paper ([39]). We separate the statement into two theorems for an easier understanding. The first one is about coverings by two (open) subspaces, the second one about adjunction spaces.

First, we say the based pair (X, A) is *connected* if A and X are path connected and for $x \in A$ the induced map of fundamental groups $\pi_1(A, x) \to \pi_1(X, x)$ is surjective, or, equivalently, using the homotopy exact sequence, when $\pi_1(X, A, x) = 0$.

Having in mind that all pairs are based but not including the base point in the statement, we have:

Theorem 2.3.1 *Let A, U_1, and U_2 be subspaces of X such that the total space X is covered by the interiors of U_1 and U_2. We define $U_{12} = U_1 \cap U_2$, and $A_\nu = A \cap U_\nu$ for $\nu = 1, 2, 12$. If the pairs (U_ν, A_ν) are connected for $\nu = 1, 2, 12$, then:*

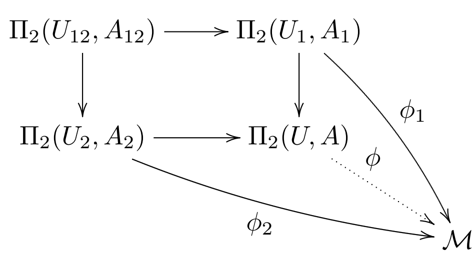
(Con) *The pair (X, A) is connected.*

(Iso) *The following diagram induced by inclusions*

$$\begin{array}{ccc} \Pi_2(U_{12}, A_{12}) & \longrightarrow & \Pi_2(U_2, A_2) \\ \downarrow & & \downarrow \\ \Pi_2(U_1, A_1) & \longrightarrow & \Pi_2(X, A) \end{array} \tag{2.3.1}$$

is a pushout of crossed modules.

Remark 2.3.2 Recall that this statement means that the above mentioned diagram is commutative and has the following universal property: For any crossed module $\mathcal{M}$ and morphisms of crossed modules $\phi_\nu : \Pi_2(U_\nu, A_\nu) \to \mathcal{M}$ for $\nu = 1, 2$ making the external square commutative, there is a unique morphism of crossed modules $\phi : \Pi_2(X, A) \to \mathcal{M}$ such that the diagram



commutes.

There is a slightly more general version of the theorem for adjunction spaces that can be deduced from the preceding theorem by using general mapping cylinder arguments.

Theorem 2.3.3 *Let X and Y be spaces, A a subset of X and $f : A \to Y$ a map. We consider subspaces $X_1 \subseteq X$ and $Y_1 \subseteq Y$ and define $A_1 = X_1 \cup A$ and $f_1 = f| : A_1 \to Y_1$. If the inclusions $A \subseteq X$ and $A_1 \subseteq X_1$ are closed cofibrations and the pairs $(Y, Y_1), (X, X_1), (A, A_1)$ are connected, then:*

(Con) *The pair $(X \cup_f Y, X_1 \cup_{f_1} Y_1)$ is connected.*

(Iso) *The following diagram induced by inclusions*

$$
\begin{array}{ccc}
\Pi_2(A, A_1) & \longrightarrow & \Pi_2(Y, Y_1) \\
\downarrow & & \downarrow \\
\Pi_2(X, X_1) & \longrightarrow & \Pi_2(X \cup_f Y, X_1 \cup_{f_1} Y_1)
\end{array} \qquad (2.3.2)
$$

is a pushout of crossed modules.

Remark 2.3.4 The term closed cofibration included in the hypothesis of the theorem is satisfied in a great number of useful cases. It can be intuitively interpreted as saying that the placing of A in X and of A_1 in X_1 are 'locally not wild'.

The interest in these theorems is at least seven fold:

- The theorem does have Whitehead's Theorem as a consequence (see Corollary 5.4.8).
- The theorem is a very useful computational tool and gives information unobtainable so far by other sources.
- The theorem is an example of a local-to-global theorem. Such theorems play an important rôle in mathematics and its applications.
- The theorem deals with nonabelian objects, and so cannot be proved by traditional means of algebraic topology.
- The two available proofs use groupoid notions in an essential way.
- The existence of the theorem confirms the value of the crossed module concept, and of the methods used in its proof. We should be interested in algebraic structures for which this kind of result is true.
- It shows the difficulty of homotopy theory since one has, it appears, to go through all this just to determine, as we explain in Section 5.8, the second homotopy groups of certain mapping cones.

A further point is that the proof we shall give later does not assume the general existence of pushouts of crossed modules. What it does is directly verify the required universal property in this case.

These theorems are deduced in the above mentioned paper from a more general theorem on double groupoids ([39, Theorem B]) which will be proved as Theorem 6.8.2 in its appropriate setting. Now we conclude this section by stating a more general version of Theorem 2.3.1 for general covers of a space X that can also be deduced from Theorem 6.8.2.

Let Λ be an indexing set and suppose we are given a family $\mathcal{U} = \{U_\lambda\}_{\lambda \in \Lambda}$ of subsets of X such that the interiors of the sets of $\mathcal{U}$ cover X. For each $\nu = \{\nu_1, \cdots, \nu_n\} \in \Lambda^n$, we write

$$U_\nu = U_{\nu_1} \cap \cdots \cap U_{\nu_n}.$$

Let A be a subspace of X, and define $A_\nu = U_\nu \cap A$, for any $\nu \in \Lambda^n$. Suppose also given a base point $x \in A$ which is contained in every X_λ.

Theorem 2.3.5 *Assume that for every $\nu \in \Lambda^n, n \geqslant 1$, the pair (U_ν, A_ν) is connected. Then*

(Con) *the pair (X, A) is connected, and*

(Iso) *the crossed module* $\Pi_2(X, A)$ *satisfies the following universal property: For any crossed module* $\mathcal{M}$ *and any family of morphisms of crossed modules* $\{\phi_\lambda : \Pi_2(U_\lambda, A_\lambda) \to \mathcal{M} \mid \lambda \in \Lambda\}$ *such that for any* $\lambda, \mu \in \Lambda$ *the diagram*

$$\begin{array}{ccc} \Pi_2(U_{\lambda\mu}, A_{\lambda\mu}) & \longrightarrow & \Pi_2(U_\lambda, A_\lambda) \\ \downarrow & & \downarrow \phi_\lambda \\ \Pi_2(U_\mu, A_\mu) & \xrightarrow{\phi_\mu} & \mathcal{M} \end{array}$$

commutes, there is a unique morphism of crossed modules $\phi : \Pi_2(X, A) \to \mathcal{M}$ *such that all triangles of the form*

$$\begin{array}{ccc} \Pi_2(U_\lambda, A_\lambda) & \longrightarrow & \Pi_2(X, A) \\ & \searrow & \downarrow \phi \\ & & \mathcal{M} \end{array}$$

commute.

The universal property of the theorem can be expressed as what is called a 'co-equaliser condition' (see Appendix).

Remark 2.3.6 It can be easily seen from the proof that the conditions on n-fold intersections for all $n \geqslant 1$ can be relaxed to path connectivity of all 4-fold intersections, and 1-connectivity of all pairs given by 8-fold intersections. More refinements of the arguments, using Lebesgue covering dimension, reduce these numbers to 3 and 4 respectively. These improvements were originally shown by Razak Salleh in his thesis [161].

The proof of Theorem 2.3.5 will be given later via another intermediate algebraic structure, that of double groupoids, since these have properties which are more appropriate than are those of crossed modules for expressing the geometry of the proof.

2.4 The classifying spaces of a group and of a crossed module

We are going to give in the second part of this book the construction of the classifying space of a crossed complex that includes as particular cases the classifying spaces of a group and of a crossed module.

Nevertheless, this is a good point to recall some of the properties of both. In particular we want to stress that these classifying spaces classify the 1-type and the 2-type of a space.

The *classifying space of a group* P is a functorial construction

$$B : \mathsf{Groups} \to \mathsf{Top}_*$$

assigning a reduced CW-complex BP to each group P so that

Proposition 2.4.1 *The homotopy groups of the classifying space of the group* P *are given by*

$$\pi_i(BP) \cong \begin{cases} P & \text{if } i = 1, \\ 0 & \text{if } i \geqslant 2. \end{cases}$$

This gives a natural equivalence from $\pi_1 B$ to the identity. There is also some relation between $B\pi_1$ and the identity. It is given by

Proposition 2.4.2 *Let X be a reduced CW-complex and let $\phi : \pi_1(X) \to P$ be a homomorphism of groups. Then there is a map*

$$X \to BP$$

inducing the homomorphism ϕ on fundamental groups.

As consequence we get that $B\pi_1$ captures all information on fundamental groups.

Theorem 2.4.3 *Let X be a reduced CW-complex and let $P = \pi_1(X)$. Then there is a map*

$$X \to BP$$

inducing an isomorphism of fundamental groups.

It is because of these results that groups are said to model pointed, connected homotopy 1-types.

Next, we indicate a definition and state some properties of the *classifying space of a crossed module* $\mathcal{M} = (\mu : M \to P)$ B. It is a functor

$$B : \mathsf{XMod} \to \mathsf{Top}_*$$

assigning to $\mathcal{M}$ a pointed CW-space $B\mathcal{M}$ with the following properties:

Proposition 2.4.4 *The homotopy groups of the classifying space of the crossed module $\mathcal{M}$ are given by*

$$\pi_i(B\mathcal{M}) \cong \begin{cases} Coker\,\mu & for\ i = 1 \\ Ker\,\mu & for\ i = 2 \\ 0 & for\ i > 2. \end{cases}$$

There is a twofold relation with the classifying space of a group defined before. On one hand, it is a generalisation, i.e.

Proposition 2.4.5 *If P is a group then the classifying space $B(1 \to P)$ is exactly the classifying space BP discussed before.*

On the other hand

Proposition 2.4.6 *Let $M \triangleleft P$. Then the morphism of crossed modules $(M \to P) \to (1 \to P/M)$ induces a homotopy equivalence of classifying spaces $B(M \to P) \to B(P/M)$.*

This follows from Whitehead's theorem, that a map of CW-spaces inducing an isomorphism of all homotopy groups is a homotopy equivalence.

Proposition 2.4.7 *The classifying space BP is a subcomplex of $B\mathcal{M}$, and there is a natural isomorphism of crossed modules*

$$\Pi_2(B\mathcal{M}, BP) \cong \mathcal{M}. \tag{2.4.1}$$

Theorem 2.4.8 *Let X be a reduced CW-complex, and let $\Pi_2(X, X^1)$ be the crossed module $\pi_2(X, X^1) \to \pi_1(X^1)$, where X^1 is the 1-skeleton of X. Then there is a map*

$$X \to B(\Pi_2(X, X^1)) \tag{2.4.2}$$

inducing an isomorphism of π_1 and π_2.

It is because of these results that it is reasonable to say that crossed modules model all pointed connected homotopy 2-types. This result is originally due to Mac Lane and Whitehead [137] (they use the term 3-type for what later came to be called 2-type), and with a different proof.

Later we shall give by means of crossed complexes an elegant description of the cells of the classifying space $B(M \to P)$. The existence and properties of the classifying space show that calculations of pushouts of crossed modules, such as those required by the 2-dimensional van Kampen Theorem, may also be regarded as calculations of homotopy 2-types. This is evidence that we do have in the fundamental crossed module of a pair an appropriate candidate for a 2-dimensional version of the fundamental group, as sought by an earlier generation of topologists.

The situation we have for crossed modules and pairs of spaces comes under the following format:

(2.4.3)
$$\begin{array}{ccc} \text{(topological data)} & \underset{\mathcal{B}}{\overset{\Pi}{\rightleftarrows}} & \text{(algebraic data)} \\ & {}_U\searrow \quad \swarrow_B & \\ & \mathsf{Top} & \end{array}$$

We suppose the following properties:

(i) The functor Π preserves certain colimits.

(ii) There is a natural equivalence $\Pi\mathcal{B} \simeq 1$.

(iii) $B = U\mathcal{B}$.

(iv) There is a convenient natural transformation $1 \simeq \mathcal{B}\Pi$ preserving some homotopy properties.

Property (i) is a form of the van Kampen Theorem. This enables some computations to get started.
Property (ii) shows that the algebraic data forms a reasonable mirror of the topological data.
Property (iii) allows the classifying space to be defined: U is some kind of forgetful functor.
Property (iv) is difficult to state precisely in general terms. The intention is to show that the structure $\mathcal{B}\Pi$ captures some slice of the homotopy properties of the original topological data.

We shall not use any general format of or deduction from these properties, but it should be realised that the material we give on groups and on crossed modules forms part of a much more general pattern.

Let us finish this section by giving also some indications of how to go up one dimension further. First a theorem about the behaviour of that classifying space of crossed modules functor when applied to a short exact sequence. This theorem will be deduced from a more general theorem on the classifying space of crossed complexes, where more machinery is available for the proof.

Theorem 2.4.9 *Suppose the commutative diagram*

(2.4.4)
$$\begin{array}{ccccccccc} 0 & \longrightarrow & L & \xrightarrow{i} & M & \xrightarrow{p} & N & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle\lambda} & & \downarrow{\scriptstyle\mu} & & \downarrow{\scriptstyle\nu} & & \\ 1 & \longrightarrow & K & \xrightarrow[j]{} & P & \xrightarrow[f]{} & Q & \longrightarrow & 1 \end{array}$$

is such that the vertical arrows are crossed modules, the squares are morphisms of crossed modules, and the rows are exact sequences of groups. Then the diagram of induced maps of classifying spaces

$$B(L \to K) \to B(M \to P) \to B(N \to Q)$$

is a fibration sequence.

In the above situation we say that the crossed module $L \to K$ is a kernel of the morphism (p, f) of crossed modules. Note that the groups L, K are essentially normal subgroups of M, P respectively. There is an additional property, that if $k \in K, m \in M$, then $p(m^{-1}m^{j(k)}) = 1$, so that $m^{-1}m^{j(k)} \in \operatorname{Im} i$. This gives rise to a function $h : K \times M \to L$. The properties are summarised by saying that the first square of diagram (2.4.4) is a *crossed square* [128]. This structure gives the next stage after crossed modules for modelling homotopy types, that is they model homotopy 3-types. There seem to be good reasons why the analysis of kernels should give rise to a higher order structure modelling a further level of homotopy types. These ideas are quite subtle and require notions of 'crossed squares' which cannot be pursued this book (see [53] and [158]).

2.5 Cat^1-groups.

There are several algebraic and combinatorial categories that are equivalent to the category of crossed modules. Some of these equivalences were already known to Verdier in the late 60's, but the first published account seems to have been by Brown and Spencer in 1976 [61]. Later, these equivalences have been generalised by Ellis in [83] to a more categorical setting.

Of the categories equivalent to XMod/Groups, perhaps the most used is the category Cat^1- Groups of cat^1-groups. One of its advantages is the naturality of the generalisation to higher dimensions and in this way was used for Loday in [128]. It is also useful in some cases when describing the colimits used in the van Kampen Theorem.

In this section, we explain this equivalence and some of the applications. Let us begin by trying to express the basic properties of a crossed module $\mathcal{M} = (\mu : M \to P)$ in an alternative way.

The action of P on M can be encoded using the semidirect product $P \ltimes M$. Then, the map μ gives a homomorphism $t : P \ltimes M \to P \ltimes M$ by the rule $(p, m) \mapsto (p\mu(m), 1)$. (Then t is a homomorphism of groups by CM1)).

It is a bit more difficult to find the way CM2) can be translated, but after playing for a while can be seen that it gives that the elements of $\operatorname{Ker} t$ and those of M commute in $P \ltimes M$. This is the kind of algebraic object we are going to need.

A *cat*^{1}*-group* is a triple $\mathcal{G} = (G, s, t)$ such that G is a group and $s, t : G \to G$ are group homomorphisms satisfying

CG1) $st = t$ and $ts = s$

CG2) $[\operatorname{Ker} s, \operatorname{Ker} t] = 1$.

A *homomorphism of cat*^{1}*-groups* between (G, s, t) and (G', s', t') is a homomorphism of groups $f : G \to G'$ preserving the structure, i.e. such that $s'f = fs$ and $t'f = ft$. These objects and morphisms define the category Cat^1- Groups of cat^1-groups.

Example 2.5.1 The category of groups, Groups, can be considered a full subcategory of Cat^1- Groups using the inclusion functor

$$I : \mathsf{Groups} \to \mathsf{Cat^1\text{-}Groups}$$

given by $I(G) = (G, Id, Id)$.

Having in mind the discussion at the beginning of this section, we define a functor

$$\lambda : \mathsf{XMod/Groups} \to \mathsf{Cat^1\text{-}Groups}$$

given by $\lambda(\mu : M \to P) = (P \ltimes M, s, t)$, where $s(g, m) = (g, 1)$ and $t(g, m) = (g(\mu m), 1)$.

Proposition 2.5.2 *If $\mu : M \to P$ is a crossed module, then $\lambda(\mu : M \to P)$ is a cat^1-group.*

Proof It is clear that s is a homomorphism. To check that t is also a homomorphism , let us consider elements $(g, m), (g', m') \in P \ltimes M$. Then, we have

$$\begin{aligned} t((g,m)(g',m')) &= t(gg', m^{g'}m') \\ &= (gg'\mu(m^{g'})\mu m'), 1) = (gg'g'^{-1}\mu m g'\mu m'), 1) \quad \text{by CM1)} \\ &= (g\mu m g'\mu m'), 1) = t(g,m)t(g',m'). \end{aligned}$$

It is also easy to prove that s, t satisfy CG1).

To check CG2), let us consider generic elements $(1, m) \in \text{Ker } s$ and $(\mu m', m'^{-1}) \in \text{Ker } t$. Then, we have

$$\begin{aligned} (1,m)(\mu m', m'^{-1}) &= (\mu m', m^{\mu m'} m'^{-1}) = (\mu m', m'^{-1} m m' m'^{-1}) \quad \text{by CM2)} \\ &= (\mu m', m'^{-1} m) = (\mu m', m'^{-1})(1, m). \end{aligned}$$

□

Example 2.5.3 Thus, associated to any normal subgroup M of G, we have a cat[1]-group $M \ltimes G$, where G acts on M by conjugation.

To define the functor back, let us check that all cat[1]-groups have a semidirect product decomposition.

Proposition 2.5.4 *For any cat^1-group (G, s, t):*

i) The maps s, t have the same range, i.e. $s(G) = t(G) = N$, and are the identity on N.

ii) The morphisms s and t are "projections", i.e. $t^2 = t$ and $s^2 = s$.

Proof i) As $st = t$, we have $\operatorname{Im} t \subseteq \operatorname{Im} s$ and as $ts = s$, we have $\operatorname{Im} s \subseteq \operatorname{Im} t$.

ii) We have $ss = sts = ts = s$. Similarly, $tt = t$. □

As an easy consequence, we have:

Corollary 2.5.5 *There are two split short exact sequences*

$$1 \to Ker\, s \hookrightarrow G \xrightarrow{s} N \to 1 \qquad \text{and} \qquad 1 \to Ker\, t \hookrightarrow G \xrightarrow{t} N \to 1.$$

Remark 2.5.6 Thus G is isomorphic to both semidirect products $N \ltimes \text{Ker } s$ and $N \ltimes \text{Ker } t$, where N acts on both kernel by conjugation. The map $N \ltimes \text{Ker } s \to G$ is just the product and the inverse isomorphism $G \to N \ltimes \text{Ker } s$ is given by $g \mapsto (s(g), s(g^{-1})g)$.

We can also define an inverse functor

$$\gamma : \mathsf{Cat^1\text{-} Groups} \to \mathsf{XMod/Groups}$$

given by $\gamma(G, s, t) = (t| : \text{Ker } s \to \operatorname{Im} s)$ where $\operatorname{Im} s$ acts on $\text{Ker } s$ by conjugation.

Proposition 2.5.7 *If (G, s, t) is a cat^1-group, then $\gamma(G, s, t)$ is a crossed module.*

Proof With respect to CM1), for all $g \in \operatorname{Im} s$ and $m \in \operatorname{Ker} s$, we have

$$t(m^g) = t(g^{-1}mg) = (tg)^{-1}(tm)(tg).$$

Now, since $g \in \operatorname{Im} s = \operatorname{Im} t$, by Proposition 2.5.4 we have $tg = g$. Thus, $t(m^g) = g^{-1}(tm)g$.

On the other hand, with respect to CM2) for all $m, m' \in \operatorname{Ker} s$, we have

$$m'^{(tm)} = (tm^{-1})m'(tm) = (tm^{-1})m'(tm)m^{-1}m.$$

Now, since $(tm)m^{-1} \in \operatorname{Ker} s$ and $m' \in \operatorname{Ker} s$, they commute, giving

$$m'^{(tm)} = (tm^{-1})(tm)m^{-1}m'm = m^{-1}m'm.$$

□

Proposition 2.5.8 *The functors λ and γ give an equivalence of categories.*

Proof On one hand we have $\lambda\gamma(G, s, t) = (\operatorname{Im} t \ltimes \operatorname{Ker} s, s', t')$ where $s'(g, m) = (g, 1)$ and $t'(g, m) = (gt(m), 1)$. Clearly there is a natural isomorphism of groups $\phi : G \to \operatorname{Im} t \ltimes \operatorname{Ker} s$ given by $\phi(g) = (s(g), s(g)^{-1}g)$ that is an isomorphism of cat^1-groups.

On the other hand, $\gamma\lambda(\mu : M \to P) = (\operatorname{Ker} \xrightarrow{t} \operatorname{Im} s)$ where $s : P \ltimes M \to P \ltimes M$ is given by $s(g, m) = (g, 1)$. There are obvious natural isomorphisms $\operatorname{Ker} s \cong M$ and $\operatorname{Im} s \cong P$ giving a natural isomorphism of crossed modules. □

2.6 The fundamental crossed module of a fibration

In this section the proofs will be omitted or be sketchy, since background in fibrations of spaces is needed. Throughout we assume that ‘space’ means ‘pointed space’.

In this section we are going to give a proof that for any fibration $\mathcal{F} = (F \xrightarrow{i} E \xrightarrow{p} X)$ the induced map

$$i_* : \pi_1(F) \to \pi_1(E)$$

is a crossed module $\Pi_2(\mathcal{F})$ which we call the *fundamental crossed module* of the fibration $\mathcal{F}$. This is an observation first made by Quillen and from it can be deduced the fundamental crossed module of a pair of spaces.

Perhaps it is better first to recall in some detail the action of $\pi_1(E)$ on $\pi_1(F)$ for any fibration $\mathcal{F}$.

Let us consider $[\mu] \in \pi_1(F)$ and $[\alpha] \in \pi_1(E)$. The projection to X of the loop $\alpha^{-1}\mu\alpha$ is homotopic to the constant through a homotopy of loops $H : I \times I \to X$. Since p is a fibration, using the homotopy lifting property, we get a homotopy of loops $\overline{H} : I \times I \to E$ from $\alpha^{-1}\mu\alpha$ to a loop projecting to the constant, i.e. $\operatorname{Im} \overline{H}_1 \subseteq F$. We define

$$[\mu]^{[\alpha]} = [\overline{H}_1] \in \pi_1(E).$$

We omit the proof that this action is well defined. This is a good exercise on fibration theory.

To prove that i_* is a crossed module, we proceed in a roundabout way. Clearly, it is equivalent to prove that the semidirect product $\pi_1(E) \ltimes \pi_1(F)$ given by the action just defined is a cat^1-group. Again, this is not done directly, but instead we prove that there is a natural isomorphism of groups

$$\pi_1(E \times_X E) \cong \pi_1(E) \ltimes \pi_1(F)$$

and that the former is a cat^1-group, where $E \times_X E$ is the pullback of p along p, i.e.

$$E \times_X E = \{(e, e') \in E \times E : p(e) = p(e')\}.$$

First, let us prove that $\pi_1(E \times_X E)$ decomposes in the expected semidirect product.

Proposition 2.6.1 *For any fibration $\mathcal{F} = (F \xrightarrow{i} E \xrightarrow{p} X)$, there are two splitting short exact sequences*

$$1 \to \pi_1(F) \xrightarrow{i_{1*}} \pi_1(E \times_X E) \xrightarrow{p_{1*}} \pi_1(E) \to 1 \quad \textit{and} \quad 1 \to \pi_1(F) \xrightarrow{i_{2*}} \pi_1(E \times_X E) \xrightarrow{p_{2*}} \pi_1(E) \to 1$$

where i_l is the inclusion of F in the l^{th} factor. Moreover both are natural with respect to maps of fibrations.

Proof Recall that the projection in the first factor $E \times_X E \to E$ is a fibration with fibre F since it is the pullback of p along itself. Also, the diagonal map gives a section of this fibration. Thus, its homotopy exact sequence decomposes into a sequence of splitting short exact sequences. In particular,

$$1 \to \pi_1(F) \xrightarrow{i_{1*}} \pi_1(E \times_X E) \xrightarrow{p_{1*}} \pi_1(E) \to 1$$

is a splitting short exact sequence. The same is true in the second case. □

Now, we are able to prove that $(\pi_1(E \times_X E), s, t)$ where s (resp. t) is the homomorphism induced on the fundamental groups by the composition of the projection in the first (resp. second) factor and the diagonal is a cat^1-group for any fibration $\mathcal{F}$. We shall call it the *fundamental cat^1-group of the fibration $\mathcal{F}$.*

Proposition 2.6.2 *Let $\mathcal{F} = (F \xrightarrow{i} E \xrightarrow{p} X)$ be a fibration. Then $(\pi_1(E \times_X E), s, t)$ is a cat^1-group.*

Proof It clearly satisfies CG1) since the maps s, t are essentially projections.

To prove CG2), using the exact sequence of Proposition 2.6.1, we have $\operatorname{Ker} s = \operatorname{Im} i_{1*}$ and $\operatorname{Ker} t = \operatorname{Im} i_{2*}$

Also by Proposition 2.6.1 the elements of Ker s are of the form $[(ct, \mu)]$ where μ is a loop in the fibre and the elements of Im s are of the form $[(\alpha, \alpha)]$ where α is a loop in E.

We choose elements $[(ct, \mu)] \in$ Ker s and $[(\nu, ct)] \in$ Ker t where μ and ν are loops in the fibre. The commutativity of these elements is now clear, since

$$[(\nu, ct)][(ct, \mu)] = [(ct, \mu)][(\nu, ct)] = [(\nu, \mu)].$$

□

Now, we proceed to identify the crossed module associated with $(\pi_1(E \times_X E), s, t)$.

Proposition 2.6.3 *The crossed module $(t| : Ker\, s \to Im\, s)$ associated to the cat^1-group $\pi_1(E \times_X E)$ is naturally isomorphic to $\Pi_2\mathcal{F} = (\pi_1(F) \to \pi_1(E))$.*

Proof There are natural isomorphisms $\pi_1(F) \cong \mathrm{Ker}\ s$ and $\pi_1(E) \cong \mathrm{Im}\ s$, given by $[\mu] \mapsto [(ct, \mu)]$ and $[\alpha] \mapsto [(\alpha, \alpha)]$ respectively. It remains only to check that these isomorphisms preserve actions.

The action of Ker s on Im s is given by conjugation in $\pi_1(E \times_X E)$. Under these isomorphisms the result of the action of $[\alpha] \in \pi_1(E)$ on $[\mu] \in \pi_1(F)$, is the homotopy class of any loop ν in F satisfying

$$[(ct, \nu)] = [(\alpha^{-1}\alpha, \alpha^{-1}\mu\alpha)].$$

Recalling the definition of the $\pi_1(E)$ action on $\pi_1(F)$ at the beginning of the section, we see that $[\mu]^{[\alpha]}$ is represented by just this same element. □

To define the fundamental cat^1-group functor on maps of general topological spaces we need some more homotopy theory. There is no space to develop this here in full, and so we just sketch the ideas, which are well covered in books on abstract homotopy theory, for example [120].

A standard procedure in homotopy theory is to factor any map $f : Y \to X$ through a homotopy equivalence i and a fibration $\overline{f} : \overline{Y} \to X$ where $\overline{Y} = \{(\lambda, y) \in X^I \times Y : \lambda(1) = f(y)\}$ and $\overline{f}(\lambda, y) = \lambda(0)$.

This gives a functor $Fib : f \mapsto \overline{f}$ from maps to fibrations. We define the cat^1-group functor on maps of general topological spaces by composition with the cat^1-group of fibrations functor.

Let us sketch a direct description of the composite functor

$$\mathsf{Maps} \to \mathsf{Cat}^1\text{-}\,\mathsf{Groups}$$

following ideas of Gilbert in [93].

The functor is defined by

$$(f : Y \to X) \mapsto (\pi_1(\overline{Y} \times_X \overline{Y}), p_{1*}, p_{2*}).$$

Using the homeomorphism

$$\overline{Y} \times_X \overline{Y} \equiv \{(y_1, \lambda, y_2) \in Y \times X^I \times Y : \lambda(0) = f(y_1) \text{ and } \lambda(1) = f(y_2)\}$$

the projections in the factors correspond to the maps

$$\begin{array}{rclcrcl} p_1(y_1, \lambda, y_2) & = & (y_1, \lambda_1), & \text{where} & \lambda_1(t) & = & \lambda(t/2) \\ p_2(y_1, \lambda, y_2) & = & (y_2, \lambda_2), & \text{where} & \lambda_2(t) & = & \lambda(1-(t/2)). \end{array}$$

Via the same homeomorphism, the elements of $\pi_1(\overline{Y} \times_X \overline{Y})$ correspond to homotopy classes of triples, $[(\alpha, \mu, \beta)]$, where $\mu : I \times I \to X$ maps $I \times \{0,1\}$ to the base point and $\alpha, \beta : I \to Y$ are loops on Y lifting $\mu(0, -)$ and $\mu(1, -)$ respectively. The homotopies correspond to triples, (F, H, G), the map $H : I \times I \times I \to X$ sending $I \times \{0,1\} \times I$ to the base point, and $F, G : I \times I \to Y$ being homotopies of loops, relative to the end points, lifting $H(0, -, -)$ and $H(1, -, -)$, respectively.

The description of p_{1*} and p_{2*} follows easily.

For the sake of coherence let us point out that if f is already a fibration, both definitions of the fundamental cat^1-group produce the same group up to isomorphism.

If f is a fibration, f and $\overline{f}$ are fibre homotopy equivalent. It can be checked directly that $Y \times_X Y$ and $\overline{Y} \times_X \overline{Y}$ are also homotopy equivalent, but it is also a consequence of the following cogluing theorem which is a special case of the results of [37]. The dual of this result, namely a 'gluing theorem', is proved in [30] and in an abstract setting in [120].

Theorem 2.6.4 *Suppose given maps over X*

$$\begin{array}{ccc} Y & \xrightarrow{\;i\;} & \overline{Y} \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle \overline{f}} & \\ & X & \end{array} \qquad \begin{array}{ccc} Z & \xrightarrow{\;j\;} & \overline{Z} \\ & {\scriptstyle g}\searrow \quad \swarrow{\scriptstyle \overline{g}} & \\ & X & \end{array}$$

such that $f, \overline{f}, g, \overline{g}$ are fibrations, and i, j are homotopy equivalences. Then the induced map on pullbacks

$$i \times_X j : Y \times_X Z \to \overline{Y} \times_X \overline{Z}$$

is also a homotopy equivalence, and in fact a fibre homotopy equivalence.

In the particular case in which we are mostly interested, we consider a pair of topological spaces (X, A). Associated to the inclusion $i : A \to X$ there is the fibration $\overline{A} \to X$ where $\bar{A}$ is the space of paths in X starting at some point of A and the map sends each path to its end point. The fibre of this fibration is the space

$$F_i = \{\lambda \in X^I : \lambda(0) \in A \quad \text{and} \quad \lambda(1) = *\}$$

whose homotopy groups are, by definition, those of the pair (X, A), i.e.

$$\pi_n(F_i) = \pi_{n+1}(X, A).$$

In particular, the fundamental crossed module of a pair functor

$$\Pi_2 : \mathsf{Top}^2_* \longrightarrow \mathsf{Fib} \longrightarrow \mathsf{XMod/Groups}$$

is given by

$$\Pi_2(X, A) = (\partial : \pi_2(X, A) \to \pi_1(A))$$

with the usual action already known and used by Whitehead.

Finally in this section, we mention some relations of crossed modules with algebraic K-theory, for those familiar with that area.

Let R be a ring. A basic structure for algebraic K-theory is the homotopy fibration

$$F(R) \to BGL(R) \to BGL(R)^+.$$

This yields the crossed module

$$(\pi_1(F(R)) \to \pi_1(BGL(R)))$$

which is equivalent to

$$(St(R) \to GL(R))$$

which has cokernel $K_1(R)$ and kernel $K_2(R)$.

Now let I be an ideal of R, and let $GL(R, I)$, the *congruence subgroup*, be the kernel of $GL(R) \to GL(R/I)$. By the same trick, we get a crossed module

$$St(R, I) \to GL(R, I)$$

which has cokernel $K_1(R, I)$ and kernel $K_2(R, I)$. This is Loday's definition of the relative K_2 [127]. It differs from that of Milnor [149] by relations corresponding to those of the second rule CM2) for a crossed module.

One advantage of this procedure is the generalisation to multirelative groups $K_2(R; I_1, \ldots, I_n)$ [102, 84]. The relevant algebra is that of crossed n-cubes of groups. All this was one of the motivations for the van Kampen Theorem for n-cubes of spaces [53].

2.7 The category of categories internal to groups

In this section, we study another category equivalent to XMod/Groups, namely the category of categories internal to groups, written Cat[Groups]. This category has easy generalisations both to higher dimensions and to other algebraic settings.

This category has two features that make it very interesting. On the one hand it can be used as an intermediate step to get a simplicial equivalent of crossed modules which can be generalised to crossed n-cubes. (This has been done by T. Porter in [158]).

On the other hand, we shall see that the category Cat[Groups] is formed by groupoids, being also the category of group-groupoids. This will be generalised in Chapter 6 to an equivalence from the category XMod of crossed modules over groupoids to a category of double groupoids.

First, let us recall from the Appendix that the definition of a category $\mathcal{C}$ is given by two sets, the object set, $Ob\ \mathcal{C}$, and the morphism set, $Mor\,\mathcal{C}$, and four maps, the identity i, the source and target s, t, and the composition of morphisms $\circ$, satisfying several axioms. Note that $\circ$ is considered as a function on its domain.

We say that $\mathcal{C}$ is a *category internal to* Groups, if both $Ob\,\mathcal{C}$ and $Mor\,\mathcal{C}$ have a group structure and the maps s, t, i and $\circ$ are homomorphisms of groups. Thus, a category internal to Groups is also a *group in the category of all small categories.* This principle for algebraic structure that 'an $\mathcal{A}$ in a $\mathcal{B}$ is also a $\mathcal{B}$ in an $\mathcal{A}$' is of wide applicability.

Similarly, a functor $f : \mathcal{C} \to \mathcal{C}'$ between two categories, is a pair of maps $Ob\ f$ and f commuting with the structure maps (source, target, identity and composition).

A functor between categories internal to Groups is a *functor internal to* Groups if both maps are homomorphisms of groups.

Then, Cat[Groups] is the category whose objects and morphisms are categories and functors internal to Groups.

For any object $\mathcal{C}$ in Cat[Groups], we will write the product in $Mor\,\mathcal{C}$ additively and the product in $Ob\ \mathcal{C}$ multiplicatively. Then, if 1 and 0 are the identities in $Ob\,\mathcal{C}$ and $Mor\,\mathcal{C}$, we have $i(1) = 0$, $s(0) = 1$ and $t(0) = 1$. So, the elements of $\operatorname{Ker} s$ (resp. $\operatorname{Ker} t$) are the morphisms with source 1 (target 1).

The next property shows that, for any category internal to Groups, we can define the composition of morphisms in terms of the other structure maps.

Proposition 2.7.1 *For any two composable morphisms, u and v, we have*

(i)
$$v \circ u = v - itu + u = v - isv + u,$$

(ii)
$$v \circ u = u - itu + v = u - isv + v.$$

Proof (i) We have

$$v \circ u = (v + 0) \circ (itu + (-itu + u)) = (v \circ itu) + (0 \circ (-itu + u)) = v - itu + u.$$

The second equality is immediate because the morphisms are composable, so $itu = isv$.

(ii) is proved in a similar way. □

Remark 2.7.2 Thus, to prove that a category where the objects and morphisms are groups, and the source target and identity are homomorphisms, is internal to groups, all we have to check is that the composition defined using Proposition 2.7.1 is a homomorphism.

There is also an expression for the inverse of a morphism, proving that all categories internal to groups are groupoids.

Proposition 2.7.3 *For any morphism in a category internal to groups we have*

$$u^{-1} = isu - u + itu.$$

Proof Let us define u^{-1} by this formula. We can easily check that it has the appropriate source and target and that both compositions are the identity. □

Remark 2.7.4 As a consequence of this property, a category internal to groups is a groupoid internal to groups, or, equivalently, a group in the category of groupoids.

Considering that a group is just a groupoid with only one object, we could try to study the category of "groupoids of groupoids", or "double groupoids". We shall do this in Chapter 6.

To end this section, we state the relation of $\mathsf{Cat}[\mathsf{Groups}]$ to the previous categories. The equivalence with $\mathsf{Cat}^1\text{-}\mathsf{Groups}$ is easily defined.

In one direction, we assign to the cat^1-group (G, s, t) the category having $\operatorname{Im} s = \operatorname{Im} t$ as set of objects, G as set of morphisms, s and t as source and target, identity the inclusion $\operatorname{Im} s \subseteq G$ and composition defined by $g' \circ g = g' - itg + g$, for any $g, g' \in G$ with $tg = sg'$. It can be easily checked that this gives a category internal to Groups.

In the other direction, to any category $\mathcal{C}$ internal to Groups we assign the cat^1-group $(Mor\,\mathcal{C}, i \circ s, i \circ t)$.

Thus, the categories $\mathsf{XMod/Groups}$ and $\mathsf{Cat}[\mathsf{Groups}]$ are equivalent, since both are equivalent to $\mathsf{Cat}^1\text{-}\mathsf{Groups}$. However, it is convenient to record for further use the functors giving this equivalence.

The functor one way is defined as $\mathcal{C} \mapsto (s| : \text{Ker } t \to Ob\ \mathcal{C})$, where $\mathcal{C}$ is a cat^1-group. The reverse functor assigns to any crossed module $\mathcal{M} = (\mu : M \to P)$ the category having P as set of objects, $P \ltimes M$ as set of morphisms; identity map given by the inclusion; source and target maps given by $s(g, m) = g$ and $t(g, m) = g(\mu m)$ and composition given by any of the formulae in Proposition 2.7.1.

Nevertheless, there is a simpler expression for the composition in this case. Notice first that with the definition of source and target, two morphisms $(g', m'), (g, m) \in P \ltimes M$ are composable when $g\mu m = g'$.

Proposition 2.7.5 *The composition of morphisms in* $P \ltimes M$*,*

$$\circ : P \ltimes M\,{}_s{\times}_t\, P \ltimes M \to P \ltimes M$$

is given by $(g(\mu m), m') \circ (g, m) = (g, mm')$.

Proof This is not difficult to prove using the definition of composition given in Proposition 2.7.1 i). □

With this property we can get another model of the category internal to Groups associated to a crossed module.

Proposition 2.7.6 *The map $A : Mor\,\mathcal{C}\ {}_s\!\times_t Mor\,\mathcal{C} \to P \ltimes M \ltimes M$ defined by $A((g', m'), (g, m)) = (g, m, m')$, is an isomorphism carrying the composition to the map*

$$\circ' : P \ltimes M \ltimes M \to P \ltimes M$$

sending (g, m, m') to (g, mm').

Proof Clearly A is bijective and transforms the composition to the afore mentioned map. It remains to check that A is a homomorphism and that is left as an exercise. □

Let us consider now the composite functor

$$\mathsf{Fib} \to \mathsf{Cat}^1\text{- }\mathsf{Groups} \to \mathsf{Cat}[\mathsf{Groups}]$$

i.e., mapping $\mathcal{F}$ to the category internal to Groups associated to the cat^1-group $\pi_1(E \times_X E)$.

Using the isomorphism $\mathrm{Im}\, p_{i*} \cong \pi_1(E)$, this category is isomorphic to the category that has $\pi_1(E)$ as objects, $\pi_1(E \times_X E)$ as morphisms, source and target given by projections, identity given by the diagonal and composition the only one possible to make this a category internal to groups.

As seen before, this category is also isomorphic to the one associated to $\pi_1(E) \ltimes \pi_1(F)$, that has $\pi_1(E)$ as objects, $\pi_1(E) \ltimes \pi_1(F)$ as morphisms, $([\alpha], [\mu]) \mapsto [\alpha]$ and $([\alpha], [\mu]) \mapsto [\alpha] * i_*([\mu])$ as source and target maps and composition given by

$$([\alpha] * i_*[\mu], [\mu']) \circ ([\alpha], [\mu]) = ([\alpha][\mu * \mu']).$$

We finish by stating a description of the composition in $\pi_1(E \times_X E)$.

Proposition 2.7.7 *Let $[(\alpha, \beta)], [(\beta', \gamma')] \in \pi_1(E \times_X E)$ be such that $[\beta] = [\beta']$, i.e. there is a homotopy $G : \beta' \cong \beta$. Since p is a fibration there is a homotopy H lifting pG and starting with γ'. Then*

$$[(\beta', \gamma')] \circ [(\alpha, \beta)] = [(\alpha, H_1)].$$

Proof It is clear that $[(\beta', \gamma')]$ and $[(\beta, H_1)]$ are homotopic using the homotopy (G, H). Then, $[(\beta', \gamma')] \circ [(\alpha, \beta)] = [(\beta, H_1)] \circ [(\alpha, \beta)]$. So, we only have to consider the composition in the case $[(\beta, \gamma)] \circ [(\alpha, \beta)]$. Using that $\mathcal{F}$ is a fibration there are unique $[\mu], [\mu'] \in \pi_1(F)$ with

$$[(\alpha, \beta)] = A([\alpha], [\mu]) = [(\alpha * ct, \alpha * \mu)]$$

and

$$[(\beta, \gamma)] = A([\beta], [\mu']) = [(\beta * ct, \beta * \mu')].$$

Clearly, $[\beta] = [\alpha] * i_*([\mu])$, and

$$\begin{aligned}
[(\beta, \gamma)] \circ [(\alpha, \beta)] &= A([\beta], [\mu']) \circ A([\alpha], [\mu]) \\
&= [(\beta * ct, \beta * \mu')] \circ [(\alpha * ct, \alpha * \mu)] \\
&= [(\alpha * ct, \alpha * \mu' * \mu)] \\
&= [((\alpha * ct) * ct, \alpha * \mu' * \mu)] \\
&= [(\alpha * ct, \beta * \mu')] \\
&= [(\alpha, \gamma)].
\end{aligned}$$

□

This proof is related to a proof in [50] which shows that in the construction of a double homotopy groupoid of a map of spaces, a composition defined geometrically agrees with that derived from Generalised Galois Theory.

We can also describe easily the functor

$$\mathsf{Maps} \to \mathsf{Cat}[\mathsf{Groups}].$$

Notice that $\pi_1(\overline{Y})$ is isomorphic to $\pi_1(Y)$ under the projection. So the associated category internal to groups is equivalent to the one having $\pi_1(Y)$ as objects, $\pi_1(\overline{Y} \times_X \overline{Y})$ as morphisms, source and target given by $[(\alpha, \mu, \beta)] \to [\alpha]$ and $[(\alpha, \mu, \beta)] \to [\beta]$, and composition given by $[(\beta, \mu', \gamma)] \circ [(\alpha, \mu, \beta)] = [(\alpha, \mu' * \mu, \gamma)]$.

Note that if ν is an homotopy from β to β', the composition of $[(\alpha, \mu, \beta)]$ with $[(\beta', \mu', \gamma)]$ is given by $[(\alpha, \mu' * \nu * \mu, \gamma)]$ since $[(\beta', \mu', \gamma)] = [(\beta, \mu' * \nu, \gamma)]$.

Chapter 3

Basic algebra of crossed modules

In this chapter we analyse what historically was the second source of crossed modules over groups: identities among relations in presentations of groups.

A central problem in mathematics is the representation of infinite objects in manipulable, and preferably finite, terms. One method of doing this is by what is called a *resolution*. There is not a formal definition of this, but we can see several examples.

This notion first arose in the 19th century study of invariants. *Invariant theory* deals with subalgebras of polynomial algebras $\Lambda = k[x_1, \dots, x_n]$, where k is a ring. Consider for example, the subalgebra A of $\mathbb{Z}[a, b, c, d]$ generated by

$$a^2 + b^2,\ c^2 + d^2,\ ac + bd,\ ad - bc.$$

It is called an *invariant subalgebra* since it is invariant under the action of $\mathbb{Z}_2$ which switches the variables a, b and c, d. As pointed out in [92, p.247], "these generators satisfy the relation

$$(ac + bd)^2 + (ad - bc)^2 = (a^2 + b^2)(c^2 + d^2)$$

which is classically called a *syzygy*, and the algebra A of invariant polynomials turns out to be the homomorphic image of the polynomial algebra in four variables given by the quotient algebra

$$\mathbb{Z}[x, y, z, w]/(z^2 + w^2 - xy).$$

In particular, the algebra is finitely generated by four explicit polynomials, and the ideal of relations is finitely generated by a single explicit relation."

On [92, p.253-4] we have: "Since the second main problem had succumbed so easily, it was natural to turn to chains of syzygies, studying relations among the generating set of relations and so on. More precisely, this work involved the sequence of finitely generated $k[x_1, \dots, x_n]$-modules

$$\begin{array}{ccccccccc} 0 & \longrightarrow & J_1 & \longrightarrow & F_1 & \longrightarrow & B_1 & \longrightarrow & 0 \\ 0 & \longrightarrow & J_2 & \longrightarrow & F_2 & \longrightarrow & J_1 & \longrightarrow & 0 \\ & & \cdots & & \cdots & & \cdots & & \\ 0 & \longrightarrow & J_q & \longrightarrow & F_q & \longrightarrow & J_{q-1} & \longrightarrow & 0, \end{array}$$

where the F_i are free with rank equal to the minimal number of generators of the i'th syzygy J_i. Hilbert's main theorem on the chains of syzygies says that if k is a field then $J_q = 0$ if $q > n$. In effect, this launched the theory of homological dimension of rings."

It was also natural to splice the morphisms $F_q \to J_{q-1} \to F_{q-1}$ together to get a sequence

$$\cdots \xrightarrow{\partial_{q+1}} F_q \xrightarrow{\partial_q} F_{q-1} \xrightarrow{\partial_{q-1}} \cdots \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} B_1$$

which was *exact* in the sense that

$$\operatorname{Ker} \partial_q = \operatorname{Im} \partial_{q+1}$$

for all q. This sequence was called a *free resolution* of the module B_1.

A natural question was the independence of this sequence on the choices made. It was found that given any two such free resolutions $F_* \to B_1$, $F'_* \to B_1$, then there was a morphism $F_* \to F'_*$ and any two such morphisms were *homotopic*. It was also later found that the condition 'free' could conveniently be replaced by the condition *projective*.

Another source for homological algebra was the homology and cohomology theory of groups. As pointed out in [134], the starting point for this was the 1942 paper of Hopf [112]. Let X be an aspherical space (i.e. connected and with $\pi_i X = 0$ for $i > 1$), and let

$$1 \to R \to F \to \pi_1 X \to 1$$

be an exact sequence of groups with F free. Hopf proved the formula

$$H_2 X \cong (R \cap [F, F])/[F, R].$$

We shall see in Section 5.5 that this formula follows from our van Kampen Theorem for crossed modules. Thus we see the advantage for homotopy theory of having a 2-dimensional algebraic model of homotopy types.

Later work of Eilenberg-Mac Lane [80] found an algebraic formula for $H_n X$, $n \geqslant 2$ as follows. Produce sequences of $\mathbb{Z}G$-modules

$$\begin{array}{ccccccccc} 0 & \longrightarrow & J_1 & \longrightarrow & F_1 & \longrightarrow & \mathbb{Z} & \longrightarrow & 0 \\ 0 & \longrightarrow & J_2 & \longrightarrow & F_2 & \longrightarrow & J_1 & \longrightarrow & 0 \\ & & \cdots & & \cdots & & \cdots & & \\ 0 & \longrightarrow & J_q & \longrightarrow & F_q & \longrightarrow & J_{q-1} & \longrightarrow & 0, \end{array}$$

in which $\mathbb{Z}$ is the trivial $\mathbb{Z}G$-module, and each F_n is a free $\mathbb{Z}G$-module. Splice these together to give a *free resolution of* $\mathbb{Z}$:

$$F_* : \ldots \to F_n \to F_{n-1} \to \ldots \to F_2 \to F_1 \to \mathbb{Z}.$$

Form the chain complex $C = F \otimes_{\mathbb{Z}G} \mathbb{Z}$. Then $H_n C \cong H_n X$. Using particular choices of the F_n, the Hopf formula may be deduced [20, p.46].

Thus we see an input from the homotopy and homology theory of spaces into the development of homological algebra. The use of homological methods across vast areas of mathematics is a feature of 20th century mathematics. It seems the solution of Fermat's last theorem depended on it, but it has also been applied in differential equations, coding theory and theoretical physics.

In its 20th century form, homological algebra is primarily an abelian theory. There is considerable work on nonabelian homological algebra, but this is only beginning to link with work in homotopical algebra, differential topology, and related areas. This book has an aim of showing one kind of start to a more systematic background to such an area.

Now the elementary, computational and example-oriented approach to groups considers presentations $\langle X; R \rangle$ of a group Q: that is X is a subset of Q and there is an exact sequence

$$1 \to N \to FX \xrightarrow{p} Q \to 1 \qquad (*)$$

where FX is the free group on generators $[x],\ x \in X$; p is defined by $p[x] = x,\ x \in X$; and R is a set of generators of N as normal subgroup of FX. Thus, each element of N is a *consequence*

$$c = (r_1^{\epsilon_1})^{u_1} \dots (r_n^{\epsilon_n})^{u_n},$$

$r_i \in R, \epsilon_i = \pm 1, u_i \in FX$ and $a^b = b^{-1}ab$. However, this representation of elements of N, and the persistent use of N and FX as non-abelian groups (rather than of modules derived from them) plays a small role in the homological algebra of groups. One would expect, *a priori*, that the sequence (*) would be the beginning of a "nonabelian resolution" of the group Q. We will show that this is so in a later chapter.

Another curiosity is that there are a number of results in homotopy theory which are satisfactory for 1-connected spaces, but for which no formulation has been given when this assumption has been dropped, particularly when some non-abelian group has to be described. As long as interest was focussed on high-dimensional, or stable, problems, this restriction seemed not to matter. In many problems of current interest (for example low-dimensional topology, low-dimensional homology of groups, algebraic K-theory) this restriction has proved irksome, but few appropriate constructions have been generally seen to be available. This is one of the reasons for promoting the subject matter of this book.

In section 3.1 we recall what is a presentation $\langle X \mid \omega \rangle$ of a group P, and show that the 'identities among the relations' can be seen as the elements of the kernel of a morphism $\theta : F(R \times P) \to P$ which satisfies CM1) in the definition of crossed modules.

This gives good reason to relax the concept of crossed module. In Section 3.3 we define precrossed modules in terms of axiom CM1) and also the functor that associates to every precrossed module a crossed module. This construction $(-)^{\mathrm{cr}}$ is adjoint to the inclusion of categories $\mathsf{XMod/Groups} \hookrightarrow \mathsf{PXMod/Groups}$.

The morphism $\theta : F(R \times P) \to P$ has some extra freeness properties, making it what is called a 'free precrossed module'. These are studied in Section 3.4.

The chapter ends with the definition of a category of algebraic objects equivalent to that of precrossed modules and generalising the equivalence defined in Section 2.5.

3.1 Presentation of groups and identities among relations.

We now show how crossed modules arise in combinatorial group theory, following to some extent the exposition in [49].

A group G is of course defined as a set with a multiplication satisfying certain axioms. In some cases this multiplication can be specified by a formula involving the elements: notable examples are certain matrix groups, such as the Heisenberg group H of matrices of the form

$$\begin{bmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{bmatrix}$$

for $x, y, z \in \mathbb{Z}$. Thus the elements of H are given by triples (x, y, z) of integers with multiplication

$$(x, y, z)(u, v, w) = (x + u, y + v + xw, z + w).$$

This is known as a 'polynomial group law'. So we have a formula for the elements of the group H and for the multiplication.

The reader should not be surprised that this could raise difficulties in other cases. Part of the problem may be to give a useful formula for the elements of the group, let alone a formula for the multiplication. In mathematics as a whole, the question of 'presenting' information on a structure is often a key part of a problem.

An often useful way of representing the elements of a group is by giving generators for the group.

Example 3.1.1 Let D_4 be the dihedral group of order 8, i.e. the group of symmetries of the square. This group is generated by the elements x, y where x is rotation anticlockwise through $90°$ and y is reflection in a vertical bisector of the square. The elements of D_4 can then be written as

$$1, x, x^2, x^3, y, yx, yx^2, yx^3$$

and this is quite a convenient labelling of the elements. However if you try to work out the multiplication table in terms of this labelling you find you need more information, namely *relations* among the generators, for example

$$x^4 = 1,\ y^2 = 1,\ xyxy = 1.$$

If you are not already familiar with these, particularly the last one, then you are expected to verify them using some kind of model of a square. It turns out that every relation you might need in working out the multiplication table is a consequence only of these three. Thus we can specify the group completely also in terms of what we call a 'presentation'

$$\mathcal{P} = \langle x, y \mid x^4, y^2, xyxy \rangle.$$

If there is any need, we shall write $D_4 = gp\,\mathcal{P}$. We need a definition of this idea of a presentation.

The first thing to note is that the term x^4 in the presentation $\mathcal{P}$ is not an element of the group D_4, since the 4th power of the element x in D_4 is 1. Rather, as is common with the mathematical use of $=$, one side of the $=$ sign in $x^4 = 1$ is in fact an instruction: 'multiply x by itself 4 times', while the other side tells you what will be the result. A convenient language to express both an 'instruction for a procedure' and the result of the procedure is that of a morphism defined on a free group.

A *free group* $F(X)$ on a set X is intuitively a group $F(X)$ together with an inclusion mapping $i : X \to F(X)$ such that X generates the group $F(X)$ and 'there are no relations among these generators'. There are two useful ways of expressing this precisely.

One of them is to give what is called a 'universal property': this is that a morphism $g : F(X) \to G$ to a group G is entirely determined by its values on the set X. Put in another way, given any function $f : X \to G$, there is a unique morphism $g : F(X) \to G$ such that $gi = f$. This 'external' definition thus defines a free group by its relation to all other groups, and is a model for the notion of 'freeness' in other algebraic situations. A set X generating a free group plays a rôle similar to that of a basis for a vector space, and we also talk about X as a basis for the free group $F(X)$. However, unlike vector spaces, not every group is free. The simplest example is the group $\mathbb{Z}_2$ with two elements: it is not free because there is only one morphism $\mathbb{Z}_2 \to \mathbb{Z}$, the zero morphism.

The other 'internal' way of specifying a free group is to specify its elements and the multiplication, and this can be done in terms of 'reduced words': every non identity element of $F(X)$ is uniquely expressible in the form

$$x_1^{r_1} x_2^{r_2} \dots x_n^{r_n} \tag{3.1.1}$$

where $n \geqslant 1, x_i \in X, r_i \in \mathbb{Z}, r_i \neq 0$, and for no i is $x_i = x_{i+1}$, i.e. no cancellation in the expression (3.1.1) is possible. In this specification, work is needed to give the multiplication since adjoining two reduced words often yields a non reduced word, and the reduction process has to be given. Accounts of this are in many books on combinatorial group theory, for example [118, 131, 67]. Reduced words are commonly used to store elements of a free group in computer implementations of combinatorial group theory.

We assume now that we have free groups, and this allows us to give our first definition of a presentation of a group Q.

Definition 3.1.2 A *presentation* $\mathcal{P} = \langle X \mid R \rangle$ of a group Q consists of a set X and a subset R of the free group $F(X)$ together with a surjective morphism $\phi : F(X) \to Q$ such that Ker ϕ is the normal closure in $F(X)$ of the set R.

If there is any need, we shall write $D_4 = gp\,\mathcal{P}$.

We explain in more detail the notion of normal closure, since this gives a useful model of an important general process, and we will use a more general form later for presentations of groupoids. First recall that for any normal subgroup $K \lhd P$, the group P acts on the group K by conjugation: we write k^p for $p^{-1}kp, k \in K, p \in P$. A basic aspect of group theory is that a normal subgroup is a kernel of a morphism (in this case, for example, of the quotient morphism $P \to P/K$), and that the kernel of any morphism from P to a group is normal in P.

If R is a subset of the group P then the *normal closure* $N^P(R)$ of R in P is the smallest normal subgroup of P containing R. We write conjugation of p by q as $p^q = q^{-1}pq$ for all $p, q \in P$. The elements of $N^P(R)$ are all *consequences* of R in P, namely all products

(3.1.2) $$c = (r_1^{\varepsilon_1})^{p_1} \dots (r_m^{\varepsilon_m})^{p_m}$$

where $r_i \in R$, $\varepsilon_i = \pm 1$, $p_i \in P$ and $m \geqslant 1$. An important point is that if $\phi : P \to Q$ is any morphism to a group Q such that $\phi(R) = \{1\}$, then $\phi(N^P(R)) = \{1\}$, since Ker ϕ is normal. Thus ϕ factors as $P \to P/N^P(R) \to Q$ where the first morphism is the quotient morphism.

Now we can see that there might be *identities among consequences.* Intuitively, such an identity is a 'formal' product such as (3.1.2) which is 1 when evaluated in the group P. A definition is given below. Here we consider some examples.

Example 3.1.3 For any elements r, s of R, we have the identities

$$r^{-1}s^{-1}rs^r = 1,$$
$$rs^{-1}r^{-1}s^{(r^{-1})} = 1.$$

These identities hold always, whatever R.

Example 3.1.4 Suppose $r \in R, p \in P$ and $r = p^m, m \in \mathbb{Z}$. Then $rp = pr$, i.e. p belongs to the centraliser $C(r)$ of r in P. We have the identity

(3.1.3) $$r^{-1}r^p = 1.$$

It is known that if the group P is free and $r \in R$ then there is a unique element p of P such that $r = p^m$ with $m \in \mathbb{N}$ maximal and then $C(r)$ is the infinite cyclic group generated by p. This element p is called the *root* of r and if $m > 1$ then r is called a *proper power.*

Example 3.1.5 Suppose the commutators $[p, q] = p^{-1}q^{-1}pq, [q, r], [r, p]$ are among the elements of R. Then the well known rule

(3.1.4) $$[p, q][p, r]^q\,[q, r][q, p]^r\,[r, p][r, q]^p = 1$$

is an identity among the consequences of R, since $[q, p] = [p, q]^{-1}$.

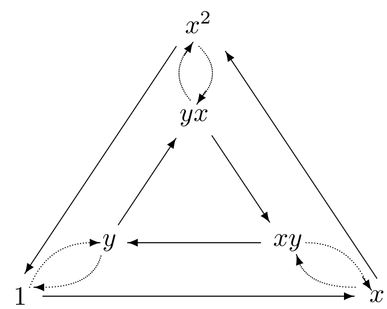
Example 3.1.6 Let S_3 be the symmetric group on three letters with presentation $\langle x, y \mid r, s, t \rangle$ where $r = x^3, s = y^2, t = xyxy$. The fact that each relation is a proper power gives rise to three identities among relations, namely

$$r^{-1}r^x, s^{-1}s^y, t^{-1}t^{xy}.$$

However there is also a fourth identity namely

$$(s^{-1})^{x^{-1}}ts^{-1}(r^{-1})^{y^{-1}}t^{x}(s^{-1})^{x}r^{-1}t^{x^{-1}}.$$

We leave it to you to verify that this is an identity among relations by writing out the formula in the free group on x, y. This identity can also be interpreted as a kind of composition of 2-cells in the following picture:



The Cayley graph of S_3

We shall discuss this a bit more in the next Section in terms of van Kampen diagrams.

Note that in all these examples conjugation is crucial. This is related to the fact that the kernel K of a morphism from a group P should be thought of not just as a subgroup K of the group P but also as a subgroup with an action of P on K. This principle, that a kernel in nonabelian situations has more structure than just that of subobject, is of general applicability. It is of direct applicability to the definition of an identity among the consequences of a subset R of the group P.

One extra formality is needed. We wish to allow for the consideration of repeated elements of R. One reason for this is that we may have some difficulty in recognising that two specified elements of P are in fact the same. In the context of presentations, we wish to allow for repeated relations. In the geometric context, we allow repeated attaching of cells by the same map (for example a constant map). Therefore we replace the subset R of P by a function $\omega : R \to P$ and denote a presentation as $\mathcal{P} = \langle X \mid \omega \rangle$. Nevertheless, we keep the notation $\langle X \mid R \rangle$ whenever $R \subseteq F(X)$ and ω is the inclusion.

Now in order to say that an identity among consequences is a *formal* product such as (3.1.2) which is 1 when evaluated in the group P, we need to define the free object in which such a 'formal product' should lie.

We adopt a more general notation and define the *free P-group on R* to be the free group on the set $R \times P$. We denote this P-group by H. The action of P on $R \times P$ is given by the product, i.e. by

$$(r, p)^q = (r, pq)$$

and this determines an action of P on the free group H. By another use of the universal property of a free group there is a morphism $\theta : H \to P$ defined on generators by

$$\theta(r, p) = p^{-1}\omega(r)p.$$

It is easy to see that the image of θ is the normal closure in P of $\omega(R)$. In symbols:

$$\theta(H) = N^P(\omega(R)).$$

It is clear also that the map θ preserves the action of P: that is, for any $h \in H, p \in P$

(3.1.5) $$\theta(h^p) = p^{-1}(\theta h)p.$$

You will recognise this as the axiom CM1) for a crossed module; however H with θ does not necessarily satisfy axiom CM2).

The elements of H will be called *formal consequences* of $\omega : R \to P$ in P.

There is an alternative description of H which we give for those familiar with the group theoretic background.

Proposition 3.1.7 *The group H is isomorphic to the normal closure of R in the free product $P * F(R)$.*

Proof This is a simple consequence of the Kurosch subgroup Theorem for free products. □

Our first definition is that an *identity among the consequences* of $\langle X \mid \omega \rangle$ in P is an element of $E = \mathrm{Ker}\ \theta$. Equivalently, an identity among consequences is a formal consequence which gives 1 when evaluated as an actual consequence in P.

The idea of specifying an identity among consequences is thus very similar to that of specifying a relation as an element of the free group FX, but taking the action into account. That is, we have to work with the appropriate concept of 'free'. However, we are not yet at our final position.

It is easy to see that certain identities are always present in E. We define the *basic Pfeiffer elements* to be the elements of E of the form

$$a^{-1}b^{-1}ab^{\theta(a)}$$

where $a, b \in R \times P$. Note that

$$(r', p')^{\theta(r,p)} = (r', p'p^{-1}(wr)p).$$

More generally, if $h, k \in H$ we will write

$$[\![h, k]\!] = h^{-1}k^{-1}hk^{\theta(h)}$$

and call such an element a *Peiffer element*. These should be thought of as 'twisted commutators'. In this spirit, there is a 'Peiffer commutator calculus' whose study has been advanced considerably by Baues and Conduché [13].

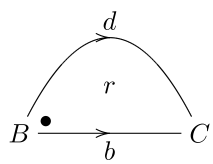
3.2 van Kampen diagrams

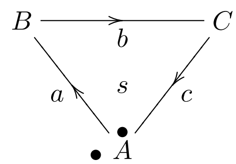
These diagrams give a geometric method of deducing consequences of relations, and can, as we shall see, be used to show exactly how to write a word as a consequence of the relations. We do not give a general definition or description, but illustrate it with examples. The idea has been used extensively in some sophisticated theorems in combinatorial group theory. For our purposes, the idea illustrates geometric aspects of the use of crossed modules.

The idea of these diagrams come from the fact that a relation in a presentation can be represented by a based cell whose sides are labeled by the letters of the relation in such way that when they are read clockwise from the base point we get the relation.

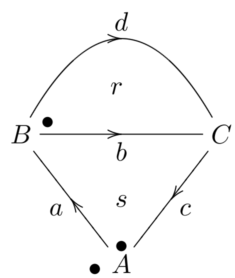
Then, we can get new relations by gluing two or more of these cell along some common sides. Let us consider a simple case.

Example 3.2.1 Suppose for a given presentation we have the relations $r = bd^{-1}$ and $s = abc$. They can be represented as based cells as follows:





We write $\delta s = abc$, $\delta t = db^{-1}$. Now, we glue r and s alongside b getting



The boundary of this new cell is

$$adc = abc \cdot c^{-1}b^{-1} \cdot db^{-1} \cdot bc = (\delta s)(\delta(t^{bc})).$$

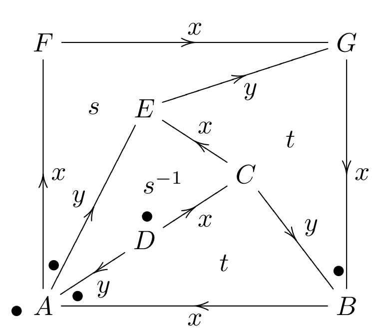
Of course t^{bc} makes sense in the context of crossed modules of groupoids, since t is based at B whereas t^{bc} is based at A.

Here is a more complex example.

Example 3.2.2 The quaternion group of order 8 is usually presented in the form

$$Q_8 = gp\,\langle x, y \mid x^4, x^2y^{-2}, y^{-1}xyx\rangle.$$

However the following diagram shows that the relation x^4 is a consequence of the other two relations. Set $r = x^4, s = x^2y^{-2}, t = y^{-1}xyx$ and consider the drawing



In this diagram, each cell has a base point, represented by a •, which is where the reading of the boundary starts in clockwise direction. This explains why we have an s and s^{-1}, since the latter is s read counterclockwise.

Now we have to show how we can deduce from this diagram the expression we want.

We take the outside loop starting from A (which has a base point for the outside 'cell') and then change it to traverse the boundary of each internal cell, obtaining the rule which you can easily verify:

$$xxy^{-1}y^{-1}\cdot yx^{-1}x^{-1}y\cdot y^{-1}xyx\cdot x^{-1}\cdot y^{-1}xyx\cdot x=x^4.$$

This can be reread as:

$$s\cdot yx^{-1}x^{-1}y\cdot t\cdot x^{-1}\cdot t\cdot x=x^4.$$

But $yx^{-1}x^{-1}y = yx^{-1}x^{-1}\cdot yyx^{-1}x^{-1}\cdot xxy^{-1} = (s^{-1})^{xxy^{-1}}$. So our final result is that

$$s\cdot (s^{-1})^{xxy^{-1}}\cdot t\cdot t^x\cdot r^{-1}$$

is an identity among relations, or, alternatively, shows in a precise way how x^4 is a consequence of the other relations.

One context for van Kampen diagrams is clarified by the notion of *shelling* of such a diagram. This is a sequence of 2-dimensional subcomplexes $K_0, K_1, \ldots, K_n$ each of which is formed of 2-dimensional cells, with K_0 consisting of a chosen basepoint $*$, K_1 being a 2-cell s_1 with $*$ on its boundary, and such that for $i = 2, \ldots, n$, K_i is obtained from K_{i-1} by adding a 2-cell s_i such that $s_i \cap K_{i-1}$ is a non empty union of 1-cells which form a connected and 1-connected set, i.e. a path. Such a shelling will yield a formula for the boundary of K_n in terms of the boundaries of each individual cell, provided each cell is given a base point and orientation.

Here is a clear way of getting the formula (explained to us by Chris Wensley):

Choose $* = K_0$ as base point for all the K_i. The relation for K_0 is the trivial word. If B_1 is the base point for s_1 and P_1 is the anticlockwise path around s_1 from B_1 to $*$ and w_1 is the word in the generators read off along P_1, then the relation for K_1 is $\delta({s_1}^{w_1})$. For $i \geqslant 2$, let B_i be the base point for s_i, and let U_i, V_i be the first and last vertices in the intersection $s_i \cap K_{i-1}$ met when traversing the boundary of K_{i-1} in a clockwise direction (so that the intersection is a path $U_i \ldots V_i$). Then if B_i lies on $U_i \ldots V_i$ let P_i be the path $B_i \ldots U_i \ldots *$, otherwise let P_i be the path $B_i \ldots V_i \ldots U_i \ldots *$ (traversing the boundary of s_i in an anticlockwise direction and the boundary of K_{i-1} clockwise). If w_i is the word in the generators read off along P_i then

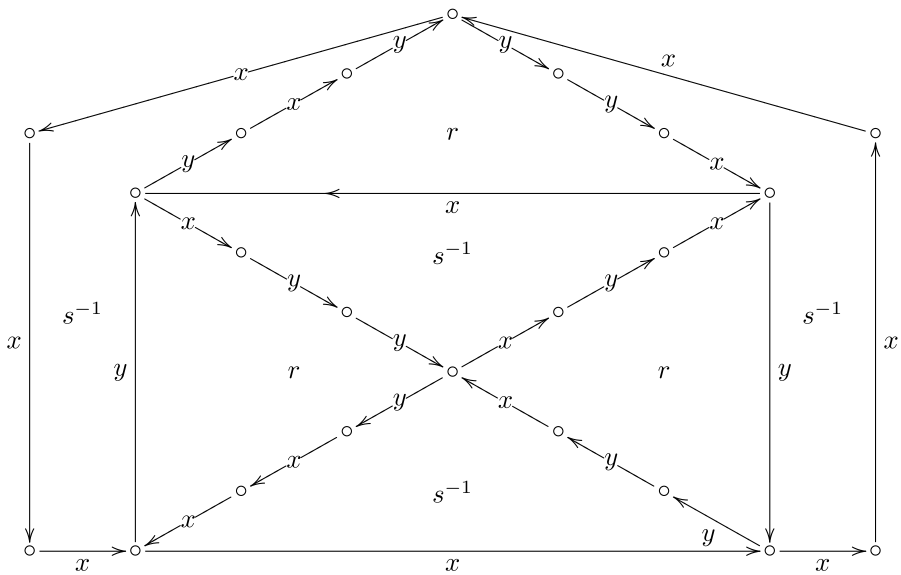
$$(\text{relation for } K_i) = (\text{relation for } K_{i-1}).\delta({s_i}^{w_i}).$$

We finish this short section by a more involved example

Example 3.2.3 Let us prove the non obvious fact that the relations

$$r = x^2yxy^3, \;\; s = y^2xyx^3$$

have x^7 as a consequence using the next picture. We leave it as an exercise to check that base points and orientations can be assigned, and, harder, to give x^7 as a consequence of r, s.



These examples are from David Johnson's book [118]. Other examples on van Kampen diagrams are in that book, and may also be found by a web search. The geometric and metric analysis of van Kampen diagrams has proved important in aspects of combinatorial group theory.

Here is a more formal definition of a van Kampen diagram.

A *complete generalised van Kampen diagram* is a finite regular CW-structure K on a compact subset of the sphere S^2. *Regularity* here means that each attaching map $f_s : (S^1, 1) \to (K^1, K^0)$ of a 2-cell s is a homeomorphism into. By omitting one 2-cell s_∞ from K and using stereographic projection we can also regard $K \setminus s_\infty$ as a subset of the plane $\mathbb{R}^2$. The projection of $K \setminus s_\infty$ gives a planar van Kampen diagram.

Whitehead's Theorem (Corollary 5.4.8) says essentially that $\Pi(K, K^1, K^0)$ is the free crossed $\pi_1(K^1, K^0)$-module on the characteristic maps of the 2-cells of K.

3.3 Precrossed and crossed modules

Following the concepts introduced in the first section, it seems a good idea to study morphisms having the same formal properties as $\theta : H \to P$. One way of describing the distinctive feature of θ is to say that θ is a morphism of P-groups, where P acts on itself by conjugation.

So, let M and P be groups such that P acts on M on the right and let $\mu : M \to P$ be a homomorphism of groups. We say that $\mathcal{M} = (\mu : M \to P)$ is a *pre-crossed module* if it satisfies CM1) of section 2.2, that is:

CM1) $\mu m^p = p^{-1} \mu m p = (\mu m)^p$ for all $m \in M$ and $p \in P$,

i.e., μ is a morphism of P-groups when P acts on itself by conjugation.

A *morphism* between two precrossed modules $\mathcal{M} = (\mu : M \to P)$ and $\mathcal{N} = (\nu : N \to Q)$ is a pair (g, f) of homomorphisms of groups $g : M \to N$ and $f : P \to Q$ such that

i) the diagram

$$\begin{array}{ccc} M & \xrightarrow{g} & N \\ {\scriptstyle\mu}\downarrow & & \downarrow{\scriptstyle\nu} \\ P & \xrightarrow[f]{} & Q \end{array}$$

commutes, i.e. $f\mu = \nu g$, and

ii) the actions are preserved, i.e. $g(m^p) = (gm)^{fp}$ for any $p \in P$ and $m \in M$.

The above objects and morphisms define the category $\mathsf{PXMod/Groups}$ of precrossed modules and morphisms.

Example 3.3.1 It is easy to see that if $\langle X \mid R \rangle$ is a presentation of a group, then using the notation of Section 3.1, $\theta : H \to F(X)$ is a precrossed module.

Analogously to our method in this example, we can define Peiffer elements in any precrossed module. Let $\mathcal{M} = (\mu : M \to P)$ be a precrossed module and let m, m' be elements of M. Their *Peiffer commutator* is defined as

$$[\![m, m']\!] = m^{-1}m'^{-1}mm'^{\mu m}.$$

The precrossed modules in which all Peiffer commutators are trivial are precisely the crossed modules. Thus the category of crossed modules is the full subcategory of the category of precrossed modules whose objects are crossed modules.

Since the Peiffer elements are always defined in a precrossed module, it is a natural idea to factor out by the normal subgroup that they generate and consider the induced map from the quotient. Let us check that this produces a crossed module.

The *Peiffer subgroup* $[\![M, M]\!]$ of M is the subgroup of M generated by all Peiffer commutators. We now prove that this subgroup inherits the P-action and is a normal subgroup.

Theorem 3.3.2 *For any precrossed module $\mu : M \to P$, the Peiffer subgroup $[\![M, M]\!]$ of M is a P-invariant normal subgroup.*

Proof The Peiffer subgroup is P-invariant since for any $m, m' \in M$ and $p \in P$, we have

$$\begin{aligned} [\![m, m']\!]^p &= (m^{-1}m'^{-1}mm'^{\mu m})^p \\ &= (m^p)^{-1}(m'^p)^{-1}m^p m'^{(\mu m)p} \\ &= (m^p)^{-1}(m'^p)^{-1}m^p m'^{p(\mu m)^p} \\ &= (m^p)^{-1}(m'^p)^{-1}m^p m'^{p(\mu m^p)} \\ &= [\![m^p, m'^p]\!]. \end{aligned}$$

It is also normal since for any $m, m', n \in M$ we have

$$\begin{aligned} n^{-1}[\![m, m']\!]n &= n^{-1}m^{-1}m'^{-1}mm'^{\mu m}n \\ &= n^{-1}m^{-1}m'^{-1}m(nm'^{\mu mn}(m'^{-1})^{\mu mn}n^{-1})m'^{\mu m}n \\ &= ((mn)^{-1}m'^{-1}mnm'^{\mu mn})(((m'^{\mu m})^{\mu n})^{-1}n^{-1}m'^{\mu m}n) \\ &= [\![mn, m']\!][\![n, m'^{\mu m}]\!]^{-1}. \end{aligned}$$

□

Now for any precrossed module $\mu : M \to P$ we define

$$M^{\mathrm{cr}} = M/[\![M, M]\!].$$

By the previous property, M^{cr} is a P-group. Let us see that the homomorphism μ induces a crossed module.

Proposition 3.3.3 *For any precrossed module $\mu : M \to P$, the induced map gives a crossed module*

$$\mathcal{M}^{\mathrm{cr}} = (\mu^{\mathrm{cr}} : M^{\mathrm{cr}} \to P)$$

which we call the crossed module associated to μ.

Proof It is easy to see that for each $m, m' \in M$, $\mu[\![m, m']\!] = 1$, so μ induces a homomorphism of groups μ^{cr}.

Clearly μ^{cr} satisfies CM1) because it was already satisfied by μ. It also satisfies CM2) because all Peiffer commutators have been quotiented out. □

The association of the crossed module $M^{\mathrm{cr}} \to P$ to a precrossed module $M \to P$ gives a functor

$$(-)^{\mathrm{cr}} : \mathsf{PXMod/Groups} \to \mathsf{XMod/Groups}.$$

That is, a morphism (g, f) of precrossed modules yields a morphism (g^{cr}, f) of crossed modules, and this association satisfies the usual functorial rules.

Moreover let us prove that $(-)^{\mathrm{cr}}$ is a left adjoint of the inclusion $\mathsf{XMod/Groups} \hookrightarrow \mathsf{PXMod/Groups}$ by checking the appropriate universal property.

Proposition 3.3.4 *Let $\mathcal{M} = (\mu : M \to P)$ be a precrossed module. For any crossed module $\mathcal{N} = (\nu : N \to Q)$ and any morphism of precrossed modules $(g, f) : \mathcal{M} \to \mathcal{N}$ there is a unique morphism of crossed modules*

$$(g^{\mathrm{cr}}, f) : (\mu^{\mathrm{cr}} : M^{\mathrm{cr}} \to P) \longrightarrow (\nu' : N \to Q)$$

such that $g = g^{\mathrm{cr}} \circ \theta$ where θ is the quotient homomorphism $\theta : M \to M^{\mathrm{cr}}$.

Proof Obviously, g^{cr} can only be the homomorphisms induced by g on the quotient, and this is well defined since $g[\![m, m']\!] = 1$ for any elements m, m' of M. □

For future computations it is interesting to have a set of generators of the Peiffer subgroup as small as possible. The following property taken from Brown-Huebschmann [49] is useful for this.

Proposition 3.3.5 *Let $\mu : M \to P$ be a precrossed module and let V be a subset of M which generates M as a group and is also P-invariant. Then the Peiffer subgroup $[\![M, M]\!]$ of M is the normal closure in M of the set of Peiffer commutators*

$$\{[\![a, b]\!] \mid a, b \in V\}.$$

Proof Let Z be the normal closure of $W = \{[\![a, b]\!] \mid a, b \in V\}$. Since $[\![M, M]\!]$ is normal and contains W, it is clear that $Z \subseteq [\![M, M]\!] \subseteq \mathrm{Ker}\ \mu$. On the other hand W is P-invariant since $[\![a, b]\!]^p = [\![a^p, b^p]\!]$ as was proved in

Theorem 3.3.2. So Z is also P-invariant. Thus μ induces a homomorphism of groups $\overline{\mu} : M/Z \to P$ which is P-invariant, so that we have a precrossed module. Let us check that it is also a crossed module.

Let $\overline{V}$ be the image of V in M/Z, i.e. $\overline{V}$ is the set of cosets of all elements in V. Notice that we have

$$y^{\overline{\mu}x} = x^{-1}yx \qquad (**)$$

for any x and y lying in $\overline{V}$, which is a set of generators of M/Z. It is easy to see that for a fixed x in M/Z the set P_x of y's satisfying this equation (**) is a subgroup containing $\overline{V}$ so P_x has to be all of M/Z.

Consider now the set Q_x of x in M/Z satisfying (**) for all y in M/Z. It is closed under multiplication (since

$$y^{xx'} = (y^x)^{x'} = (x^{-1}yx)^{x'} = (x^{-1})^{x'}y^{x'}x^{x'} = x'^{-1}x^{-1}x'x'^{-1}yx'x'^{-1}xx' = x'^{-1}x^{-1}yxx')$$

and also under inversion (since if $w = y^{x^{-1}}$, we have $w^x = y$ and $w^x = x^{-1}wx$, so that $x^{-1}wx = y$ and $w = xyx^{-1}$). So $Q_x = M/Z$ and thus $\overline{\mu} : M/Z \to P$ is a crossed module. It follows that $[\![M, M]\!] \subseteq Z$. □

Corollary 3.3.6 *Let $\omega : R \to P$ be a function to the group P and let $\theta : H \to P$ be the associated precrossed module. Then the Peiffer subgroup $[\![H, H]\!]$ of H is the normal closure in H of the basic Peiffer elements $[\![a, b]\!] = a^{-1}b^{-1}ab^{\theta a}$ where $a, b \in R \times P$.*

3.4 Free precrossed and crossed modules

Another crucial property satisfied by the precrossed module associated to a presentation of a group is that it is, in some sense, free. We need to make this property explicit.

As explained in the Appendix, a free construction in a category is usually the left adjoint of some forgetful functor. The appropriate forgetful functor in this case goes from the category of crossed modules to the category of *sets over a group* forgetting the algebra of the top group and considering only the underlying boundary map. We recall the appropriate categories.

Let P be a group. We have defined the category XMod/P of crossed P-modules in Section 2.2. In a similar way, we define the category PXMod/P by restricting to precrossed modules over P.

Let P be a set. We define Sets/P to be the category whose objects are P-sets, i.e. maps $\nu : S \to P$, and whose morphisms are P-maps, i.e. maps $\alpha : S \to S'$ making commutative the diagram

$$\begin{array}{ccc} S & \xrightarrow{\alpha} & S' \\ & {\scriptstyle\nu}\searrow \quad \swarrow{\scriptstyle\nu'} & \\ & P & \end{array} .$$

We have a forgetful functor

$$U : \mathsf{XMod}/P \to \mathsf{Sets}/P.$$

Thus, the *free crossed module* construction is a functor

$$F : \mathsf{Sets}/P \to \mathsf{XMod}/P$$

such that for any P-set $\mathcal{S} = (\nu : S \to P)$ and for any crossed P-module $\mathcal{M} = (\mu : M \to P)$ there is a natural bijection

$$(\mathsf{Sets}/P)(\mathcal{S}, U\mathcal{M}) \cong (\mathsf{XMod}/P)((F\mathcal{S}, \mathcal{M}),$$

i.e. there is a P-inclusion $i : S \to FS$, corresponding to the morphism Id_{FS} of crossed P-modules such that for any P-map $f : S \to M$ there exist a unique extension to a morphism $f' : FS \to M$ of crossed P-modules.

In the same way we may define the *free precrossed module* using the forgetful functor between PXMod/P and Sets/P.

To determine when a crossed module is *free*, let $\mathcal{M} = (\mu : M \to P)$ be a crossed module, R a set and $\omega : R \to M$ an injective map (equivalently, let $\{m_r \mid r \in R\}$ be an indexed family of elements of M). We say that $\mathcal{M}$ is the *free crossed P-module* on ω (also, that ω is a *basis* of M) if the unique morphism of crossed modules $F\omega : FR \to M$ extending ω is an isomorphism, or, equivalently, if it satisfies the following universal property: for any crossed module $\mathcal{M}' = (\mu' : M' \to P)$ and map $\omega' : R \to M'$ such that $\mu\omega = \mu'\omega'$ there exists a unique morphism $h : \mathcal{M} \to \mathcal{M}'$ of P-crossed modules such that $h\omega = \omega'$.

There is a similar definition of the free precrossed module on ω. As always in universal constructions, the free crossed and precrossed P-modules on ω are unique up to isomorphism. We study now its existence.

We consider a group P and an injective map $\omega : R \to P$, or equivalently, an indexed family $\{p_r \mid r \in R\}$ of elements of P. First we create the *free P-group with basis R*. To this end, we define $E = F(R \times P)$. It is the free group on the formal elements $\{r^p \mid r \in R, p \in P\}$. We think of (r, p) as r^p. Then, any element of E can be seen as a formal product

$$(r_1^{p_1})^{\varepsilon_1} \cdots (r_n^{p_n})^{\varepsilon_n}$$

with $n \in \mathbb{N}$, $\varepsilon_i = \pm 1$, $p_i \in P$ and $r_i \in R$.

This representation makes clear the definition of the P-action on generators, since to be an action it has to satisfy $(r^p)^{p'} = r^{pp'}$. Thus, we define a P-action on E by

$$(r, p)^{p'} = (r, pp')$$

on generators and we extend it in the only possible way.

We define a map $\theta : E \to P$ by the only possible definition to make it a P-map, i.e. $\theta(r, p) = p^{-1}\omega(r)p$ on generators.

Let us check that the map θ just constructed gives the free precrossed module.

Proposition 3.4.1 *$\mathcal{E} = (\theta : E \to P)$ is the free precrossed module on $\omega : R \to E$ where $\omega(r) = (r, 1)$.*

Proof It is clear that P acts on E and also that θ is a homomorphism by the way they are defined.

It is easy to check that $\theta : E \to P$ is a precrossed module,

$$\theta(r, p)^{p'} = \theta(r, pp') = p'^{-1}p^{-1}\omega(r)pp' = p'^{-1}\theta(r, p)p'.$$

To prove the universal property, consider $\mathcal{M}' = (\mu' : M' \to P)$ a precrossed P-module and a map $\omega' : R \to M'$. We can define the map

$$\begin{array}{ccc} R \times P & \to & M' \\ (r, p) & \mapsto & (\omega' r)^p \end{array}$$

that extends to a unique homomorphism $h : E \to M'$ that is a morphism of precrossed modules since

i) $\mu' h(r, p) = \mu'(\omega' r)^p = p^{-1}(\mu'\omega' r)p = \theta(r, p)$ and

ii) $h(r, p)^{p'} = h(r, pp') = (\omega' r)^{pp'} = ((\omega' r)^p)^{p'} = h(r, p)^{p'}$.

Actually, this is the only possible definition of h to make it a map of P-groups. So h is unique. □

Corollary 3.4.2 *The crossed P-module $\mathcal{E}^{\mathrm{cr}} = (\theta^{\mathrm{cr}} : E^{\mathrm{cr}} \to P)$ is the free crossed module on $\omega : R \to E^{\mathrm{cr}}$.*

Proof Obviously $\mathcal{E}^{\mathrm{cr}}$ is a crossed module. Let us check the universal property.

For any crossed P-module $\mathcal{M}' = (\mu' : M' \to P)$ and any map $\omega' : R \to M'$ there is a unique morphism of precrossed modules $\alpha : E \to M'$ satisfying $\omega'\alpha = \omega$. Thus, the induced map $\alpha^{\mathrm{cr}} : E^{\mathrm{cr}} \to M'$ is the only morphism of crossed modules satisfying $\omega'\alpha = \omega$. □

Remark 3.4.3 For any crossed module $\mathcal{M} = (\mu : M \to P)$ such that $\mu(M)$ is a free group, there is a section $s : \mu M \to M$ which is a homomorphism of groups. Then, the Proposition 2.2.4 applies.

So if $\mu : M \to P$ is the free crossed P-module associated to a presentation $(\omega : R \to P)$ of a group G then there is a short exact sequence of G-modules

$$0 \to \pi = \mathrm{Ker}\ \mu \longrightarrow M^{\mathrm{ab}} \xrightarrow{\mu^{\mathrm{ab}}} (\mu M)^{\mathrm{ab}} \to 0.$$

From the construction of the free precrossed module as a free group, it is clear that $\omega : R \to E$ is injective. It is not so clear that $\omega : R \to E^{\mathrm{cr}}$ is also injective. This is a consequence of the following property:

Proposition 3.4.4 *Given a free crossed P-module $\mathcal{M} = (\mu : M \to P)$ on $\omega : R \to M$, with G the cokernel of μ, then M^{ab} is a free G-module with basis $\omega^{\mathrm{ab}} : R \to M^{\mathrm{ab}}$.*

Proof We know by Proposition 2.2.3 ii) that M^{ab} is a G-module. To see that M^{ab} is free we will prove that it satisfies the universal property of a free G-module.

Let M' be a G-module. The projection $P \times M' \to P$ becomes a crossed P-module when P acts on $P \times M'$ by conjugation on P and the G action on M'. For any map $v : R \to M'$ we define $v' = (\mu\omega, v) : R \to P \times M'$. Since $\mu : M \to P$ is a free crossed P-module we get a unique morphism of P-crossed modules $\phi : M \to P \times M'$ such that $v' = \phi\omega$. The composite $M \to M'$ factors through a G-morphism $\overline{\phi} : M^{\mathrm{ab}} \to M'$ which is the only morphism of G-modules satisfying $\overline{\phi}\omega^{\mathrm{ab}} = v$. □

We now give an example due to Whitehead which illustrates some of the difficulties of working with free crossed modules.

Example 3.4.5 Let $(\partial : C(R) \to F(X))$ be the free crossed module on the subset R of $F(X)$ and suppose that Y is a subset of X, and S a subset of R. Let M be the subgroup of $C(R)$ generated by $F(Y)$ operating on S, and assume that $\partial(M) \subseteq F(Y)$. Let $\mathcal{M}' = (\partial' : M \to F(Y))$ be the crossed module given by restricting ∂ to M. Then $\mathcal{M}'$ is not necessarily a free crossed module. Whitehead in [176] gives the following example and proposition.

Let $X = Y = \{x\}, R = \{a, b\}, S = \{b\}$ be such that $\partial a = x, \partial b = 1$. Since $\partial b = 1$, we have $ab = ba$, whence $b^x b^{-1} = a^{-1}bab^{-1} = 1$. Therefore $\mathcal{M}'$ is not a free crossed module.

Proposition 3.4.6 *Let G, G' be the cokernels of ∂, ∂' respectively, and let $\eta : G \to G'$ be the morphism induced by the inclusion $i : F(Y) \to F(X)$. If η is injective, then $\mathcal{M}'$ is the free crossed $F(Y)$-module on S.*

Proof Let $d : C(S) \to F(Y)$ be the free crossed $F(Y)$-module on S. It is clear that $d(C(S)) = \partial(M)$. Let $j : C(S) \to M$ be the morphism of crossed $F(Y)$-modules. Clearly j is surjective, and the result is proved when we have shown that j is injective.

Suppose that $u \in C(S)$ and $j(u) = 1$. Then $d(u) = 1$. Let $k : C(S)^{\mathrm{ab}} \to C(R)^{\mathrm{ab}}$ be the induced morphism of the abelianised groups. These abelianised groups are in fact free modules over G, G' respectively on the bases S, R respectively. Since η is injective, it follows that k is injective. Let $\overline{u}$ denote the class of u in $C(S)^{\mathrm{ab}}$. Then $k\overline{u} = 0$, and hence $\overline{u} = 0$. But the morphism $C(S) \to C(S)^{\mathrm{ab}}$ is injective on Ker d. It follows that $u = 1$. □

3.5 Pre Cat^1-groups and the existence of colimits

In the two previous section we have seen that when working with crossed modules it is sometimes convenient to consider the weaker structure of precrossed modules and see the category XMod/Groups as a full subcategory of PXMod/Groups.

In Section 2.5 we have seen that the category Cat$^1-$ Groups of cat^1-groups is equivalent to the category XMod/Groups. It is an easy exercise to put both together and construct a category bigger than and equivalent to PXMod/Groups.

So, a *pre-cat^1-group* is a triple (G, s, t) where G is a group and $s, t : G \to G$ are endomorphisms satisfying $st = t$ and $ts = s$. Thus we are omitting CG2) from the axioms of a cat^1-group, i.e. we do not impose commutativity between elements of $\mathrm{Ker}\, s$ and $\mathrm{Ker}\, t$.

As before, a *morphism* between pre-cat^1-groups is just a homomorphism of groups commuting with the s's and t's. These objects and morphisms define the category PCat$^1-$ Groups. It contains Cat$^1-$ Groups as a full subcategory.

Proposition 3.5.1 *The categories* PCat$^1-$ Groups *and* PXMod/Groups *are equivalent, by an equivalence extending that between* Cat$^1-$ Groups *and* XMod/Groups.

Proof The definitions of both functors are the same as in Section 2.5, namely

$$\lambda : \mathsf{PXMod/Groups} \to \mathsf{PCat^1\!-Groups}$$

is given by $\lambda(\mu : M \to P) = (P \ltimes M, s, t)$, s and t being defined as before, and

$$\gamma : \mathsf{PCat^1\!-Groups} \to \mathsf{PXMod/Groups},$$

is defined by $\gamma(G, s, t) = (t| : \mathrm{Ker}\, s \to \mathrm{Im}\, s)$.

It is easily checked that both functors are well defined and both compositions are naturally equivalent to the identity. □

As in the Section 3.3, we may define a functor associating to each pre-cat^1-group a cat^1-group

$$(-)^{cat} : \mathsf{PCat^1\!-Groups} \to \mathsf{Cat^1\!-Groups}$$

defined by $(G, s, t)^{cat} = (G/N, s', t')$, where $N = [\mathrm{Ker}\ s, \mathrm{Ker}\ t]$.

It is easy to see that the functor $(-)^{cat}$ corresponds through the equivalences of categories to

$$(-)^{\mathrm{cr}} : \mathsf{PXMod/Groups} \to \mathsf{XMod/Groups}.$$

Then, it follows

Proposition 3.5.2 *The functor $(-)^{cat}$ is a left adjoint of the inclusion.*

Using this last property we can prove the existence of colimits in $\mathsf{Cat}^1-\mathsf{Groups}$.

Since left adjoint functors preserve colimits (see [133] or Appendix A.6), for any indexed family $\mathcal{G}_\lambda = (G_\lambda, s_\lambda, t_\lambda)$ of cat^1-groups and morphisms between them, we have

$$\mathrm{colim}_{cat}\{\mathcal{G}_\lambda\} = (\mathrm{colim}_{pre}\{\mathcal{G}_\lambda\})^{cat}.$$

So, the existence of colimits in $\mathsf{Cat}^1-\mathsf{Groups}$ has been reduced to the existence of colimits in $\mathsf{PCat}^1-\mathsf{Groups}$.

It is not difficult now to check that in $\mathsf{PCat}^1-\mathsf{Groups}$ the colimits are as expected, i.e. for an indexed family $\{\mathcal{G}_\lambda \mid \lambda \in \Lambda\}$ of pre-cat^1-groups $\mathcal{G}_\lambda = (G_\lambda, s_\lambda, t_\lambda)$ and morphisms between them,

$$\mathrm{colim}_{pre}\{\mathcal{G}_\lambda\} = (\mathrm{colim}_{gr}\{G_\lambda\}, \mathrm{colim}_{gr}\{s_\lambda\}, \mathrm{colim}_{gr}\{t_\lambda\}).$$

From the existence of colimits in $\mathsf{Cat}^1-\mathsf{Groups}$ follows the existence of colimits in $\mathsf{XMod/Groups}$ using the equivalence between both categories.

Remark 3.5.3 We have just added another way of computing colimits of crossed modules. So, if we have an indexed family of crossed modules $\{\mu_\lambda : M_\lambda \to P_\lambda\}$, we construct the associated family of cat^1-groups $\{(M_\lambda \ltimes P_\lambda, s_\lambda, t_\lambda)\}$ getting their colimit (G, s, t) and the colimit crossed module is $t| : \mathrm{Ker}\, s \to \mathrm{Im}\, t$.

Even if it seems a long way around, it is worthwhile because for example $M_\lambda \ltimes P_\lambda$ may be finitely generated, even if M_λ and P_λ are not. Also, there are some efficient computer-assisted ways of getting colimits, kernels and images of finitely generated groups and homomorphisms.

3.6 Implementation of crossed modules in GAP

Nowadays is almost impossible to make any serious computational work in group theory without use of a computational group theory package. Some of these packages have evolved to accommodate more structures becoming veritable computational discrete algebra packages. The one we have been using along the book is GAP (see [99] for more information). The package GAP has been developed primarily for combinatorial group theory, and has the significant advantage of free availability of the library code, thus enabling the user to modify a function so as to return additional information.

Work at Bangor (in particular by M.Alp and C.D. Wensley) has produced the module XMOD which includes a number of constructions on crossed modules, cat^1-groups and their morphisms. In particular: derivations, kernels and images; the Whitehead group; cat^1-groups and their relation with crossed modules; induced crossed modules.

This package has already been in use for some time, and has been incorporated into GAP4. We note that Alp and Wensley have in [6] used this programme to list many finite cat^1-groups.

In Section 5.9 we shall show XMOD has been used to determine explicitly some induced crossed modules which do not follow from general theorems and seem too hard to compute by hand.

Chapter 4

Coproducts of crossed P-modules

In this chapter we start to show how the van Kampen theorem in dimension 2 and the algebra of crossed modules allows specific nonabelian calculations in homotopy theory in dimension 2. To this end, we study the coproduct of crossed modules (mainly of two crossed modules) over the same group P. We construct the coproduct of crossed P-modules, check some properties and, using the van Kampen Theorem, we apply these general results to some topological cases.

In the first section (4.1) we construct the coproduct of crossed P-modules. First, we see what the definition of coproduct in a general category means in this case, and then we prove its existence by a two step procedure. As a first step, we prove that the free product of groups gives the coproduct in the category of precrossed P-modules. Then, using the fact that the functor $(-)^{\mathrm{cr}}$ preserves coproducts, we see that its associated crossed P-module is the coproduct in the category of P-modules.

This procedure is a bit complicated to implement because the free product is always a very big group (it is normally infinite even if all groups are finite). So in Section 4.2 we give an alternative description of the coproduct of two crossed P-modules. This is obtained by dividing the construction of the associated crossed module in this case into two steps, of which the first gives a semidirect product. Thus the coproduct of two crossed P-modules is a quotient of a semidirect product. Hence we can get presentations of the coproduct using the known presentations of the semidirect product.

This has some topological bearings as explained in Section 4.3. First, we know that the coproduct of two crossed P-modules is just the pushout of these two crossed modules with respect to the trivial crossed module $1 \to P$. Thus in the case that we have a topological space X that is the union of two open subsets U_1, U_2 such that both (U_i, U_{12}) are 1-connected, the fundamental crossed module $\Pi_2(X, U_{12})$ is the coproduct $\Pi_2(U_1, U_{12}) \circ \Pi_2(U_2, U_{12})$ (Theorem 4.3.1) and we can use the previous results to get information on the second homotopy group of some spaces. We end this section by studying some consequences in this case.

In the last section (4.4) we study the coproduct in a particular case that we shall use later. We begin with two crossed P-modules $\mathcal{M} = (\mu : M \to P)$ and $\mathcal{N} = (\nu : N \to P)$ satisfying the condition

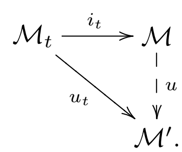
$$(*) \ : \ \nu(N) \subseteq \mu(M) \text{ and there is an equivariant section of } \mu.$$

In this case, we get a description of their coproduct using the displacement subgroup N_M (Theorem 4.4.8). This case is not uncommon and we get some topological applications when the space X is got from Y by attaching a cone CA, that is, X is a mapping cone. We finish this last section with a description of the coproduct for an arbitrary set of indexes satisfying the above condition (*). This result will be used at the end of the next Chapter (see Section 5.8).

4.1 The coproduct of crossed P-modules

We give a construction of coproducts in the category XMod/P of crossed modules over the group P. We do this for a general family of indices since this causes no more difficulty than the case of two crossed modules.

From the general definition of the coproduct in a category given in the Appendix, we see that the coproduct of a family $\{\mathcal{M}_t \mid t \in T\}$ of crossed modules over P is given by a crossed module $\mathcal{M}$ and a family of morphisms of crossed P-modules $\{i_t : \mathcal{M}_t \to \mathcal{M} \mid t \in T\}$ satisfying the following universal property: for any family $\{u_t : \mathcal{M}_t \to \mathcal{M}' \mid t \in T\}$ of morphisms of crossed modules over P, there is a unique morphism $u : \mathcal{M} \to \mathcal{M}'$ of crossed modules over P such that $u_t = ui_t$ for each $t \in T$. Diagrammatically, there exists a unique dashed arrow such that the following diagram commutes:



As with any universal construction, the coproduct is unique up to isomorphism.

As we have seen in Section 3.3, the functor $(-)^{\mathrm{cr}}$ from precrossed modules to crossed modules, obtained by factoring out the Peiffer subgroup, is left adjoint to the inclusion of crossed modules into precrossed modules, and so takes coproducts into coproducts. Thus to construct the coproduct of crossed P-modules we construct the coproduct in PXMod/P, the category of precrossed modules over the group P and apply the functor crs to it. The coproduct in PXMod/P is simply obtained using the coproduct in the category Groups of groups, and this is the well known free product $*_t G_t$ of a family $\{G_t\}$ of groups [131].

Proposition 4.1.1 *Let T be an indexing set and, for each $t \in T$ let $\mathcal{M}_t = (\mu_t : M_t \to P)$ be a precrossed P-module. We define $*_t M_t$ to be the free product of the groups M_t, $t \in T$. There is an action of P on $*_t M_t$ defined by the action of P on each M_t. Consider the morphism*

$$*_t \mathcal{M}_t = (\partial' : *_t M_t \to P),$$

together with the homomorphisms $i_t : M_t \to *_t M_t$ given by the inclusion in the free product, and where $\partial' = *_t \mu_t$ is the homomorphism of groups induced from the homomorphisms μ_t using the universal property of the coproduct of groups. Then the above defined $*_t \mathcal{M}_t$ is a precrossed P-module and the homomorphisms i_t are morphisms of precrossed modules over P giving the coproduct in the category PXMod/P.

Proof Let $\mathcal{M} = *_t \mathcal{M}_t$. If we represent by $p_\#$ the action by $p \in P$, then the action $p_\# : \mathcal{M} \to \mathcal{M}$ of p is defined by the composite morphisms $\mathcal{M}_t \xrightarrow{p_\#} \mathcal{M}_t \xrightarrow{i_t} \mathcal{M}$.

In terms of the normal form of an element of the free product, this means that the action is given by the formula

$$(m_{t_1} \dots m_{t_n})^p = (m_{t_1})^p \dots (m_{t_n})^p, \ m_{t_i} \in M_{t_i}.$$

As already pointed out, the homomorphisms μ_t extend uniquely to a homomorphism $*_t \mu_t$. So

$$\begin{aligned}(*_t \mu_t)((m_{t_1} \dots m_{t_n})^p) &= (*_t \mu_t)(m_{t_1}^p \dots m_{t_n}^p) \\ &= (\mu_{t_1}(m_{t_1}^p)) \dots (\mu_{t_n}(m_{t_n}^p)) \\ &= p^{-1}(\mu_{t_1} m_{t_1})p \dots p^{-1}(\mu_{t_n}(m_{t_n}))p \\ &= p^{-1}((\mu_{t_1} m_{t_1}) \dots (\mu_{t_n}(m_{t_n}))p\end{aligned}$$

and $*_t\mu_t$ is a precrossed module.

The verification of the universal property is easy. □

We now easily obtain:

Corollary 4.1.2 *If $\mathcal{M}_t = (\mu_t : M_t \to P), t \in T$ is a family of crossed P-modules, then applying the functor $(-)^{\mathrm{cr}}$ to $*_t\mathcal{M}_t$ to give*

$$\partial'^{\mathrm{cr}} : (*_t\mathcal{M}_t)^{\mathrm{cr}} \to P$$

with the morphisms $j_t : M_t \xrightarrow{i_t} *_t\mathcal{M}_t \to (*_t\mathcal{M}_t)^{\mathrm{cr}}$, where the second morphism is the quotient homomorphism, gives the coproduct of crossed P-modules.

We denote this coproduct by

$$\bigcirc_t\mathcal{M}_t = (\partial : \bigcirc_t M_t \to P)$$

where the morphisms $j_t : \mathcal{M}_t \to \bigcirc_t\mathcal{M}_t$ are understood to be part of the structure. These morphisms need not be injective. In the case $T = \{1, 2, \ldots, n\}$, this coproduct will be written $M_1 \circ \cdots \circ M_n \to P$. As is standard for coproducts in any category, the coproduct in XMod/P is associative and commutative up to natural isomorphisms.

4.2 The coproduct of two crossed P-modules

Throughout this section we suppose given two crossed P-modules $\mathcal{M} = (\mu : M \to P)$ and $\mathcal{N} = (\nu : N \to P)$, and we develop at some length the study of their coproduct in XMod/P

$$\mathcal{M} \circ \mathcal{N} = (\mu \circ \nu : M \circ N \to P)$$

and the canonical morphisms from M, N into $M \circ N$. This is the case that has been analysed more deeply in the literature. Most of the results of this section were in print for the first time in a paper by Brown ([27]). Further results were obtained in [94], and some more applications and results are also given in [110]. However this construction as a quotient of the free product really goes back to Whitehead [179].

The basic observation in [27] is that $M \circ N$ may be obtained as a quotient of the semidirect product group $M \ltimes N$ where M operates on N *via* P. This result makes the coproduct of two crossed modules computable and from this we get some topological computations.

For convenience, we assume M, N are disjoint. To study $M \circ N = (M * N)^{\mathrm{cr}}$ in some detail we should have a closer look at $[\![M * N, M * N]\!]$, the Peiffer subgroup of $M * N$. As seen in Section 3.3, $[\![M * N, M * N]\!]$ is the subgroup of $M * N$ generated by all Peiffer commutators

$$[\![k, k']\!] = k^{-1}k'^{-1}kk'^{(\mu*\nu)k}$$

for all k, $k' \in M * N$.

Notice that by Proposition 3.3.5, $[\![M * N, M * N]\!]$ is also the normal subgroup generated by the Peiffer commutators of any given P-invariant set of generators. Now $M \cup N$ generates $M * N$ and is P-invariant. Since $\mathcal{M}$ and $\mathcal{N}$ are crossed modules, redwe have $[\![m, m']\!] = 1$ and $[\![n, n']\!] = 1$, for all $m, m' \in M$ and $n, n' \in N$. Thus $[\![M * N, M * N]\!]$ is the normal subgroup of $M * N$ generated by the elements

$$r(m, n) = n^{-1}m^{-1}nm^n, \text{ and } s(m, n) = m^{-1}n^{-1}mn^m$$

for all $m \in M$, $n \in N$.

It is useful to divide the process of quotienting out by the Peiffer subgroup into two steps. First, we consider the quotient of $M * N$ by the group U generated by $\{s(m,n) \mid m \in M, n \in N\}$ all Peiffer commutators of the second kind. A useful observation already developed in [27] is that this quotient is the well known semidirect product.

Proposition 4.2.1 *The precrossed P-module*

$$\frac{\mathcal{M} * \mathcal{N}}{U} = (\mu * \nu : (M * N)/U \to P)$$

where U is the normal P-invariant subgroup generated by the set $\{m^{-1}n^{-1}mn^m \mid m \in M, n \in N\}$ is isomorphic to

$$\mathcal{M} \ltimes \mathcal{N} = (\mu \ltimes \nu : M \ltimes N \to P)$$

where the semidirect product is associated to the action of M on N via μ and the P-action.

Proof The inclusions $M \to M \ltimes N$ and $N \to M \ltimes N$ extend to a homomorphism of groups

$$\varphi : M * N \to M \ltimes N.$$

Let us check that $\varphi(U) = 1$ by computing φ on all generators,

$$\begin{aligned} \varphi(m^{-1}n^{-1}mn^m) &= (m^{-1},1)(1,n^{-1})(m,1)(1,n^m) \\ &= (m^{-1},n^{-1})(m,n^m) \\ &= (m^{-1}m,(n^{-1})^m n^m) \\ &= (1,1). \end{aligned}$$

So we have an induced homomorphism of P-groups

$$\overline{\varphi} : (M * N)/U \to M \ltimes N.$$

We define a homomorphism in the other direction

$$\psi : M \ltimes N \to (M * N)/U$$

by $\psi(m,n) = [mn]$ the equivalence class of the element $mn \in M * N$. To check the homomorphism property, we compute

$$\begin{aligned} \psi(m',n')^{-1}\psi(m,n)^{-1}\psi((m,n)(m',n')) &= [{n'}^{-1}{m'}^{-1}][n^{-1}m^{-1}]\psi(mm',n^{m'}n') \\ &= [{n'}^{-1}{m'}^{-1}n^{-1}m^{-1}mm'n^{m'}n'] \\ &= [{n'}^{-1}({m'}^{-1}n^{-1}m'n^{m'})n'] \\ &= [1] \end{aligned}$$

since ${m'}^{-1}n^{-1}m'n^{m'} \in U$.

Clearly $\overline{\varphi}\psi = 1$. Since $\psi\overline{\varphi}$ is a homomorphism, to prove that it is 1 it is enough to check this on the generators $\psi\overline{\varphi}[mn]$, $m \in M$, $n \in N$, and this is clear.

It now follows, as may be proved directly, that $\mu \ltimes \nu : M \ltimes N \to P$, $(m,n) \mapsto (\mu m)(\nu n)$ is a homomorphism which with the action of P given by $(m,n)^p = (m^p,n^p)$ is a precrossed P-module. □

So $\mathcal{M} \ltimes \mathcal{N}$ is a precrossed module containing $\mathcal{M}$ and $\mathcal{N}$ as submodules. Let us see that it satisfies a universal property with respect to maps of the crossed modules $\mathcal{M}$ and $\mathcal{N}$ to any given crossed module $\mathcal{M}'$.

Proposition 4.2.2 *Let $\mathcal{M}' = (\mu' : M' \to P)$ be a crossed P-module and let $f : M \to M'$ and $g : N \to M'$ be morphisms of crossed P-modules. Then there is a unique map of precrossed P-modules extending f and g, namely $f \ltimes g : M \ltimes N \to M'$, $(m, n) \mapsto (fm)(gn)$.*

Proof Uniqueness is obvious.

To prove existence we have to check that the morphism of precrossed P-modules

$$f * g : M * N \to M'$$

sends all elements of U to 1, where U is the subgroup specified in Proposition 4.2.1. On generators of U we have

$$(f * g)(m^{-1}n^{-1}mn^m) = f(m^{-1})g(n^{-1})f(m)g(n^m) = g(n^{-1})^{\mu' fm}g(n)^{\mu m} = 1$$

since $\mu' : M' \to P$ is a crossed module and $\mu' f = \mu$. □

Therefore it is clear that the coproduct of two crossed P-modules $\mu : M \to P$ and $\nu : N \to P$ is the crossed module associated to the precrossed module $\mu \ltimes \nu : M \ltimes N \to P$, i.e.

$$\mathcal{M} \circ \mathcal{N} = ((\mu \ltimes \nu)^{\mathrm{cr}} : (M \ltimes N)^{\mathrm{cr}} \to P) = (M \circ N \to P).$$

This has some striking consequences.

Remark 4.2.3 If we have two crossed P-modules such that M and N are finite groups (resp. finite p-groups), then so also is the semidirect product $M \ltimes N$ and hence their coproduct as crossed modules $M \circ N$ is also a finite group (resp. a finite p-group). This result was not clear at all from previous descriptions of the coproduct of crossed P-modules.

Remark 4.2.4 If $(\mu : M \to P)$, $(\nu : N \to P)$ are crossed P-modules such that each of M, N act trivially on the other via P, then $M \ltimes N = M \times N$ and $\partial : M \times N \to P$, where $\partial(m, n) = (\mu m)(\nu n)$ is the coproduct where $(m, n)^p = (m^p, n^p)$.

We now study the Peiffer subgroup $[\![M \ltimes N, M \ltimes N]\!]$ of $M \ltimes N$, which we shall write $\{M, N\}$. As we have seen, it is the subgroup generated by the Peiffer commutators of all elements of $M \ltimes N$. Alternatively, $\{M, N\}$ is generated by the images by φ of $r(m, n)$, i.e. by

$$\{\{n, m\} \mid m \in M, n \in N\}$$

Lemma 4.2.5 *The elements $\{n, m\}$ satisfy*

$$\{n, m\} = ([m, n], [n, m]),$$

where $[m, n] = m^{-1}m^n$ and $[n, m] = n^{-1}n^m$.

Proof Notice that any m, $m' \in M$ and $n \in N$ satisfy the relation

$$n'^{(m^n)} = ((n'^{n^{-1}})^m)^n = n^{-1}(nn'n^{-1})^m n = n^{-1}n^m n'^m (n^{-1})^m n \qquad (*)$$

Thus,

$$\begin{aligned}\{n,m\} &= n^{-1}m^{-1}nm^n \\ &= (1,n^{-1})(m^{-1},1)(1,n)(m^n,1) \\ &= (m^{-1},(n^{-1})^{m^{-1}})(m^n,(n^m)^n) \\ &= (m^{-1}m^n,((n^{-1})^{m^{-1}})^{m^n}(n^m)^n) \\ &= (m^{-1}m^n,n^{-1}n^m) \qquad \text{using } (*).\end{aligned}$$

Finally, we have

$$\{n,m\} = ([m,n],[n,m]).$$

□

Using the previous result and some well known facts on the semidirect product, we get a presentation of the coproduct of two crossed modules as follows. First, recall that the semidirect product has a presentation with generators the elements $(m,n) \in M \times N$ and relations

$$(m,n)(m',n') = (mm',n^{m'}n')$$

for all $m,m' \in M$ and $n,n' \in N$. The set of relations may equivalently be expressed as

$$(m,n^{m'^{-1}})(m',n') = (mm',nn').$$

To get a presentation of $M \circ N$ we add the relations corresponding to the Peiffer subgroup $\{M,N\}$. By the preceding property the relation $\{m,n\} = 1$ is equivalent to $[m,n] = [n,m]^{-1}$, giving $(m^n)^{-1}m = n^{-1}n^m$, or $n(m^{-1})^n = (n^{m^{-1}})^{-1}m^{-1}$. This may be expressed, taking $m' = m^{-1}$,

$$nm'^n = (n^{m'})^{-1}m'$$

suggesting the next proposition.

Theorem 4.2.6 *The group $M \circ N$ has a presentation with generators $\{m \circ n \mid m \in M, n \in N\}$, and relations*

$$mm' \circ nn' = (m \circ n^{m'^{-1}})(m' \circ n') = (m \circ n)(m'^n \circ n'),$$

for all $m,m' \in M$ and $n,n' \in N$.

Proof Let K be the group with this presentation. Then P acts on K by $(m \circ n)^p = m^p \circ n^p$, and the map

$$\xi : K \to P,\ m \circ n \mapsto (\mu m)(\nu n),$$

is a well defined homomorphism. It is routine to verify the crossed module rules for this structure.

It is also not difficult to check that this crossed module together with the morphisms $i : M \to K,\ m \mapsto m \circ 1$ and $j : N \to K,\ n \mapsto 1 \circ n$ satisfy the universal property of the coproduct. We omit further details. □

We describe some extra facts about $\{M,N\}$. In particular, the expression of the products and inverses of the elements $\{n,m\}$.

Proposition 4.2.7 *For any $m,m' \in M$ and $n,n' \in N$ we have*

$$\{n,m\}\{n',m'\} = ([m,n][m',n'],[n',m'][n,m]).$$

Proof

$$\begin{aligned}\{n,m\}\{n',m'\} &= ([m,n],[n,m])([m',n'],[n',m']) \\ &= ([m,n][m',n'],[n,m]^{[m',n']}[n',m'],\end{aligned}$$

and

$$\begin{aligned}[n,m]^{[m',n']}[n',m'] &= (n^{-1}n^{m})^{m'^{-1}m'^{n'}}n'^{-1}n'^{m'} \\ &= ((n^{-1})^{m'^{-1}}(n^{m})^{m'^{-1}})^{m'^{n'}}n'^{-1}n'^{m'} \\ &= n'^{-1}n'^{m'}n^{-1}n^{m}(n'^{-1})^{m'}n'n'^{-1}n'^{m'} \qquad \text{using (*) in Lemma 4.2.5} \\ &= n'^{-1}n'^{m'}n^{-1}n^{m} \\ &= [n',m'][n,m].\end{aligned}$$

Thus

$$\{n,m\}\{n',m'\} = ([m,n][m',n'],[n',m'][n,m])$$

as indicated. □

Remark 4.2.8 This result extends to any finite product of elements $\{n_i, m_i\}$ with $m_i \in M, n_i \in N$.

Corollary 4.2.9 *For any $m \in M$ and $n \in N$ we have*

$$\{n,m\}^{-1} = \{n^{-1}, m^{n}\}.$$

The proof is left to the reader.

Finally for this section, and in preparation for the next, we express the universal property of the coproduct of two crossed P-modules in another way.

Proposition 4.2.10 *If $(\mu : M \to P)$, $(\nu : N \to P)$ are crossed P-modules then the following diagram*

$$\begin{array}{ccc} (1 \to P) & \longrightarrow & (N \to P) \\ \downarrow & & \downarrow \\ (M \to P) & \longrightarrow & (M \circ N \to P) \end{array} \tag{4.2.1}$$

is a pushout in the category XMod/P *and also in the category* $\mathsf{XMod}/\mathsf{Groups}$.

The equivalence of the pushout property in the category XMod/P with the universal property of the coproduct is easy to verify. We defer the proof of the pushout property in the category $\mathsf{XMod}/\mathsf{Groups}$ until we have introduced in Section 5.2 the pullback functor $f^* : \mathsf{XMod}/Q \to \mathsf{XMod}/P$ for a morphism $f : P \to Q$ of groups.

4.3 The coproduct and the van Kampen theorem

One of the interesting features of the coproduct of crossed P-modules is its topological applications. The van Kampen Theorem as stated in Theorem 2.3.1 involved a kind of generalised pushout (a coequaliser, in fact). One of the simpler cases is the following.

Theorem 4.3.1 *Suppose that the connected space X is the union of the interior of two connected subspaces U_1, U_2, with connected intersection U_{12}. Suppose that the pairs (U_1, U_{12}) and (U_2, U_{12}) are 1-connected. Then the pair (X, U_{12}) is 1-connected and the morphism*

$$\Pi_2(U_1, U_{12}) \circ \Pi_2(U_2, U_{12}) \to \Pi_2(X, U_{12})$$

induced by inclusions is an isomorphism of crossed $\pi_1(U_{12})$-modules.

Proof We apply Theorem 2.3.1 to the cover of X given by U_1 and U_2 with $A = U_{12}$. The connectivity result is immediate. Also by the same theorem the following diagram is a pushout of crossed modules:

$$\begin{array}{ccc} \Pi_2(U_{12}, U_{12}) & \longrightarrow & \Pi_2(U_1, U_{12}) \\ \downarrow & & \downarrow \\ \Pi_2(U_2, U_{12}) & \longrightarrow & \Pi_2(X, U_{12}) \end{array}$$

Since $\Pi_2(U_{12}, U_{12}) = (1 \to \pi_1(U_{12}))$, the result follows from Proposition 2.1.3. □

We would like to extract from this result some information on the absolute homotopy group $\pi_2(X)$. Consider the following part of the homotopy exact sequence of the pair (X, U_{12}) stated in 2.1.3,

$$\cdots \to \pi_2(U_{12}) \xrightarrow{i_*} \pi_2(X) \xrightarrow{j_*} \pi_2(X, U_{12}) \xrightarrow{\partial} \pi_1(U_{12}) \to \cdots .$$

It is clear that we have an isomorphism

(4.3.1)
$$\frac{\pi_2(X)}{i_*(\pi_2(U_{12}))} \cong \operatorname{Ker} \partial = \operatorname{Ker} (\partial_1 \circ \partial_2).$$

Notice than, in particular, this result gives complete information on $\pi_2(X)$ when $\pi_2(U_{12}) = 0$.

It would be a good thing to be able to identify the kernel of the coproduct of two crossed P-modules in a more workable way. To do this, let us introduce the pull back of crossed P-modules. Given two crossed modules $\mathcal{M} = (\mu : M \to P)$, $\mathcal{N} = (\nu : N \to P)$ we form the pullback square

(4.3.2)
$$\begin{array}{ccc} M \times_P N & \xrightarrow{p_1} & M \\ {\scriptstyle p_2}\downarrow & & \downarrow{\scriptstyle \mu} \\ N & \xrightarrow[\nu]{} & P \end{array}$$

where $M \times_P N = \{(m, n) \in M \times N \mid \mu(m) = \nu(n)\}$, p_1 and p_2 are the projections. Obviously $M \times_P N$ is a P-group (P acts diagonally).

Proposition 4.3.2 *$M \times_P N$ is isomorphic as P-group to* $\operatorname{Ker}(\mu \ltimes \nu)$.

Proof Let

$$\phi : M \times_P N \to M \ltimes N$$

be defined as $\phi(m, n) = (m, n^{-1})$.

To check that it is a homomorphism of groups we compute for all $m, m' \in M$ and $n, n' \in N$

$$\begin{aligned}\phi(m,n)\phi(m',n') &= (m, n^{-1})(m', n'^{-1}) \\ &= (mm', (n^{-1})^{m'} n'^{-1}) \\ &= (mm', (n^{-1})^{n'} n'^{-1}) \\ &= (mm', n'^{-1} n^{-1} n' n'^{-1}) \\ &= (mm', (nn')^{-1}) \\ &= \phi(mm', nn').\end{aligned}$$

Clearly, ϕ is a bijection onto $\mathrm{Ker}\,(\mu \ltimes \nu)$ that preserves the P-actions. □

Now, to any $m \in M$ and $n \in N$ we associate an element of $M \times_P N$ defined as

$$\langle m, n\rangle = (m^{-1}m^n, (n^{-1})^m n). \tag{4.3.3}$$

If we write $\langle M, N\rangle$ for the normal subgroup of $M \times_P N$ generated by $\{\langle m, n\rangle | m \in M, n \in N\}$, we have seen that $\phi(\langle M, N\rangle) = \{M, N\}$.

Thus, there is an induced map

$$\overline{\phi} : \frac{M \times_P N}{\langle M, N\rangle} \longrightarrow \frac{M \ltimes N}{\{M, N\}} = M \circ N.$$

We deduce immediately from the proposition

Corollary 4.3.3 *The map $\overline{\phi}$ gives an isomorphism of P-modules*

$$\overline{\phi} : \frac{M \times_P N}{\langle M, N\rangle} \cong \mathrm{Ker}(\mu \circ \nu).$$

Remark 4.3.4 Notice that this result has some purely algebraic consequences. Since $\mathcal{M} \circ \mathcal{N}$ is a crossed module, Ker $(\mu \circ \nu)$ is abelian; so $\langle M, N\rangle$ contains the commutator subgroup of $M \times_P N$.

Now we can translate this algebraic result into a topological one.

Theorem 4.3.5 *If (U_1, U_{12}) and (U_2, U_{12}) are 1-connected and $\pi_2(U_{12}) = 0$, we have,*

$$\pi_2(X) \cong \frac{\pi_2(U_1, U_{12}) \times_{\pi_1(U_{12})} \pi_2(U_2, U_{12})}{\langle \pi_2(U_1, U_{12}), \pi_2(U_2, U_{12})\rangle}.$$

Proof Since $\pi_2(U_{12}) = 0$, from the equation (4.3.1), we have $\pi_2(X) \cong \mathrm{Ker}\,(\partial_1 \circ \partial_2)$ and the result follows from the corollary before. □

Let us study some other algebraic way of computing Ker $(\mu \circ \nu)$ or, equivalently, the quotient

$$\frac{M \times_P N}{\langle M, N\rangle}.$$

We may also define a homomorphism of groups $k : M \times_P N \to P$ by the formula $k(m, n) = \mu(m) = \nu(n)$. This gives the following result.

Proposition 4.3.6 *There is an exact sequence of P-groups*

$$0 \to \operatorname{Ker}\mu \oplus \operatorname{Ker}\nu \to M \times_P N \xrightarrow{k} \mu(M) \cap \nu(N) \to 1.$$

Proof It is immediate that $k(M \times_P N) = \mu(M) \cap \nu(N)$. It remains to check that Ker $k =$ Ker $\mu \oplus$ Ker ν; but this is clear since

$$\operatorname{Ker} k = \{(m,n) \mid \mu(m) = \nu(n) = 0\}$$

and all $m \in$ Ker μ and $n \in$ Ker ν commute. □

Bringing again the subgroup $\langle M, N\rangle$ into the picture, it is immediate that $k(\langle m, n\rangle) = [\mu(m), \nu(n)]$. Then we have $k(\langle M, N\rangle) = [\mu(M), \nu(N)]$ giving a homomorphism $\overline{k}$ onto the quotient. This gives directly the next result.

Corollary 4.3.7 *There is an exact sequence of P-modules*

$$0 \to (\operatorname{Ker}\mu \oplus \operatorname{Ker}\nu) \cap (\langle M, N\rangle) \to \operatorname{Ker}\mu \oplus \operatorname{Ker}\nu \to \frac{M \times_P N}{\langle M, N\rangle} = Ker\,(\mu \circ \nu) \xrightarrow{\overline{k}} \frac{\mu(M) \cap \nu(N)}{[\mu(M), \nu(N)]} \to 0.$$

Remark 4.3.8 An easy consequence is that $\mu \circ \nu$ is injective if and only if

i) Ker μ $\oplus$ Ker $\nu \subset \langle M, N\rangle$ and

ii) $[\mu(M), \nu(N)] = \mu(M) \cap \nu(N)$.

As before, we can apply this result to the topological case, getting a way to compute the second homotopy group of a space in some cases.

Theorem 4.3.9 *If (U_1, U_{12}) and (U_2, U_{12}) are 1-connected and $\pi_2(U_{12}) = 0$, the following sequence of groups and homomorphisms is exact*

$$0 \to (\pi_2(U_1) \oplus \pi_2(U_2)) \cap \langle \pi_2(U_1, U_{12}), \pi_2(U_2, U_{12})\rangle \to \pi_2(U_1) \oplus \pi_2(U_2) \to \pi_2(X) \to \frac{R_1 \cap R_2}{[R_1, R_2]} \to 1,$$

where $R_l = \operatorname{Ker}(\pi_1(U_{12}) \to \pi_1(U_l))$ for $l = 1, 2$.

If further $\pi_2(U_1) = \pi_2(U_2) = 0$, then there is an isomorphism

$$\pi_2(X) \cong \frac{R_1 \cap R_2}{[R_1, R_2]}.$$

Proof Let us consider the crossed modules $\partial_l : \pi_2(U_l, U_{12}) \to \pi_1(U_{12})$. Recall from (2.1.3) that the homotopy exact sequence of the pair (U_l, U_{12}) is

$$\cdots \to \pi_2(U_{12}) \xrightarrow{i_{l*}} \pi_2(U_l) \xrightarrow{j_{l*}} \pi_2(U_l, U_{12}) \xrightarrow{\partial_l} \pi_1(U_{12}) \to \cdots .$$

Directly from this exact sequence, we have

$$\operatorname{Im}\partial_l = R_l.$$

On the other hand,

$$\operatorname{Ker}\,\partial_l = \pi_2(U_l)$$

using the same homotopy exact sequence and $\pi_2(U_{12}) = 0$.

Thus the result is a translation of Corollary 4.3.7.

□

Remark 4.3.10 Whenever U_1, U_2 are based subspaces of X with intersection U_{12} there is always a natural map

$$\sigma : \pi_2(U_1, U_{12}) \circ \pi_2(U_2, U_{12}) \to \pi_2(X, U_{12})$$

determined by the inclusions, but in general σ is not an isomorphism. Bogley and Gutierrez in [17] have had some success in describing Ker σ and Coker σ in the case when all the above spaces are connected.

4.4 Some special cases of the coproduct

We end this chapter by giving a careful description of the coproduct of crossed P-modules in the particular case of two crossed P-modules $\mu : M \to P, \nu : N \to P$ in a useful special case, i.e. when $\nu(N) \subseteq \mu(M)$ and there is a P-equivariant section $\sigma : \mu M \to M$ of μ. Notice that this includes the case when $M = P$ and μ is the identity. These results were first published in [73].

This case is important because of the topological applications and also because it is useful in Section 5.6 for describing as a coproduct the crossed module induced by a monomorphism.

We start with some general results that will be used several times in this book.

Definition 4.4.1 If M acts on the group N we define $[N, M]$ to be the subgroup of N generated by the elements, often called *pseudo-commutators*, $n^{-1}n^m$ for all $n \in N$, $m \in M$. This subgroup is called the *displacement subgroup* and measures how much N is moved under the M-action.

The following result is analogous to a standard result on the commutator subgroup.

Proposition 4.4.2 *The displacement subgroup $[N, M]$ is a normal subgroup of N.*

Proof It is enough to prove that the conjugate of any generator of $[N, M]$ lies also in $[N, M]$.

Let $m \in M, n, n_1 \in N$. We easily check that

$$n_1{}^{-1}(n^{-1}n^m)n_1 = ((nn_1)^{-1}(nn_1)^m)(n_1{}^{-1}n_1{}^m)^{-1}$$

and the product on the right hand side belongs to $[N, M]$ since both factors are generators. So we have proved $n_1{}^{-1}[N, M]n_1 \subseteq [N, M]$, whence $[N, M]$ is a normal subgroup of N. □

Definition 4.4.3 The quotient of N by the displacement subgroup is written $N_M = N/[N, M]$. The class in N_M of an element $n \in N$ is written $[n]$. It is clear that N_M is a trivial M-module since $[n^m] = [n]$.

Proposition 4.4.4 *Let $\mu : M \to P$, $\nu : N \to P$ be crossed P-modules, so that M acts on N via μ. Then P acts on N_M by $[n]^p = [n^p]$. Moreover this action is trivial when restricted to μM.*

Proof To see that the P-action on N induces one on N_M, we have to check that [N,M] is a P-invariant subgroup and this follows because $(n^{-1}n^m)^p = (n^{-1})^p(n^m)^p = (n^p)^{-1}(n^p)^{m^p}$ for all $n \in N,\ m \in M,\ p \in P$.

The action of μM is trivial since $[n]^{\mu m} = [n^{\mu m}] = [n^m] = [n]$. □

Now we study the homomorphism

$$\xi : M \times N_M \to P,\ (m, [n]) \mapsto \mu m.$$

We have just seen that N_M is a P-group.

Proposition 4.4.5 *With P acting on $M \times N_M$ by the diagonal action, $\xi : M \times N_M \to P$ is a precrossed P-module.*

Proof If $m \in M, n \in N, p \in P$ then

$$\xi((m, [n])^p) = \xi(m^p, [n^p]) = \mu(m^p) = p^{-1}(\mu m)p = p^{-1}(\xi(m, [n])^p))p.$$

□

Remark 4.4.6 In general it is not a crossed module. Nevertheless when N_M is abelian, the actions of both factors on each other are trivial. In this case it follows from Remark 4.2.4 that $\xi : M \times N_M \to P$ is a crossed module. (It is an easy exercise to prove this directly.)

We shall study now a condition first stated in [94] that implies that N_M is abelian.

Proposition 4.4.7 *Let $\mu : M \to P,\ \nu : N \to P$ be crossed P-modules such that $\nu N \subseteq \mu M$. Then N_M is abelian and therefore $\xi : M \times N_M \to P$ is a crossed P-module.*

Proof Let $n, n_1 \in N$. Choose $m \in M$ such that $\nu n_1 = \mu m$. Then by the crossed module rule CM2)

$$n_1{}^{-1} n n_1 = n^{\nu n_1} = n^{\mu m}$$

and so in the quotient $[n_1]^{-1}[n][n_1] = [n^{\mu m}] = [n]$. □

We now study the case where there is also a P-equivariant section $\sigma : \mu M \to M$ of μ defined on μM. We will see that in this case $\xi : M \times N_M \to P$ is isomorphic to the coproduct $\mathcal{M} \circ \mathcal{N}$ of crossed P-modules. We shall follow the later proof given by Brown and Wensley in [63]. This contains the main result of [94] but it is stronger in the sense that it determines explicitly the coproduct structure. Since we shall use this structure for later results, we give the proof in detail.

Theorem 4.4.8 *Let $\mu : M \to P,\ \nu : N \to P$ be crossed P-modules with $\nu N \subseteq \mu M$ and let $\sigma : \mu M \to M$ be a P-equivariant section of μ. Then the morphisms of crossed P-modules*

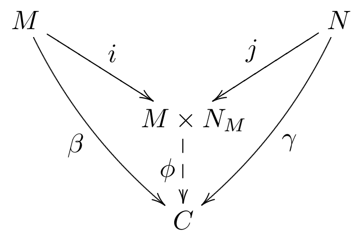
$$\begin{aligned} i : M &\to M \times N_M, & j : N &\to M \times N_M, \\ m &\mapsto (m, 1) & n &\mapsto (\sigma\nu n, [n]) \end{aligned}$$

give a coproduct of crossed P-modules. Hence the canonical morphism of crossed P-modules

$$M \circ N \to M \times N_M$$

given by $m \circ n \mapsto (m(\sigma\nu n), [n])$ is an isomorphism.

Proof We need to verify that the pair (i, j) satisfies the universal property of the coproduct of crossed P-modules. Consider an arbitrary crossed P-module $\chi : C \to P$ and morphisms of crossed P-modules $\beta : M \to C$, and $\gamma : N \to C$. We have the following diagram:



and we want to prove that there is a unique $\phi : M \times N_M \to C$ determining a morphism of crossed P-modules closing the diagram i.e. such that $\phi i = \beta$, and $\phi j = \gamma$.

Let us consider uniqueness. For any $m \in M, n \in N$, since ϕ has to be a homomorphism, we have

$$\begin{aligned}\phi(m, [n]) &= \phi((m, 0)(\sigma\nu n, 0)^{-1}(\sigma\nu n, [n])) \\ &= (\beta m)(\beta\sigma\nu n)^{-1}(\gamma n).\end{aligned}$$

This proves uniqueness of any such a ϕ. We now prove that this formula gives a well-defined morphism.

It is immediate from the formula that $\phi : M \times N_M \to C$ has to be β on the first factor and is defined on the second one by the map $[n] \mapsto (\beta\sigma\nu n)^{-1}(\gamma n)$. We have to check that this latter map is a well defined homomorphism.

We define the function

$$\psi : N \to C$$

by $n \mapsto (\beta\sigma\nu n^{-1})(\gamma n)$ and prove in turn the following statements.

4.4.9 *$\psi(N) \subseteq Z(C)$, the centre of C, and $\chi(C)$ acts trivially on $\psi(N)$.*

Proof of 4.4.9 Since $\chi\beta = \mu$ and $\chi\gamma = \nu$, it follows that $\chi\psi = 0$ and $\psi(N) \subseteq \text{Ker } \chi$. Since C is a crossed module, $\chi(C)$ acts trivially on Ker χ and Ker $\chi \subseteq Z(C)$. □

4.4.10 *ψ is a morphism of crossed P-modules.*

Proof of 4.4.10 We have to prove that ψ is a morphism and is P-equivariant. The latter is clear, since $\beta, \gamma, \sigma, \nu$ are P-equivariant. So let $n, n_1 \in N$. Then

$$\begin{aligned}\psi(nn_1) &= (\beta\sigma\nu n_1^{-1})(\beta\sigma\nu n^{-1})(\gamma n)(\gamma n_1) \\ &= (\beta\sigma\nu n_1^{-1})(\psi n)(\gamma n_1) \\ &= (\psi n)(\beta\sigma\nu n_1^{-1})(\gamma n_1) && \text{by (4.4.9)} \\ &= (\psi n)(\psi n_1).\end{aligned}$$

□

Note that even if σ is not P-equivariant, ψ is still a group homomorphism.

4.4.11 *M acts trivially on $\psi(N)$.*

Proof of 4.4.11 Let $m \in M,\ n \in N$. Note that $(\beta\sigma\mu m)(\beta m^{-1})$ lies in Ker χ, and so belongs to $Z(C)$. Hence

$$\begin{aligned}(\psi n)^m &= (\beta\sigma\nu n^m)^{-1}(\gamma n)^{\mu m}\\ &= \beta\sigma((\mu m^{-1})(\nu n)(\mu m))^{-1}(\gamma n)^{\chi\beta m}\\ &= (\beta\sigma\mu m^{-1})(\beta\sigma\nu n^{-1})(\beta\sigma\mu m)(\beta m^{-1})(\gamma n)(\beta m)\\ &= (\beta\sigma\mu m^{-1})(\beta\sigma\mu m)(\beta m^{-1})(\beta\sigma\nu n^{-1})(\gamma n)(\beta m)\\ &= (\beta m^{-1})(\psi n)(\beta m)\\ &= \psi n \qquad \text{by (4.4.9)}\end{aligned}$$ □

It follows that ψ induces a morphism $\psi' : N_M \to C,\ [n] \mapsto \psi n$, and so we define

$$\phi = (\beta, \psi') : M \times N_M \to C$$

by $(m, [n]) \mapsto (\beta m)(\psi n)$. Since ψn commutes with βm we easily verify that ϕ is a homomorphism, $\phi i = \beta,\ \phi j = \gamma$ and $\chi\phi = \xi$. Thus the pair of morphisms $i : M \to M \times N_M,\ j : N \to M \times N_M$ satisfies the universal property of a coproduct. This completes the proof of the theorem. □

A standard consequence of the existence of a homomorphism $\sigma : \mu M \to M$ which is a section of μ on μM is that M is isomorphic to the semidirect product $\mu M \ltimes \text{Ker}\ \mu$, where μM acts on Ker μ by conjugation, i.e. $m'^{\mu m} = m^{-1}m'm$. Moreover, in the case when μ is a crossed module and σ is P-equivariant, the isomorphism is as crossed P-modules. Thus we have a third expression for the coproduct.

Proposition 4.4.12 *Let $\mu : M \to P,\ \nu : N \to P$ be crossed P-modules with $\nu N \subseteq \mu M$ and let $\sigma : \mu M \to M$ be a P-equivariant section of μ. There is an isomorphism of crossed P-modules*

$$M \circ N \cong (\mu M \ltimes Ker\mu) \times N_M$$

given by $m \circ n \mapsto (m(\sigma\mu m)^{-1}, (\mu m)(\nu n), [n])$.

We now give a topological application.

Corollary 4.4.13 *Let (Y, A) be a connected based pair of spaces, and let $X = Y \cup CA$ be obtained from Y by attaching a cone on A. Then there is an isomorphism of crossed $\pi_1(A)$-modules*

$$\pi_2(X, A) \cong \pi_1(A) \times \pi_2(Y, A)_{\pi_1(A)}.$$

Proof We apply Theorem 4.3.1 with $U_1 = CA, U_2 = Y$, so that $U_{12} = A$. Then $\pi_2(CA, A) \cong \pi_1(A)$, by the exact sequence of the pair (CA, A), so that we have $\pi_2(X, A) \cong \pi_1(A) \circ \pi_2(Y, A)$. The result now follows from Theorem 4.4.8. □

As another application of Theorem 4.4.8, we analyse the symmetry of the coproduct in a special case.

The symmetry morphism $\tau : M \circ N \to N \circ M$ is, as usual for a coproduct, given by the pair of canonical morphisms $M \to N \circ M,\ N \to N \circ M$. Hence τ is given by $m \circ n \mapsto (1 \circ m)(n \circ 1) = n \circ m^n$.

Proposition 4.4.14 *Let $\mu : M \to P$ be a crossed module where μ is an inclusion of a normal subgroup of the group P. Then the isomorphism of crossed P-modules*

$$\begin{aligned}\theta : M \circ M &\to M \times M^{\text{ab}}\\ \theta(m \circ n) &= (mn, [n])\end{aligned}$$

transforms the twist isomorphism $\tau : M \circ M \to M \circ M$ *to the isomorphism*

$$\begin{array}{rcl} \theta^{-1}\tau\theta : M \times M^{\mathrm{ab}} & \to & M \times M^{\mathrm{ab}} \\ (m, [n]) & \mapsto & (m, [n^{-1}m]). \end{array}$$

Proof Notice that in this case $M^{\mathrm{ab}} = M_M$. The isomorphism $\theta : M \circ M \to M \times M^{\mathrm{ab}}$ is given in theorem 4.4.8. The twist isomorphism is transformed into the composition

$$(m, [n]) \mapsto mn^{-1} \circ n \mapsto n \circ (mn^{-1})^n = n \circ n^{-1}m \mapsto (m, [n^{-1}m]).$$

□

For an application in the next section, we now extend the last results to more general coproducts. We first prove:

Proposition 4.4.15 *Let* T *be an indexing set, and let* $\mu : M \to P$ *and* $\nu_t : N_t \to P$, $t \in T$, *be crossed* P*-modules. Let*

$$N = \bigcirc_{t \in T} N_t.$$

Suppose that $\nu_t N_t \subseteq \mu M$ *for all* $t \in T$*. Then there is an isomorphism of* P*-modules*

$$N_M \cong \bigoplus_{t \in T} (N_t)_M.$$

Proof Since $N = \bigcirc_{t \in T} N_t$ is the quotient of the free product $*N_t$ by the Peiffer relations, N_M can be presented as the same free product with the Peiffer relations $n_s^{-1} n_t^{-1} n_s n_t^{\nu_s n_s} = 1$ and the relations $n_t^{\mu m} = n_t$ for all $n_s \in N_s,\ n_t \in N_t,\ m \in M$.

These relations are equivalent to the commutator relations $[n_s, n_t] = 1$ together with $n_t^{\mu m} = n_t$ for all $n_s \in N_s,\ n_t \in N_t,\ m \in M$. □

Corollary 4.4.16 *Suppose in addition that the restriction* $\mu| : M \to \mu M$ *of* μ *has a* P*-equivariant section* σ*. Then there are isomorphisms of crossed* P*-modules between*

(i) $M \circ (\bigcirc_{t \in T} N_t)$,

(ii) $\xi : M \times \bigoplus_{t \in T} (N_t)_M \to P,\ \xi(m, n) = \mu m$,

(iii) $\xi\eta^{-1} : \mu M \times Ker\mu \times \bigoplus_{t \in T} (N_t)_M \to P$.

Under the first isomorphism, the coproduct injections $i : M \to M \circ (\bigcirc_{t \in T} N_t),\ j_t : N_t \to M \circ (\bigcirc_{t \in T} N_t)$ *are given by* $m \mapsto (m, 0),\ n_t \mapsto (\sigma\nu_t n_t, [n_t])$.

When T is well-ordered, we may also obtain explicit isomorphisms by writing a typical element of $\bigcirc_{t \in T} N_t$ as $\bigcirc_{t \in T} n_t$, and by writing a finite product of elements $\nu_t n_t \in P$ as $\prod_{t \in T} \nu_t n_t$.

Corollary 4.4.17 *When* T *is well-ordered, the rules*

$$m \circ (\bigcirc_{t \in T} n_t) \mapsto (\, m(\prod_{t \in T} (\sigma\nu_t n_t)), \bigoplus_{t \in T} [n_t]\,) \mapsto (\, m(\sigma\mu m^{-1}),\ (\mu m)(\prod_{t \in T} \nu_t n_t),\ \bigoplus_{t \in T} [n_t])\,)$$

define isomorphisms $M \circ (\bigcirc_{t \in T} N_t) \cong M \times \bigoplus_{t \in T} (N_t)_M, \cong \mu M \times Ker\mu \times \bigoplus_{t \in T} (N_t)_M$.

Chapter 5

Induced crossed modules

Here we give a full account of another construction which allows detailed computations of non abelian homotopical information in dimension 2, namely the induced crossed modules. These arise topologically from a pushout of pairs of spaces of the form

$$\begin{array}{ccc} (A, A) & \longrightarrow & (X, A) \\ \downarrow & & \downarrow \\ (Y, Y) & \longrightarrow & (X \cup_f Y, Y) \end{array}$$

on applying the 2-dimensional van Kampen Theorem. The above diagram in fact gives a format for what is known topologically as *excision*, since if all the maps are closed inclusions then $X \cup_f Y$ with Y cut out, or excised, is the same as X with A excised. In the case of homology, and under suitable conditions, we end up with isomorphisms $H_n(X, A) \to H_n(X \cup_f Y, Y)$.

This is by no means so for relative homotopy groups, and this illustrates the complication of 2-dimensional algebra. The result we give on induced crossed modules shows how crossed modules cope with this complication. There are many implications.

We also find as a consequence of these methods that we obtain the relative Hurewicz theorem in dimension 2 and also a famous formula of Hopf on the second homology of an aspherical space. This formula was one of the starting points of the important theory of the cohomology of groups. These applications give a model for higher dimensional results.

The induced construction illustrates a feature of homotopy theory, that identifications in low dimensions can influence strongly high dimensional homotopy. Applications of generalised van Kampen theorems give information, though in a limited range of dimensions and under restrictive conditions, on how this influence is controlled.

The constructions in this chapter are quite elaborate and in places quite technical. This illustrates the complications of the geometry. We are illustrating the complications of 2-dimensional homotopy theory, and also that the algebra can cope with this.

Also the crossed module "induced" by a homomorphism of groups $f : P \to Q$ may be seen as one of the family of "change of base" functors of algebraic categories that have proved interesting in many fields from algebraic geometry to homological algebra.

The construction of the induced crossed module follows a natural pattern. Given the morphism f as above and a crossed P-module $\mu : M \to P$, we need to construct from M and f a new group N on which Q acts so as to be a candidate for a crossed Q-module. Therefore we need new elements of the form m^q for $m \in M, q \in Q$. Since these do not for the moment exits, we construct them by taking the free group on pairs (m, q) and then adding appropriate relations. This is done in detail in Section 5.3.

In Section 5.1 we describe the pullback of a crossed module $(f^*(\mathcal{M}))$. This is quite easy to construct and the existence of the induced crossed module $(f_*(\mathcal{M}))$ defined in Section 5.2 follows from the existence of an adjoint to the pullback construction. We prove by the universal property that the free crossed module of Section 3.4 is a particular case of the induced crossed module and that an induced crossed module is the pushout of $\mathcal{M}$ and the trivial crossed module $1 \to Q$ over the trivial crossed module $1 \to P$.

That leaves the induced crossed module ready to be used in some applications of the van Kampen Theorem. In Section 5.4 we prove that when X is a topological space having a decomposition in two sets U_1, U_2 such that both pairs (U_2, U_{12}) are 1-connected, then the fundamental crossed module $\Pi_2(X, U_1)$ is the crossed module induced from $\Pi_2(U_2, U_{12})$ by the homomorphism induced by the inclusion (Theorem 5.4.1). As a consequence we get some homotopical results, in particular Whitehead's Theorem.

The second part of the Chapter is devoted to study the construction of the induced crossed module in a more useful guise. Since the direct construction is in general enormous (the first step uses a free group), it is interesting to get a more manageable way of producing induced crossed modules. One fruitful idea is to study separately the case when f is surjective and the case when f is injective and this is done in the next two sections.

The surjective case (Section 5.5) is quite direct and we prove that $f_*(\mathcal{M})$ is the quotient of M by the displacement subgroup $[M, \text{Ker } f]$. This case has some interesting topological applications, in particular the relative Hurewicz's Theorem in dimension 2 and Hopf's formula for the second homology group of a group.

The case when f is injective, i.e. a monomorphism (Section 5.6), is essentially the inclusion of a subgroup. This case is much more intricate and we need the concept of the copower construction M^{*T} where T is a transversal of P in Q. We get a description of the induced crossed module as a quotient of the copower (Corollary 5.6.6). Both the group and the action have alternative descriptions that can be used to develop some examples, so obtaining in particular a bound for the number of generators and relations for an induced crossed module.

It is also proved (in Section 5.7) that the induced crossed module is finite when both M and the index $[P : Q]$ are finite. This suggests the problem of explicit computation, and in the last section of the chapter we explain some computer calculations in the finite case obtained using the package GAP.

The next Section (5.8) is quite technical but contains a detailed description of the induced crossed module in a useful special case, with many interesting examples, namely when P and M are both normal subgroups of Q. We start by studying the induced crossed module when P is a normal subgroup of Q, getting a description in terms of the coproduct $M^{\circ T}$. Then we use the description of the coproduct given in the last Section of the preceding Chapter to derive just from the universal property both the action (Theorem 5.8.6) and the map (Theorem 5.8.7). When M is just another normal subgroup included in P, we get some more concrete formulas.

This leaves many finite examples not covered by the previous theorems: the last section gives some computer calculations.

The results of this chapter are taken mainly from [39, 62, 63, 64].

5.1 Pullbacks of precrossed and crossed modules.

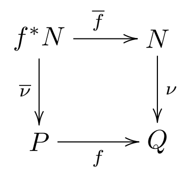
The work of this section can be done both for crossed and for precrossed modules. We shall state only the crossed case but, if nothing is said, it is understood that a similar result is true for precrossed modules. We shall not repeat the statement, but we only shall give indications of the differences.

Let us start by defining the functor that is going to be the adjoint of the induced crossed module, the "pullback". This is an important construction which, given a morphism of groups $f : P \to Q$, enables us to move from crossed Q-modules to crossed P-modules.

Definition 5.1.1 Let $f : P \to Q$ be a homomorphism of groups and let $\mathcal{N} = (\nu : N \to Q)$ be a crossed module. We define the subgroup of $N \times P$

$$f^*N = N \times_Q P = \{(n, p) \in N \times P \mid \nu n = fp\}.$$

This is the usual pullback in the category Groups. There is a commutative diagram



where $\overline{\nu} : (n, p) \mapsto p$, $\overline{f} : (n, p) \mapsto n$. Then P acts on f^*N via f and the diagonal, i.e. $(n, p)^{p'} = (n^{fp'}, {p'}^{-1}pp')$. It is easy to see that this gives a P-action. The *pullback crossed module* is

$$f^*\mathcal{N} = (\overline{\nu} : f^*N \to P)$$

It is also called the pullback of $\mathcal{N}$ along f and it is easy to see that $f^*\mathcal{N}$ is a crossed module.

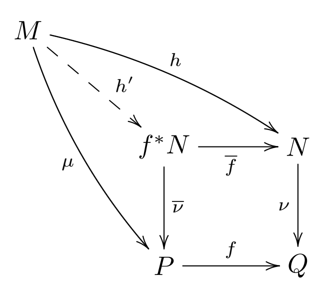
This construction satisfies a crucial universal property, analogous to that of the pullback of groups. To state it, we use also the morphism of crossed modules

$$(\overline{f}, f) : f^*\mathcal{N} \longrightarrow \mathcal{N}.$$

Theorem 5.1.2 *For any crossed module $\mathcal{M} = (\mu : M \to P)$ and any morphism of crossed modules*

$$(h, f) : \mathcal{M} \longrightarrow \mathcal{N}$$

there is a unique morphism of crossed P-modules $h' : \mathcal{M} \to f^*\mathcal{N}$ such that the following diagram commutes



Proof The existence and uniqueness of the homomorphism h' follows from the fact that f^*N is the pullback in the category of groups. It is defined by $h'(m) = (h(m), \mu(m))$. So we have only to prove that h' is a morphism of crossed P-modules. This can be checked directly. □

Using this universal property, it is not difficult to see that this construction gives a functor

$$f^* : \mathsf{XMod}/Q \to \mathsf{XMod}/P.$$

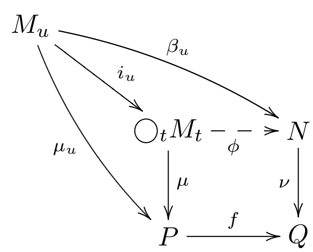
Moreover, these functors are 'natural' in the sense that there are natural equivalences $f^* f'^* \simeq (f'f)^*$ for any homomorphisms $f : P \to Q$ and $f' : Q \to R$.

In the last chapter, we dealt with the coproduct of crossed P-modules, which satisfied a universal property in the category XMod/P of crossed P-modules. We shall need an extension of this property in Section 5.8. It gives the existence and uniqueness of a morphism of crossed modules associated to a family of morphisms of crossed modules $\{(\beta_t, f)\}$ over the same homomorphism $f : P \to Q$. The standard universal property of the coproduct is just the particular case $f = \mathrm{Id}$. The argument we give uses the above pullback functor f^* and can be seen in a more general categorical light. You may skip this part until the result is needed. The proof takes time to write out but is in essence quite direct.

Proposition 5.1.3 *Let $\mathcal{M}_t$, $t \in T$ be a family of crossed P-modules. Let $f : P \to Q$ be a homomorphism of groups, let $\mathcal{N} = (\nu : N \to Q)$ be an arbitrary crossed Q-module, and for each $u \in T$ let $\beta_u : M_u \to N$ be a homomorphism giving a morphism of crossed modules over f. Then there exists a unique crossed module morphism $\phi : \bigcirc_t M_t \to N$ over f such that $\phi i_u = \beta_u$ for all $u \in T$.*

Proof The proof can be summarised by saying that we use the universal property of the pullback functor to show that the universal property for the coproduct in the category XMod/P extends to the more general case.

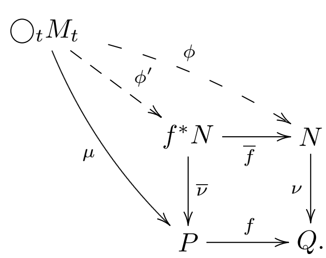
This general universal property asks for the existence and uniqueness of the dashed homomorphism ϕ in the diagram



such that the diagram commute and (ϕ, f) is a morphism of crossed modules.

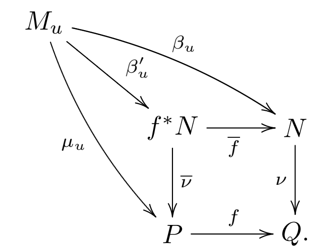
As happens many times, uniqueness is immediate from the fact that $\bigcup i_t(M_t)$ generates $\bigcirc_t M_t$.

By construction of the pullback of groups, if the homomorphism ϕ exists, it has to factor through f^*N giving a commutative diagram

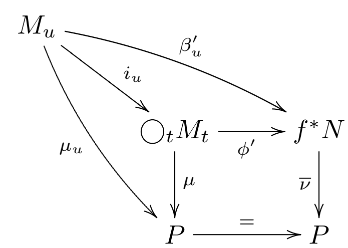


So we just have to construct a homomorphism ϕ' that gives a morphism of crossed P-modules.

By the universal property of pullbacks, for each u there is a unique homomorphism $\beta'_u : M_u \to f^*N$ such that $\overline{f}\beta'_u = \beta_u$. Moreover, β'_u is a morphism of crossed P-modules and makes the diagram commutative:



By the universal property of coproducts of crossed modules over P, there is a unique morphism of crossed P-modules $\phi' : \bigcirc_t M_t \to f^*N$ such that for all $u \in T$ the diagrams



commute.

The composite morphism $\phi = \overline{f}\phi'$ is the unique morphism satisfying $\phi i_u = \beta_u$ for all $u \in T$. □

5.2 Induced precrossed and crossed modules

Now we define a functor f_* left adjoint to the pullback f^* of the previous section. In particular we prove that the free crossed module is a particular case of an induced crossed module. Then we apply this to the topological case to get Whitehead's Theorem (Corollary 5.4.8).

The "induced crossed module" functor is defined by the following universal property, adjoint to that of pullback.

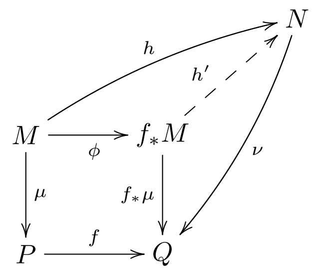
Definition 5.2.1 For any crossed P-module $\mathcal{M} = (\mu : M \to P)$ and any homomorphism $f : P \to Q$ the crossed module *induced* by f from μ should be given by:

i) a crossed Q-module $f_*\mathcal{M} = (f_*\mu : f_*M \to Q)$;

ii) a morphism of crossed modules $(\phi, f) : \mathcal{M} \longrightarrow f_*\mathcal{M}$, satisfying the dual universal property that for any morphism of crossed modules

$$(h, f) : \mathcal{M} \longrightarrow \mathcal{N}$$

there is a unique morphism of Q-crossed modules $h' : f_*M \to N$ such that the diagram



commutes.

Now we prove that this functor if it exists, forms an adjoint pair with the pullback functor. Using general categorical considerations, we can deduce the existence of the induced crossed module functor

$$f_* : \mathsf{XMod}/P \to \mathsf{XMod}/Q$$

and, also, that they satisfy the 'naturality condition' that there is a natural equivalence of functors $f'_*f_* \simeq (f'f)_*$.

Theorem 5.2.2 *For any homomorphism of groups $f : P \to Q$, f_* is the left adjoint of f^*.*

Proof From the naturality conditions expressed earlier, it is immediate that for any crossed modules $\mathcal{M} = (\mu : M \to P)$ and $\mathcal{N} = (\nu : N \to Q)$ there are bijections

$$(\mathsf{XMod}/P)(\mathcal{N}, f^*\mathcal{N}) \cong \{h : M \to N \mid (h, f) : \mathcal{M} \to \mathcal{N} \text{ is a morphism of crossed modules}\},$$

as proved in Proposition 5.1.2 , and

$$(\mathsf{XMod}/Q)(f_*\mathcal{M}, \mathcal{N}) \cong \{h : M \to N \mid (h, f) : \mathcal{M} \to \mathcal{N} \text{ is a morphism of crossed modules}\}$$

as given in the definition.

Their composition gives the bijection needed for adjointness. □

We end this section by comparing the universal properties defining the induced crossed module and two other constructions. The first one is the free crossed module on a map. Using the induced crossed module, we get an alternative description of the free crossed module.

Proposition 5.2.3 *Let P be a group and $\{\omega_r \mid r \in R\}$ be an indexed family of elements of P, or, equivalently, a function $\omega : R \to P$. Let F be the free group generated by R and $f : F \to P$ the homomorphism of groups such that $f(r) = \omega_r \in P$. Then the crossed module $f_*(1_F) : f_*F \to P$ induced from $1_F = (Id_F : F \to F)$ by f is the free crossed P-module on $\{(1, r) \in f_*F \mid r \in R\}$.*

Proof Both universal properties assert the existence of morphisms of crossed P-modules commuting the appropriate diagrams. Let us check that the data in both constructions are equivalent.

The data in the induced crossed module are a crossed module $\mathcal{N}$ and a morphism of crossed modules $(h, f) : 1_F \to \mathcal{N}$. The data in the free crossed module are a crossed module $\mathcal{N}$ and a map $\omega' : R \to N$ lifting

ω. Since F is the free group on R, the map ω' is equivalent to a homomorphism of groups $h : F \to N$ lifting ω (i.e. $h(r) = \omega'(r)$). Moreover, h satisfies

$$h(r^{r'}) = h(r'^{-1}rr') = h(r')^{-1}h(r)h(r') = (hr)^{\nu h(r')} = (hr)^{f(r')} \tag{5.2.1}$$

for all $r, r' \in R$. So h preserves the action and (h, f) is a morphism of crossed modules.

Thus the data in both cases are equivalent. □

Remark 5.2.4 It is clear that the proof in Proposition 5.2.3 does not work for precrossed modules since in proving the equality (5.2.1) we have used axiom CM2). It is easy to see that the precrossed module induced from Id $_F : F \to F$ is not the free precrossed module but its quotient with respect to the normal subgroup generated by all relations

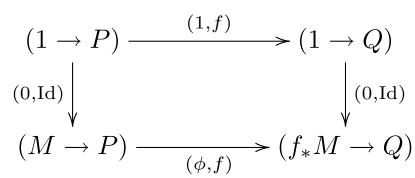
$$(p, r^{r'}) = (p\omega(r), r')$$

when $p \in P$ and $r, r' \in R$.

It is a nice exercise to find a crossed module $L \to F$ such that the free precrossed module is the induced from L.

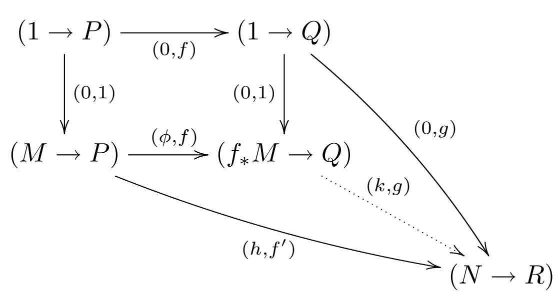
We now give an important re-interpretation of induced crossed modules in terms of pushout. This is how we can show that induced crossed modules arise from a van Kampen theorem. The proof is obtained by relating the two universal properties.

Proposition 5.2.5 *For any crossed module $\mathcal{M} = (\mu : M \to P)$ and any homomorphism $f : P \to Q$, the induced crossed module $f_*\mathcal{M}$ is such that the commutative diagram of crossed modules*



is a pushout of crossed modules.

Proof To check that the diagram satisfies the universal property of the pushout, let $\mathcal{N} = (\nu : N \to R)$ be a crossed module, and $(h, f') : \mathcal{M} \to \mathcal{N}$ and $(1, g) : 1_Q \to \mathcal{N}$ morphisms of crossed modules, such that the diagram of full arrows commutes. We have to construct the dotted morphism of crossed modules (k, g):



It immediate that $f' = gf,\ k\phi = h$. So we can transform morphisms in turn

$$(M \to P) \xrightarrow{(k\phi, gf)} (N \to R)$$
$$(M \to P) \xrightarrow{(\overline{k\phi}, 1)} ((gf)^* N \to P)$$
$$(M \to P) \xrightarrow{(\overline{k\phi}, 1)} (f^* g^* N \to P)$$
$$(f_* M \to Q) \xrightarrow{(\bar{\phi}, 1)} (g^* N \to Q)$$
$$(f_* M \to Q) \xrightarrow{(k, g)} (N \to R)$$

as required. □

5.3 Induced crossed modules: Construction in general.

We now give a simple construction of the induced crossed module, thus showing its existence. This construction is not particularly useful for computations, and this problem is dealt with later.

We are going to construct the induced crossed module in two steps, producing first the induced precrossed module and then from this the associated crossed module by quotienting out by its Peiffer subgroup.

Let us start with a homomorphism of groups $f : P \to Q$ and a crossed module $(\mu : M \to P)$. We construct

$$F = F(M \times Q),$$

the free group generated by the elements of $M \times Q$ (to make things easier to remember, we think of (m, q) as "m^q").

There is an obvious Q-action on F given on generators by

$$(m, q)^{q'} = (m, qq')$$

for any $q, q' \in Q$ and $m \in M$. Also, the map

$$\tilde{\mu} : F \to Q$$

given on generators by $\tilde{\mu}(m, q) = q^{-1} f\mu(m) q$ for any $q \in Q$ and $m \in M$ is a precrossed module.

To get the *induced precrossed module* from this map $\tilde{\mu}$, we take into the picture both the multiplication and the P-action on the first factor, and so make a quotient by the appropriate normal subgroup. Let S be the normal subgroup generated by all the relations of the two following types:

(5.3.1) $$(m, q)(m', q) = (mm', q)$$

(5.3.2) $$(m^p, q) = (m, f(p)q)$$

for any $m, m' \in M, p \in P, q \in Q$. We define $E = F/S$. It is easy to see that the action of Q on F induces one on E. Also, $\tilde{\mu}$ induces a precrossed module

$$\hat{\mu} : E \to Q.$$

There is a map

$$\phi : M \to E$$

got by projecting the map on F defined as $\phi(m) = (m, 1)$. This map is a morphism of groups thanks to the relations of type (5.3.1), while (ϕ, f) is a morphism of precrossed modules thanks to the relations of (5.3.2).

Theorem 5.3.1 *The precrossed module $\hat{\mu} : E \to Q$ is that induced from μ by the homomorphism f.*

Proof We have only to check the universal property.

For any morphism of precrossed modules

$$(h, f) : (\mu : M \to P) \longrightarrow (\nu : N \to Q)$$

there is a unique morphism of precrossed Q-modules $h' : E \to N$ such that $h = h'\phi$ because the only way to define this homomorphism is by $h'(m, q) = (hm)^q$ on generators. It is a very easy exercise to check that this definition maps S to 1, and that the induced homomorphism gives a morphism of crossed modules. □

Remark 5.3.2 If $\mathcal{M} = (\mu : M \to P)$ is a crossed module, there are two equivalent ways to obtain the induced crossed module $f_*\mathcal{M} = (f_*M \to Q)$. One way is to get the associated crossed module to the one above. The second way is to quotient out F, not only by the relations of the above two kinds, but also adding the Peiffer relations

$$(m_1, q_1)^{-1}(m_2, q_2)(m_1, q_1) = (m_2, q_2 q_1^{-1} f\mu(m_1) q_1)$$

for any $q_1,\ q_2 \in Q$ and $m_1,\ m_2 \in M$.

There are much easier descriptions of the induced crossed module in the particular cases that f is either surjective or injective and they go back to [39]. They give an alternative way of constructing the induced crossed module since every map decomposes as the product of an injection and a surjection. These are given later in Sections 5.5 and 5.6.

5.4 Induced crossed modules and the van Kampen Theorem

The relation between induced crossed module and the pushout of crossed modules suggests that the induced crossed module may appear in some cases when using the van Kampen Theorem 2.3.1. After looking to the statement of the theorem for general subspaces $A, U_1, U_2 \subseteq X$ it is easy to see that this case occurs when $A = U_1$, and this situation ia also known as 'excision'. We should give some background to this idea.

In the situation where $X = U_1 \cup U_2$, the inclusion of pairs

$$E : (U_1, U_1 \cap U_2) \to (X, U_2)$$

is known as the 'excision map' because the smaller pair is obtained by cutting out or 'excising' $X \setminus U_2$ from the larger pair. It is a theorem of homology (The Excision Theorem) that if U_1, U_2 are open in X then the excision map induces an isomorphism of relative homology groups. This is one of the basic results which make homology groups readily computable.

Here we get a result that can be interpreted as a limited form of Excision Theorem for homotopy, but it shows that the excision map is in general not an isomorphism even for second relative homotopy groups. Lack of excision is one of the reasons for the difficulty of computing homotopy groups of spaces.

Theorem 5.4.1 *Let X be a space which is the union of the interior of two subspaces U_1 and U_2 and define $U_{12} = U_1 \cap U_2$. If all spaces are connected and (U_2, U_{12}) is 1-connected, then (X, U_1) is also 1-connected and the morphism of crossed modules*

$$\Pi_2(U_2, U_{12}) \to \Pi_2(X, U_1)$$

realises the crossed module $\Pi_2(X, U_1)$ *as induced from* $\Pi_2(U_2, U_{12})$ *by the homomorphism induced by the inclusion* $\pi_1(U_{12}) \to \pi_1(U_1)$.

Proof Following the notation of Theorem 2.3.1 with $A = U_1$ we have

$$A_1 = A \cap U_1 = U_1,\ A_2 = A \cap U_2 = U_{12} \text{ and } A_{12} = A \cap U_{12} = U_{12}.$$

It is clear that the hypothesis of Theorem 2.3.1 are satisfied since $(U_1, A_1) = (U_1, U_1)$, $(U_2, A_2) = (U_2, U_{12})$ and $(U_{12}, A_{12}) = (U_{12}, U_{12})$ are 1-connected. The consequence is that the diagram of crossed modules

(5.4.1)
$$\begin{array}{ccc} \Pi_2(U_{12}, U_{12}) & \longrightarrow & \Pi_2(U_2, U_{12}) \\ \downarrow & & \downarrow \\ \Pi_2(U_1, U_1) & \longrightarrow & \Pi_2(X, U_1) \end{array}$$

is a pushout.

Proposition 5.2.5 now implies the result. □

As in the case of Theorem 2.3.1, using standard mapping cylinder arguments, we can prove a slightly more general statement.

Corollary 5.4.2 *Let* (X, A) *be a pair and* $f : A \to Y$ *a continuous map. If all spaces are connected, the inclusion* $i : A \to X$ *is a closed cofibration and the pair* (X, A) *is 1-connected, then the pair* $(Y \cup_f X, Y)$ *is also 1-connected and* $\Pi_2(Y \cup_f X, Y)$ *is the crossed module induced from* $\Pi_2(X, A)$ *by* $f_* : \pi_1(A) \to \pi_1(Y)$.

Proof This can either be deduced from the proceeding theorem by use of mapping cylinder arguments, or can be seen as a particular case of Theorem 2.3.3 when $U_1 = A$ and $Y_1 = Y$. □

This last corollary has as a consequence a homotopical Excision Theorem for closed subsets under weak conditions.

Corollary 5.4.3 *Let* X *be a space that is the union of two closed subspaces* U_1 *and* U_2 *and let* $U_{12} = U_1 \cap U_2$. *If all spaces are connected, the inclusion* $U_1 \to X$ *is a cofibration, and the pair* (U_2, U_{12}) *is connected, then the pair* (U_1, X) *is also connected and the crossed module* $(\pi_2(X, U_1) \to \pi_1(U_1))$ *is the one induced from* $(\pi_2(U_2, U_{12}) \to \pi_1(U_{12}))$ *by the morphism* $\pi_1(U_{12}) \to \pi_1(U_1)$ *induced by the inclusion.*

Before proceeding any further, we consider the case of a space X that is the homotopy pushout of classifying spaces.

Theorem 5.4.4 *Let* $\mathcal{M} = (\mu : M \to P)$ *be a crossed module, and let* $f : P \to Q$ *be a morphism of groups. Let* $\beta : BP \to B\mathcal{M}$ *be the inclusion. Consider the pushout diagram*

$$\begin{array}{ccc} BP & \xrightarrow{\beta} & B\mathcal{M} \\ {\scriptstyle Bf}\downarrow & & \downarrow \\ BQ & \xrightarrow[\beta']{} & X. \end{array}$$

i.e. $X = BQ \cup_{Bf} B\mathcal{M}$. *Then the fundamental crossed module* $\Pi_2(X, BQ)$ *is isomorphic to the induced crossed module* $f_*\mathcal{M}$.

Further, there is a map of spaces $X \to Bf_*\mathcal{M}$ *inducing an isomorphism of the corresponding* π_1, π_2.

Proof This first part immediate from Corollary 5.4.2.

The last statement requires a generalisation of Proposition 2.4.8, in which the 1-skeleton is replaced by a subcomplex Z with the property that $\pi_2(Z) = 0$ and the induced map $\pi_1(Z) \to \pi_1(X)$ is surjective. (In our case $Z = BQ$.) This result is proved in Chapter 9. □

Remark 5.4.5 The most striking consequence of the last theorem is that we have determined completely a non trivial homotopy 2-type of a space. That is, we have replaced geometric constructions by corresponding algebraic ones. As we shall see, induced crossed modules are computable in many cases, and so we can obtain many explicit computations of homotopy 2-types. The further surprise is that all this theory is needed for just this example. This shows the difficulty of homotopy theory, in that new ranges of algebraic structures are required to explain what is going on.

In the next sections, we will be able to obtain some explicit calculations as a consequence of the last results.

Remark 5.4.6 An interesting special case of the last theorem is when $\mathcal{M}$ is an inclusion of a normal subgroup, since then $B\mathcal{M}$ has the homotopy type of $B(P/M)$ by Proposition 2.4.6. So we have determined the fundamental crossed module of (X, BR) when X is the homotopy pushout

$$\begin{array}{ccc} BP & \xrightarrow{Bp} & BR \\ {\scriptstyle Bf}\big\downarrow & & \big\downarrow \\ BQ & \xrightarrow[p']{} & X \end{array}$$

in which $p : P \to R$ is surjective. In this case $\mathcal{M} = (\text{Ker } p \to P)$.

To end, we consider the case where the space we are attaching is a cone.

Theorem 5.4.7 *Let* $f : A \to Y$ *be a continuous map between connected spaces. Then the pair* $(CA \cup_f Y, Y)$ *is 1-connected and* $\Pi_2(CA \cup_f Y, Y)$ *is the crossed module induced from the identity crossed module* $1_{\pi_1(A)}$ *by* $f_* : \pi_1(A) \to \pi_1(Y)$.

Proof Using part of the homotopy exact sequence of the pair (CA, A),

$$\pi_2(CA, x) = 0 \to \pi_2(CA, A, x) \to \pi_1(A, x) \to \pi_1(CA, x) = 0$$

we get an isomorphism of $\pi_1(A, x)$ groups that transforms the fundamental crossed module $\Pi_2(CA, A)$ in $1_{\pi_1(A,x)}$.

Now, we can use Corollary 5.4.2 and identify the induced crossed module with the free module by Proposition 5.2.3. □

As a consequence we get Whitehead's theorem on free crossed modules [177].

Corollary 5.4.8 (Whitehead Theorem) *Let Y be a space constructed from X by gluing cells of dimension two. Then the map $\pi_1(X) \to \pi_1(Y)$ is surjective and $\Pi_2(Y, X)$ is the free crossed module on the characteristic maps of the 2-cells.*

As before, we apply the results just obtained to the case of a space X that is the pushout of classifying spaces.

Theorem 5.4.9 *Let $f : P \to Q$ be a morphism of groups. Then the crossed module $\Pi_2(BQ \cup_{Bf} CBP, BQ)$ is isomorphic to the induced crossed module $f_*(1_P)$.*

Proof Taking in the preceding remark $R = 1$, its classifying space is contractible. Thus, we can take CBP as equivalent to the classifying space BR. □

5.5 Calculation of induced crossed modules: the epimorphism case.

Let us consider now the case where $f : P \to Q$ is an epimorphism. Then $\operatorname{Ker} f$ acts on M via the map f and the induced crossed module f_*M may be seen as M quotiented out by the normal subgroup appropriate for trivialising the action of $\operatorname{Ker} f$ (since Q is isomorphic to $P/\operatorname{Ker} f$), i.e. by quotienting out the displacement subgroup (recall 4.4.1 to 4.4.7).

Proposition 5.5.1 *If $f : P \to Q$ is an epimorphism and $\mu : M \to P$ is a crossed module, then*

$$f_*M \cong \frac{M}{[M, \operatorname{Ker} f]}$$

where $[M, \operatorname{Ker} f]$ is the displacement subgroup, i.e. the subgroup of M generated by $\{m^{-1}m^k \mid m \in M, k \in \operatorname{Ker} f\}$.

Proof Let us recall that by Proposition 4.4.7 the quotient $M/[M, \operatorname{Ker} f]$ is a Q-crossed module with the Q-action on $M/[M, \operatorname{Ker} f]$ given by $[m]^q = [m^p]$ for $m \in M, q \in Q, q = f(p), p \in P$, and the homomorphism

$$\overline{f\mu} : \frac{M}{[M, \operatorname{Ker} f]} \to Q,$$

is induced by the composition $\mu f : M \to Q$.

It remains only to prove that this $\overline{f\mu}$ satisfies the universal property. Let

$$(h, f) : (\mu : M \to P) \longrightarrow (\nu : N \to Q)$$

be a morphism of crossed modules. We have to prove that there exists a unique homomorphism of groups

$$h' : \frac{M}{[M, \operatorname{Ker} f]} \longrightarrow N$$

such that

$$(h', f) : (\overline{f\mu} : \frac{M}{[M, \operatorname{Ker} f]} \to P) \longrightarrow (\nu : N \to Q)$$

is a morphism of crossed modules and $h'\phi = h$ where ϕ is the natural projection. Equivalently, we have to prove that h induces a homomorphism of groups h' and that (h', f) is a morphism of crossed modules.

Since $h(m^p) = (hm)^{f(p)}$ for any $m \in M$ and $p \in P$, we have $h[M, \operatorname{Ker} f] = 1$. Then, h induces a homomorphism of groups h' as above such that $h'\phi = h$.

We have only to check that h' is a map of Q-crossed modules. But

$$\nu h'[m] = \nu h(m) = f\mu(m) = \overline{f\mu}[m],$$

so the square commutes, and

$$h'([m]^q) = h'[m^p] = h(m^p) = (hm)^{f(p)} = (h'[m])^q$$

so h' preserves the actions. □

This description gives as a topological consequence a version of the relative Hurewicz Theorem.

Theorem 5.5.2 (Relative Hurewicz Theorem in dimension 2) *Consider a 1-connected pair of spaces (Y, A) such that the inclusion $i : A \to Y$ is a closed cofibration. Then the space $Y \cup C(A)$ is simply connected and its second homotopy group $\pi_2(Y \cup C(A))$ and the singular homology group $H_2(Y \cup C(A))$ are each isomorphic to $\pi_2(Y, A)$ factored by the action of $\pi_1(A)$.*

Proof It is clear from the classical van Kampen Theorem that the space $Y \cup C(A)$ is 1-connected.

Using the homotopy exact sequence of the pair $(Y \cup C(A), C(A))$,

$$\cdots \to 0 = \pi_2(C(A)) \to \pi_2(Y \cup C(A)) \to \pi_2(Y \cup C(A), C(A)) \to 0 = \pi_1(C(A)) \to \cdots$$

we have

$$\pi_2(Y \cup C(A)) \cong \pi_2(Y \cup C(A), C(A)).$$

Now we can apply Corollary 5.4.2 to show that the crossed module

$$\pi_2(Y \cup C(A), C(A)) \to \pi_1(C(A)) = 1$$

is induced from $\pi_2(Y, A) \to \pi_1(A)$ by the map given by the morphism $\pi_1(A) \to 1$ induced by the inclusion $A \to CA$.

Moreover, since the map $i_* : \pi_1(A) \to \pi_1(Y)$ is onto, by Proposition 5.5.1 we have

$$\pi_2(Y \cup C(A), C(A)) \cong \pi_2(Y, A)/[\pi_2(Y, A), \pi_1(A)].$$

This yields the result on the second homotopy group.

The absolute Hurewicz theorem for $Y \cup C(A)$ yields the result on the second homology group. □

Remark 5.5.3 Note that we obtain immediately a result on the second absolute homotopy group of $Y \cup C(A)$ without using any homology arguments. This is significant because the setting up of singular homology, proving all its basic properties, and proving the absolute Hurewicz theorem takes a considerable time. An exposition of the Hurewicz theorems occurs on pages 166-180 of G. Whitehead's text [175], assuming the properties of singular homology.

Corollary 5.5.4 *The first two homotopy groups of S^2 are given by $\pi_1(S^2) = 0, \pi_2(S^2) \cong \mathbb{Z}$.*

Proof This is the case of Theorem 5.5.2 when $A = S^1, Y = E^2_+$, where E^2_+ denotes the top hemisphere of the 2-sphere S^2. Then $\pi^2(Y, A) \cong \mathbb{Z}$ with trivial action by $\pi_1(A) \cong \mathbb{Z}$. □

Actually we have a more general result.

Corollary 5.5.5 *If A is a connected space, and $SA = CA \cup_A CA$ denotes the suspension of A, then SA is simply connected and*

$$\pi_2(SA) \cong \pi_1(A)^{\mathrm{ab}}.$$

Proof This is simply the result that $\pi_1(A)^{\mathrm{ab}} = \pi_1(A)/[\pi_1(A), \pi_1(A)]$. □

One interest in this result is the method, which extends to other situations where the notion of abelianisation is not so clear [52].

Example 5.5.6 Let $f : A \to Y$ be as in Theorem 5.4.7, let $Z = Y \cup_f CA$, and suppose that $f_* : \pi_1(A) \to \pi_1(Y)$ is surjective with kernel K. An application of Proposition 5.5.1 to the conclusion of Theorem 5.4.7 gives $\pi_2(Z) = \pi_1(A)/[\pi_1(A), K]$, and it follows from the homotopy exact sequence of the pair (Z, Y) that there is an exact sequence

$$\pi_2(Y) \to \pi_2(Z) \to K/[\pi_1(A), K] \to 0. \tag{5.5.1}$$

It follows from this exact sequence that if $A = BP$ and $Y = BQ$, so that we have an exact sequence $1 \to K \to P \to Q \to 1$ of groups, then $\pi_2(Z) \cong K/[P, K]$. Now we assume some knowledge of homology of spaces. In particular, the homology $H_i(P)$ of a group P is defined to be the homology $H_i(BP)$ of the space BP, $i \geqslant 0$. Since Z is simply connected, we get the same value for $H_2(Z)$, by the absolute Hurewicz theorem. Now the homology exact sequence of the cofibre sequence $A \to Y \to Z$ gives an exact sequence

$$H_2(P) \to H_2(Q) \to K/[P, K] \to H_1(P) \to H_1(Q) \to 0$$

(originally due to Stallings). In particular if $P = F$, a free group, or one with $H_2(F) = 0$, then we obtain an exact sequence

$$0 \to H_2(Q) \to K/[F, K] \to F^{\mathrm{ab}} \to Q^{\mathrm{ab}} \to 0.$$

This gives the famous Hopf formula

$$H_2(Q) \cong \frac{K \cap [F, F]}{[K, F]}$$

which was one of the starting points of homological algebra.

Again, one of the reason for emphasising these kinds of results is that they arise from a uniform procedure, which involves first establishing a higher order van Kampen Theorem. This theorem has analogues for algebraic models of homotopy types more elaborate than just groups or crossed modules. This procedure has led to new results, such as a higher order Hopf formula [32], which is deduced from an $(n+1)$-adic Hurewicz Theorem [52]. The only proof known of the last result is as a deduction from a van Kampen Theorem for n-cubes of spaces [53].

5.6 The monomorphism case. Inducing from crossed modules over a subgroup

In Section 5.3 we have considered the construction of an induced crossed module for a general homomorphism, and in Section 5.5 we have got a simpler expression for the case when f is an epimorphism. Now we study the case of a monomorphism. This is essentially the same as studying the case of an inclusion in a subgroup. So in all this section we shall consider the inclusion $\iota : P \to Q$ of a subgroup P of Q.

As we shall see this case is rather involved and we get an expression of the induced crossed module that is quite complicated and in some cases very much related to the coproduct. Let us introduce some concepts that shall be helpful.

Definition 5.6.1 Let M be a group and let T be a set, we define the *copower* M^{*T} to be the free product of the groups $M_t = M \times \{t\}$ for all $t \in T$. Notice that all M_t are naturally isomorphic to M under the map $(m,t) \mapsto m$. So M^{*T} can be seen as the free product of copies of M indexed by T.

This construction satisfies the adjointness condition that for any group N there is a bijection

$$\mathsf{Sets}(T, \mathsf{Groups}(M,N)) \cong \mathsf{Groups}(M^{*T}, N)$$

natural in M, N, T. Notice also that the precrossed module induced from $\mathcal{M} : (\mu : M \to P)$ by $f : P \to Q$ is a quotient of M^{*UQ} where UQ is the underlying set of Q.

In the case where we have the inclusion of a subgroup $\iota : P \to Q$, we choose T to be a *right transversal* of P in Q, by which is meant a subset of Q including the identity 1 and such that any $q \in Q$ has a unique representation as $q = pt$ where $p \in P, t \in T$. For any crossed P-module $\mathcal{M} = (\mu : M \to P)$, the precrossed Q-module induced by ι will have the form $\hat{\mu} : M^{*T} \to Q$. Let us describe the Q-action.

Proposition 5.6.2 *Let $\iota : P \to Q$, $\mathcal{M}$, and T be as above. Then there is a Q-action on M^{*T} defined on generators using the coset decomposition by*

$$(m,t)^q = (m^p, u)$$

for any $q \in Q,\ m \in M,\ t \in T$, where p, u are the unique $p \in P,\ u \in T$, such that $tq = pu$.

Proof Let $m \in M$, $t, u, u' \in T$, $p, p' \in P$ and $q, q' \in Q$ be elements such that $tq = pu$ and $uq' = p'u'$. We have $t(qq') = puq' = pp'u'$. Therefore,

$$((m,t)^q)^{q'} = (m^p, u)^{q'} = (m^{pp'}, u') = (m,t)^{qq'}$$

and Q acts on M^{*T}. □

Remark 5.6.3 We can think of (m,t) as m^t, so the action is $(m^t)^q = (m^p)^u$ where $tq = pu$. Notice that if P is normal in Q then the Q-action induces an action of P on M_t given by $(m,t)^p = (m^{tpt^{-1}}, t)$. We shall exploit this later.

Now we define the boundary homomorphism by specifying the images of the generators

$$\hat{\mu} : M^{*T} \to Q,\ (m,t) \mapsto t^{-1}\mu(m)t.$$

Proposition 5.6.4 *Let $\iota : P \to Q$, $\mathcal{M}$ and T be as above. Then $(\hat{\mu} : M^{*T} \to Q)$ is a precrossed Q-module with the above action.*

Proof We verify axiom CM1). For any $m \in M$, $t \in T$, and $q \in Q$, we have

$$\begin{aligned}
\hat{\mu}((m,t)^q) &= \hat{\mu}(m^p, u) && \text{when } tq = pu \\
&= u^{-1}\mu(m^p)u && \text{by definition of } \hat{\mu} \\
&= u^{-1}(p)^{-1}\mu(m)pu && \text{because } \mu \text{ is a crossed module} \\
&= q^{-1}(t)^{-1}\mu(m)tq && \text{because } tq = pu \\
&= q^{-1}\hat{\mu}(m,t)q && \text{because } \mu \text{ is a crossed module.}
\end{aligned}$$

□

To complete the characterisation we now prove that in this case this precrossed module is the induced one.

Theorem 5.6.5 *If $\iota : P \to Q$ is a monomorphism, and $\mathcal{M} = (\mu : M \to P)$ is a crossed P-module then $\hat{\mu} : M^{*T} \to Q$ is the precrossed module induced by ι from μ.*

Proof We check the universal property. There is a homomorphism of groups $\phi : M \to M^{*T}$ defined by $\phi(m) = (m, 1)$ that makes commutative the square

$$\begin{array}{ccc}
M & \xrightarrow{\phi} & M^{*T} \\
{\scriptstyle\mu}\downarrow & & \downarrow{\scriptstyle\hat{\mu}} \\
P & \xrightarrow[\iota]{} & Q
\end{array}$$

and so that (ϕ, ι) is a morphism of precrossed modules.

For any morphism of precrossed modules

$$(h, \iota) : (\mu : M \to P) \longrightarrow (\nu : N \to Q)$$

the only possible definition of a homomorphism of groups $h' : M^{*T} \to N$ such that $h'\phi = h$ is the one given by $h'(m,t) = (hm)^t$ on generators. It is easy to see that it is a morphism of Q-precrossed modules. □

It is immediate that the induced crossed module is the one associated to the precrossed module $\hat{\mu}$, i.e. the quotient with respect to the Peiffer subgroup.

Corollary 5.6.6 *If $\iota : P \to Q$ is a monomorphism, and $(\mu : M \to P)$ is a crossed P-module, then the crossed module induced by ι from μ is the homomorphism induced by $\hat{\mu}$ on the quotient*

$$\hat{\mu} : \frac{M^{*T}}{[\![M^{*T}, M^{*T}]\!]} \to Q$$

together with the induced action of Q.

It is useful to have a smaller number of generators of the Peiffer subgroup $[\![M^{*T}, M^{*T}]\!]$.

Proposition 5.6.7 *Let $\iota : P \to Q$ be a monomorphism, $\mathcal{M} = (\mu : M \to P)$ be a crossed P-module and M^{*T} as before. Let S be a set of generators of M as a group, and let us define $S^P = \{s^p \mid s \in S,\ p \in P\}$. Then there is an isomorphism of the induced crossed module $\iota_*\mathcal{M} = (\iota_*M \to Q)$ to a quotient*

$$\iota_* M \cong \frac{(M^{*T})}{R}$$

where R is the normal closure in M^{*T} of the elements

$$[\![(r,t),(s,u)]\!] = (r,t)^{-1}(s,u)^{-1}(r,t)(s,u)^{\hat{\mu}(r,t)}$$

for all $r, s \in S^P$ and $t, u \in T$.

Proof By Corollary 5.6.6 we just have to prove that R is the Peiffer subgroup $[\![M^{*T}, M^{*T}]\!]$ of M^{*T}.

Now, M^{*T} is generated by the set

$$(S^P, T) = \{(s^p, t) \mid s \in S,\ p \in P,\ t \in T\}$$

and this set is Q-invariant since $(s^p, t)^q = (s^{pp'}, u)$ where $u \in T$, $p' \in P$ satisfying $tq = p'u$. Then by Proposition 3.3.5 $\{M^{*T}, M^{*T}\}$ is the normal closure of the set $\{(S^P, T), (S^P, T)\}$ of basic Peiffer commutators and this is just R. □

The next corollary gives a bound on the number of generators and relations of a presentation for the induced crossed module in terms of those of a presentation of M and the index of $\iota\mu(M)$ in Q.

Corollary 5.6.8 *Suppose $\iota : P \to Q$ is injective, M has a presentation as a group with g generators and r relations, the set of generators of M is P-invariant, and $n = [Q : \iota\mu(M)]$. Then $\iota_* M$ has a presentation with gn generators and $rn + g^2 n(n-1)$ relations.*

Proof This is just a process of counting. The transversal T has n elements, so M^{*T} has gn generators and rn relations. To get a presentation of $\iota_* M$ we just add as relations the basic Peiffer commutators of the generators and those are $g^2 n(n-1)$ relations more. □

We show how this construction works out in the case of the dihedral crossed module, which exhibits a number of typical features.

Example 5.6.9 Let us recall that the dihedral group D_{2n} has presentation

$$\langle x, y \mid x^n, y^2, xyxy \rangle.$$

We consider another copy $\tilde{D}_{2n}$ of D_{2n} with presentation $\langle u, v \mid u^n, v^2, uvuv \rangle$ and the homomorphism

$$\partial : \tilde{D}_{2n} \to D_{2n},\ u \mapsto x^2,\ v \mapsto y.$$

There is a crossed module with boundary ∂ and action of D_{2n} on $\tilde{D}_{2n}$ given on generators by the equations

$$u^y = vuv^{-1},\ v^y = v,\ u^x = u,\ v^x = vu.$$

As an exercise, check this result and also that $\partial : \tilde{D}_{2n} \to D_{2n}$ is an isomorphism if n is odd, and has kernel and cokernel isomorphic to C_2 if n is even.

Example 5.6.10 We let $Q = D_{2n}$ be the dihedral group with generators x, y, and let $M = P = C_2$ be the cyclic subgroup of order 2 generated by y. Let us denote by $\iota : C_2 \hookrightarrow D_{2n}$ the inclusion.

We have that $\mathrm{Id} : C_2 \to C_2$ is a crossed module and we are going to identify the induced crossed module

$$\hat{\mu} = \iota_*(\mathrm{Id}) : \iota_*(C_2) \longrightarrow D_{2n}.$$

A right transversal of C_2 in D_{2n} is given by the elements $T = \{x^i \mid i = 0, 1, 2, \ldots, n-1\}$.

If we apply the Proposition 5.6.7 we have that $\iota_* C_2$ has a presentation with generators $a_i = (y, x^i),\ i = 0, 1, 2, \ldots, n-1$ and relations $a_i^2 = 1,\ i = 0, 1, 2, \ldots, n-1$, together with the Peiffer relations associated to these generators.

Since the D_{2n}-action on C_2^{*T} is given by

$$a_i^x = a_{i+1} \text{ and } a_i^y = a_{n-i},$$

and

$$\hat{\mu}(a_i) = x^{-i} y x^i = y x^{2i},$$

we have $(a_i)^{\hat{\mu} a_j} = a_{2j-i}$, so that the Peiffer relations become

$$a_j^{-1} a_i a_j = a_{2j-i}.$$

In this group, we define $u = a_0 a_1,\ v = a_0$. As consequence, we have $u = a_i a_{i+1}$ and $u^i = a_0 a_i$ and it is now easy to check that $(C_2^{*T})^{\text{cr}} \cong \tilde{D}_{2n}$. Also the map $\hat{\mu}$ satisfies

$$\hat{\mu} u = \hat{\mu}(a_0 a_1) = y y x^2 = x^2,\ \hat{\mu} v = y.$$

Thus y acts on $\iota_* C_2$ by conjugation by v. However x acts by $u^x = u,\ v^x = vu$.

This crossed module is the dihedral crossed module of the previous Example 5.6.9.

It is worth pointing out that this induced crossed module is finite while the corresponding precrossed module M^{*T} is clearly infinite. We shall insist on these points in the next section.

Our last proposition determines induced crossed modules under some abelian conditions. This result has some useful applications. If M is a P-module, i.e. an abelian P-group, and T is a set we define the *copower* of M with T, written $M^{\oplus T}$, to be the sum of copies of M one for each element of T.

Proposition 5.6.11 *Let $\iota : P \to Q$ and $(\mu : M \to P)$ be as before. Moreover assume that M is abelian and $\iota\mu(M)$ is normal in Q. Then $\iota_* M$ is abelian and as a Q-module is just the induced Q-module in the usual sense.*

Proof We use the result and notation of Proposition 5.6.7. Note that if $u, t \in T$ and $r \in S$ then

$$u\hat{\mu}(r, t) = u t^{-1} \mu(r) t = \iota\mu(m) u t^{-1} t = \mu(m) u$$

for some $m \in M$, by the normality condition.

The Peiffer commutator given in Proposition 5.6.7 can therefore be rewritten as

$$(r, t)^{-1} (s, u)^{-1} (r, t)(s, u)^{\hat{\mu}(r,t)} = (r^{-1}, t)(s, u)^{-1} (r, t)(s^m, u).$$

Since M is abelian, $s^m = s$. Thus the basic Peiffer commutators reduce to ordinary commutators. Hence $\iota_* M$ is the copower $M^{\oplus T}$, and this, with the given action, is the usual presentation of the induced Q-module. □

Example 5.6.12 Let $M = P = Q$ be the infinite cyclic group, which we write $\mathbb{Z}$, and let $\iota : P \to Q$ be multiplication by 2. Then

$$\iota_* M \cong \mathbb{Z} \oplus \mathbb{Z},$$

and the action of a generator of Q on $\iota_* M$ is to switch the two copies of $\mathbb{Z}$. This result could also be deduced from well known results on free crossed modules. However, our results show that we get a similar conclusion simply by replacing each $\mathbb{Z}$ in the above by for example C_4, and this fact is new.

5.7 On the finiteness of some induced crossed modules

With the results of the previous section, we have an alternative way of constructing the induced crossed module associated to a homomorphism f. We can factor f in an epimorphism and a monomorphism and then apply the constructions. As pointed out before it is always a good thing to have as many equivalent ways as possible since then we can choose the most appropriate to some particular situation.

As we have seen in the previous section, if we have a (pre)crossed module $\mathcal{M} = (M \to P)$ in which M is generated by a finite P-set of a generators, and a group homomorphism $P \to Q$ with finite cokernel, the induced (pre)crossed module is also generated by a finite set. In this section we give an algebraic proof that a crossed module induced from a finite crossed module by a morphism with finite cokernel is also finite. The result is false for precrossed modules.

Theorem 5.7.1 *Let $\mu : M \to P$ be a crossed module and let $f : P \to Q$ be a morphism of groups. Suppose that M and the index of $f(P)$ in Q are finite. Then the induced crossed module f_*M is finite.*

Proof Factor the morphism $f : P \to Q$ as $\tau\sigma$ where τ is injective and σ is surjective. Then f_*M is isomorphic to $\tau_*\sigma_*M$. It is immediate from Proposition 5.5.1 that if M is finite then so also is σ_*M. So it is enough to assume that f is injective.

Let T be a right transversal of $f(P)$ in Q. Then there are maps

$$(\xi, \eta) : T \times Q \to f(P) \times T$$

defined by $(\xi, \eta)(t, q) = (p, u)$ where $p \in P$, $u \in T$ are elements such that $tq = f(p)u$. With this notation, the form of a basic Peiffer relation got in Corollary 5.6.6 is then of the form

$$(m, t)(n, u) = (n, u)(m^{\xi(t,u^{-1}f\mu(n)u)}, \eta(t, u^{-1}f\mu(n)u)) \tag{5.7.1}$$

where $m, n \in M$, $t, u \in T$.

We now assume that the finite set T has l elements and has been given the total order $t_1 < t_2 < \cdots < t_l$. An element of M^{*T} may be represented as a word

$$(m_1, u_1)(m_2, u_2) \dots (m_e, u_e). \tag{5.7.2}$$

Such a word is said to be *reduced* when $u_i \neq u_{i+1}$, $1 \leqslant i < e$, and to be *ordered* if $u_1 < u_2 < \cdots < u_e$ in the given order on T. This yields a partial ordering of $M * T$ where $(m_i, u_i) \leqslant (m_j, u_j)$ whenever $u_i \leqslant u_j$.

A *twist* uses the Peiffer relation (5.7.1) to replace a reduced word $w = w_1(m, t)(n, v)w_2$, with $v < t$, by $w' = w_1(n, v)(m^p, u)w_2$. If the resulting word is not reduced, multiplication in M_v and M_u may be used to reduce it. In order to show that any word may be ordered by a finite sequence of twists and reductions, we define an integer weight function on the set W_n of non-empty words of length at most n by

$$\begin{array}{rcl} \Omega_n : W_n & \longrightarrow & \mathbb{Z}^+ \\ (m_1, t_{j_1})(m_2, t_{j_2}) \dots (m_e, t_{j_e}) & \mapsto & l^e \sum_{i=1}^{e} l^{n-i} j_i. \end{array}$$

It is easy to see that $\Omega_n(w') < \Omega_n(w)$ when $w \to w'$ is a reduction. Similarly, for a twist

$$w = w_1(m_i, t_{j_i})(m_{i+1}, t_{j_{i+1}})w_2 \ \to \ w' = w_1(m_{i+1}, t_{j_{i+1}})(n, t_k)w_2$$

the weight reduction is

$$\Omega_n(w) - \Omega_n(w') = l^{n+e-i-1}(\ l(j_i - j_{i+1}) + j_{i+1} - j_k\ ) \geqslant l^{n+e-i-1},$$

so the process terminates in a finite number of moves.

We now specify an algorithm for converting a reduced word to an ordered word. Various algorithms are possible, some presumably more efficient than others, but we are not interested in efficiency here. We call a reduced word *k-ordered* if the subword consisting of the first k elements is ordered and the remaining elements are greater than these. Every reduced word is at least 0-ordered. Given a k-ordered, reduced word, find the rightmost minimal element to the right of the k-th position. Move this element one place to the left with a twist, and reduce if necessary. The resulting word may only be j-ordered, with $j < k$, but its weight will be less than that of the original word. Repeat until an ordered word is obtained.

Let $Z = M_{t_1} \times M_{t_2} \times \ldots \times M_{t_l}$ be the product of the sets $M_{t_i} = M \times \{t_i\}$. Then the algorithm yields a function $\phi : Y \to Z$ such that the quotient morphism $Y \to f_*M$ factors through ϕ. Since Z is finite, it follows that f_*M is finite. □

Remark 5.7.2 In this last proof, it is in general not possible to give a group structure on the set Z such that the quotient morphism $Y \to f_*M$ factors through a morphism to Z. For example, in the dihedral crossed module of Example 5.6.9, with $n = 3$, the set Z will have 8 elements, and so has no group structure admitting a morphism onto D_6.

So the proof of the main theorem of this section does not extend to a proof that the induced crossed module construction is closed also in the category of p-groups. Nevertheless, the result is true and there is a topological proof [62].

5.8 Inducing crossed modules by a normal inclusion

We continue the study of the crossed modules induced by the inclusion of a subgroup that we have begun in general in Section 5.6. In this section we consider the particular case when P is a normal subgroup of Q. We shall show in Theorem 5.8.4 that the coproduct of crossed P-modules described in Section 4.1 may be used to give a presentation of the crossed Q-modules induced by the inclusion $\iota : P \to Q$ analogous to known presentations of induced modules.

Let us start by digressing a bit about crossed modules constructed from a given one using an isomorphism.

Definition 5.8.1 Let $\mu : M \to P$ be a crossed P-module and let α be an automorphism of P. The crossed module $\mu_\alpha : M_\alpha \to P$ *associated to* α is defined as follows. The group M_α is just $M \times \{\alpha\}$, the morphism μ_α is given by $(m, \alpha) \mapsto \alpha\mu m$ and the action of P is given by $(m, \alpha)^p = (m^{\alpha^{-1}p}, \alpha)$.

Proposition 5.8.2 *The map $\mu_\alpha : M_\alpha \to P$ is a crossed module. Moreover this crossed module is isomorphic to μ since the map $k_\alpha : M \to M_\alpha$ given by $k_\alpha m = (m, \alpha)$ produces an isomorphism over α.*

Proof Let us check both properties of crossed module

$$\mu_\alpha(m^{\alpha^{-1}p}, \alpha) = \alpha(\mu m^{\alpha^{-1}p}) = \alpha(\alpha^{-1}(p)^{-1}\mu(m)\alpha^{-1}(p)) = p^{-1}\alpha\mu(m)p = p^{-1}\mu^\alpha(m)p$$

and

$$(m, \alpha)^{\mu_\alpha(m', \alpha)} = (m, \alpha)^{\alpha\mu(m')} = (m^{\alpha^{-1}\alpha\mu(m')}, \alpha) = (m^{\mu(m')}, \alpha) = (m', \alpha)^{-1}(m, \alpha)(m', \alpha).$$

It is immediate that the map $k_\alpha : M \to M_\alpha$ is an isomorphism. Also, the diagram

$$\begin{array}{ccc} M & \xrightarrow{k_\alpha} & M_\alpha \\ {\scriptstyle\mu}\downarrow & & \downarrow{\scriptstyle\mu_\alpha} \\ P & \xrightarrow[\alpha]{} & P. \end{array}$$

commutes and the map k_α preserves the P-action over α. □

Remark 5.8.3 Notice that if $\alpha = \mathrm{Id}$, there is a natural identification $M_{\mathrm{Id}} = M$.

We continue to assume that P is a normal subgroup of Q. In this case, for any $t \in Q$, there is an inner automorphism $\alpha_t : P \to P$ defined by $\alpha_t(p) = t^{-1}pt$. Let us write $(\mu_t : M_t \to P)$ instead of $(\mu_{\alpha_t} : M_{\alpha_t} \to P)$.

Let recall that this crossed P-module is the same $(\mu_t : M_t \to P)$ that we have used to construct $\iota_* M$ in Section 5.6, namely $M_t = M \times \{t\}$, the P-action was given by $(m,t)^p = (m^{tpt^{-1}}, t)$ and the homomorphism μ_t was defined by $\mu_t(m,t) = t^{-1}\mu m t$. We have just seen that it is a crossed P-module isomorphic to $\mathcal{M}$.

Now let T be a right transversal of P in Q. We can form the precrossed Q-module $\mathcal{M}' = (\partial' : M^{*T} \to Q)$ as in Proposition 5.6.2. Recall that the Q-action is defined on generators as follows. For any $q \in Q,\ m \in M,\ t \in T$ we define

$$(m,t)^q = (m^p, u),$$

where $p \in P$ and $u \in T$ are the only ones satisfying $tq = pu$. Also the homomorphism ∂' is defined by $\partial'(m,t) = t^{-1}pt$

We had seen in Theorem 5.6.5 that the induced crossed Q-module $\iota_*\mathcal{M}$ is the quotient of M^{*T} by the Peiffer subgroup associated to the Q-action. On the other hand, we have seen in Corollary 4.1.2 that the coproduct as crossed P-modules

$$\partial : M^{\circ T} \to P$$

is the quotient of M^{*T} with respect to the Peiffer subgroup associated to the P-action. We are going to check that they are the same.

Theorem 5.8.4 *In the situation we have just described, the homomorphism*

$$M^{\circ T} \xrightarrow{\partial} P \overset{\iota}{\hookrightarrow} Q$$

with the morphism of crossed modules

$$(i_1, \iota) \ : \ \mathcal{M} \to (\iota\partial : M^{\circ T} \to Q)$$

is the induced crossed Q-module.

Proof It is immediately checked in this case that the Peiffer subgroup is the same whether M^{*T} is considered as a precrossed P-module $M^{*T} \to P$ or as a precrossed Q-module $M^{*T} \to Q$. It can also be directly checked. We leave that as an exercise.

□

We remark that the result of Theorem 5.8.4 is analogous to well known descriptions of induced modules, except that here we have replaced the direct sum which is used in the module case by the coproduct of crossed modules. Corresponding descriptions in the non-normal case look to be considerably harder.

As a consequence we obtain easily a result on p-finiteness that can be strengthened by topological means ([62]). We prove it here for normal subgroups.

Proposition 5.8.5 *If M is a finite p-group and P is a normal subgroup of finite index in Q, then the induced crossed module $\iota_* M$ is a finite p-group.*

Proof This follows immediately from the discussion in Section 4.1. □

Now the induced module $(\iota\partial : M^{\circ T} \to Q)$ in Theorem 5.8.4 may be described using Corollary 4.4.16, if the hypotheses there are satisfied. So let P be a normal subgroup of Q and T a transversal as before, and let $(\mu : M \to P)$ be a crossed P-module.

We can divide the construction of the group $M^{\circ T}$ into two parts. We define $W = M^{\circ T'}$ the coproduct of all but $M_1 = M$. Then there is an isomorphism of crossed Q-modules

$$\iota_* \mathcal{M} \cong M \circ W.$$

To apply Corollary 4.4.16 we have to assume that for all $t \in T$ we have $\mu_t(M) \subseteq \mu(M)$, i.e. that for all $t \in T$ we have $t^{-1}\mu(M)t \subseteq \mu(M)$ (notice that this is immediately satisfied if μM is normal in Q), and that there is a section $\sigma : \mu M \to M$ of μ defined on μM. Most of the time we shall require also that σ is P-equivariant.

Then there is an isomorphism

$$\iota_* \mathcal{M} \cong M \times \bigoplus_{t \in T'} (M_t)_M$$

through which the morphisms giving the coproduct structure become

$$(i, \iota) : (\mu : M \to P) \longrightarrow (\xi = \iota\mu \,\mathrm{pr}_1 : M \times \bigoplus_{t \in T'} (M_t)_M \to Q)$$

where $i = i_1 : (m, 1) \mapsto (m, 0)$ and

$$(i_t, \iota) : (\mu : M_t \to P) \longrightarrow (\xi = \iota\mu \,\mathrm{pr}_1 : M \times \bigoplus_{t \in T'} (M_t)_M \to Q)$$

where for $t \neq 1$, $i_t(m, t) = (\sigma((\mu m)^t), [m, t])$.

Let us describe first how the Q-action is defined on this last crossed Q-module. Later we shall check the universal property.

The result we give is quite complicated, technical and non memorable. It is given principally because it shows the method, and also because we it shows that these methods give control over quite complex actions in a way which seems to be unobtainable by traditional methods, since they do not allow control of nonabelian structures.

Theorem 5.8.6 *The Q-action on the group $M \times \bigoplus_{t \in T'} (M_t)_M$ is given as follows,*

(i) For any $m \in M$, $q \in Q$

$$(m, 0)^q = \begin{cases} (m^q, 0) & \text{if } v = 1, \\ (\sigma((\mu m)^q), [m^r, v]) & \text{if } v \neq 1; \end{cases}$$

where $r \in P$ and $v, \in T$, satisfy $q = rv$ and $[m,v]$ denotes the class of (m,v) in $(M_v)_M$

(ii) If $m \in M, t \in T',\ q \in Q$ then

$$(1,[m,t])^q = \begin{cases} (1,[m^p,t]) & \text{if } v = 1, \\ (\sigma(\mu m^p)^{-1} m^p, -[\sigma((\mu m^p)^{v^{-1}}), v]) & \text{if } v \neq 1,\ u = 1, \\ (1, -[\sigma((\mu m^p)^{uv^{-1}}), v] + [m^p, u]) & \text{if } v \neq 1,\ u \neq 1, \end{cases}$$

where $p \in P,\ u \in T$ are the unique elements satisfying $tq = pu$.

Proof We use the description of the morphisms associated to the coproduct structure given above to calculate the action given by Theorem 5.8.4.

The formulae (i) and (ii) for the case $v = 1$ follow from the description of the action of P on M_t given at the beginning of this section.

The remaining cases will be deduced from the formula for the action of Q given in Theorem 5.8.4, namely if $m \in M,\ t \in T,\ q \in Q$ then

$$(i_t(m,t))^q = \begin{cases} i_1(m^p,1) = (m^p,0), & \text{if } tq = p \in P, \\ i_u(m^p,u) = (\sigma((\mu m^p)^u), [m^p,u]), & \text{if } tq = pu,\ p \in P,\ u \in T'. \end{cases}$$

We first prove (i) for $v \neq 1$. We have since $q = rv,\ v \in T'$,

$$\begin{aligned} (m,0)^q &= (i_1(m,1))^{rv} \\ &= i_v(m^r, v) \\ &= (\sigma((\mu m^r)^v), [m^r, v]). \end{aligned}$$

To prove (ii) with $v \neq 1$, first note that

$$\begin{aligned} (1,[m,t]) &= (\sigma((\mu m)^t), 0)^{-1}\, (\sigma((\mu m)^t), [m,t]) \\ &= (\sigma((\mu m)^t), 0)^{-1}\, i_t(m,t). \end{aligned}$$

But

$$\begin{aligned} (\sigma((\mu m)^t), 0)^q &= (\sigma((\mu\sigma((\mu m)^t))^q), [(\sigma((\mu m)^t))^r, v]) \text{by (i)} \\ &= (\sigma((\mu m)^{tq}), [\sigma((\mu m)^{tr}), v]) \qquad \text{since } \mu\sigma = 1, \end{aligned}$$

and, from the definition of the Q-action,

$$(i_t(m,t))^q = \begin{cases} (m^p, 0) & \text{if } u = 1, \\ (\sigma((\mu m)^{tq}), [m^p, u]) & \text{if } u \neq 1. \end{cases}$$

It follows that

$$(1,[m,t])^q = \begin{cases} (\sigma(\mu m^p)^{-1} m^p, -[\sigma((\mu m^p)^{v^{-1}}), v]) & \text{if } u = 1, \\ (1, -[\sigma((\mu m^p)^{uv^{-1}}), v] + [m^p, u]) & \text{if } u \neq 1. \end{cases}$$

□

Now we check that the universal property is satisfied.

Theorem 5.8.7 *For any crossed module $\mathcal{N} = (\nu : N \to Q)$ and any morphism of crossed modules $(\beta, \iota) : \mathcal{M} \to \mathcal{N}$, the induced morphism $\phi : M \times \bigoplus_{t \in T'} (M_t)_M \to N$ is given by*

$$\phi(m, 0) = \beta m, \quad \phi(m, [n, v]) = (\beta m)\, \beta(\sigma((\mu n)^v))^{-1}\, (\beta n)^v.$$

Proof The formula for ϕ is obtained as follows:

$$\begin{aligned} \phi(m, [n, v]) &= \phi(m, 0)\, \phi(\sigma((\mu n)^v), 0)^{-1}\, \phi(i_v(n, v)) \\ &= (\beta m)\, (\beta(\sigma((\mu n)^v))^{-1})\, (\beta n)^v \end{aligned}$$

where the definition of ϕ is taken from Theorem 5.8.4 □

We now include an example for Theorem 5.8.6 showing the action in the case $v \neq 1,\ u = 1$.

Example 5.8.8 Let n be an odd integer and let $Q = D_{8n}$ be the dihedral group of order $8n$ generated by elements $\{t, y\}$ with relators $\{t^{4n}, y^2, (ty)^2\}$. Let $P = D_{4n}$ be generated by $\{x, y\}$, and let $\iota : P \to Q$ be the monomorphism given by $x \mapsto t^2,\ y \mapsto y$. Then let $M = C_{2n}$ be generated by $\{m\}$. Define $\mathcal{M} = (\mu : M \to P)$ where $\mu m = x^2,\ m^x = m$ and $m^y = m^{-1}$. This crossed module is isomorphic to a sub-crossed module of $(D_{4n} \to \mathrm{Aut}(D_{4n}))$ and has kernel $\{1, m^n\}$.

The image μM is the cyclic group of order n generated by x^2, and there is an equivariant section $\sigma : \mu M \to M,\ x^2 \mapsto m^{n+1}$ since $(x^2)^{(n+1)} = x^2$ and $\gcd(n+1, 2n) = 2$. Then $Q = P \cup Pt$, $T = \{1, t\}$ is a transversal, M_t is generated by (m, t) and $\mu_t(m, t) = x^2$. The action of P on M_t is given by

$$(m, t)^x = (m, t), \quad (m, t)^y = (m^{-1}, t).$$

Since M acts trivially on M_t,

$$\iota_* M \;\cong\; M \times M_t \;\cong\; C_{2n} \times C_{2n}.$$

Using the section σ given above, Q acts on $\iota_* M$ by

$$\begin{aligned} (m, 0)^t &= (m^{n+1}, [m, t]), \\ (m, 0)^y &= (m^{-1}, 0), \\ (1, [m, t])^t &= (m^n, (n-1)[m, t]), \\ (1, [m, t])^y &= (1, -[m, t]). \end{aligned}$$

It is worth recalling that our objective was not only to get an easier expression of the induced crossed module, but also to have some information about the kernel of its boundary map. We can obtain some information on the later in the case where P is of index 2 in Q, even without the assumption that μM is normal in Q following [25].

Suppose then that $T = \{1, t\}$ is a right transversal of P in Q. Let the morphism $M \ltimes M_t \to P$ be given as usual by $(m, (n, t)) \mapsto (\mu m)(\mu_t(n, t)) = mt^{-1}nt$.

Write $\langle M, M_t \rangle$ for the subgroup of $M \times_P M_t$ generated by the elements

$$\langle m, (n, t) \rangle = (m^{-1} m^{t^{-1}(\mu n) t}, ((n, t)^{-1})^m (n, t)),$$

for all $m \in M,\ (n, t) \in M_t$.

Proposition 5.8.9 *Let $\mu : M \to P$ and $\iota : P \to Q$ be inclusions of normal subgroups. Suppose that P is of index 2 in Q, and $t \in Q \setminus P$. Then the kernel of the induced crossed module $(\partial : \iota_* M \to Q)$ is isomorphic to*

$$(M \cap t^{-1}Mt) \, / \, [M, t^{-1}Mt].$$

In particular, if M is also normal in Q, then this kernel is isomorphic to $M/[M,M]$, i.e. to M made abelian.

Proof By previous results $\iota_* M$ is isomorphic to the coproduct crossed P-module $M \circ M_t$ with a further action of Q. The result follows from Corollary 4.3.7 □

We now give some topological applications of the last result.

Example 5.8.10 Let $\iota : P = D_{4n} \to Q = D_{8n}$ be as in Example 5.8.8, and let $M = D_{2n}$ be the subgroup of P generated by $\{x^2, y\}$, so that $\iota M \lhd \iota P \lhd Q$ and $t^{-1}Mt$ is isomorphic to a second D_{2n} generated by $\{x^2, yx\}$. Then

$$M \cap t^{-1}Mt = [M, t^{-1}Mt]$$

(since $[y, yx] = x^2$), and both are isomorphic to C_n generated by $\{x^2\}$.

It follows from Proposition 5.8.9 that if X is the homotopy pushout of the maps

$$\begin{array}{ccc} BD_{4n} & \longrightarrow & BC_2 \\ \downarrow & & \downarrow \\ BD_{8n} & \longrightarrow & X \end{array}$$

where the horizontal map is induced by $D_{4n} \to D_{4n}/D_{2n} \cong C_2$, then $\pi_2(X) = 0$.

Example 5.8.11 Let M, N be normal subgroups of the group G, and let Q be the wreath product

$$Q = G \wr C_2 = (G \times G) \rtimes C_2.$$

Take $P = G \times G$, and consider the crossed module $(\partial : Z \to Q)$ induced from $M \times N \to P$ by the inclusion $P \to Q$. If t is the generator of C_2 which interchanges the two factors of $G \times G$, then $Q = P \cup Pt$ and $t^{-1}(M \times N)t = N \times M$. So

$$(M \times N) \cap t^{-1}(M \times N)t = (M \cap N) \times (N \cap M)$$

and

$$[M \times N, N \times M] = [M, N] \times [N, M].$$

It follows that if X is the homotopy pushout of

$$\begin{array}{ccc} BG \times BG & \longrightarrow & B(G/M) \times B(G/N) \\ \downarrow & & \downarrow \\ B(G \wr C_2) & \longrightarrow & X \end{array}$$

then

$$\pi_2(X) \cong ((M \cap N)/[M, N])^2.$$

If $([m], [n])$ denotes the class of $(m, n) \in (M \cap N)^2$ in $\pi_2(X)$, the action of Q is determined by

$$([m], [n])^{(g,h)} = ([m^g], [n^h]), \ (g, h) \in P, \quad ([m], [n])^t = ([n], [m]).$$

We end this section by giving a very concrete description of the induced crossed module in the case that both M and P are normal subgroups of Q and $M \subseteq P$. It is proved by a direct verification of the universal property for an induced crossed module.

There are two construction used in the description. The first one is the abelianisation M^{ab} of a group M. If $n \in M$, then the class of n in M^{ab} is written $[n]$.

The second construction is the augmentation ideal IQ of a group Q, which we further develop later on. For now let us say that the augmentation ideal $I(Q/P)$ of a quotient group Q/P has basis $\{\bar{t} - 1 \mid t \in T'\}$ where T is a transversal of P in Q, $T' = T \setminus \{1\}$ and $\overline{q}$ denotes the image of q in Q/P.

Theorem 5.8.12 *Let $M \subseteq P$ be normal subgroups of Q, so that Q acts on P and M by conjugation. Let $\mu : M \to P$, $\iota : P \to Q$ be the inclusions and let $\mathcal{M} = (\mu : M \to P)$. Then the induced crossed Q-module $\iota_*\mathcal{M}$ is isomorphic as a crossed Q-module to*

$$(\zeta : M \times (M^{\mathrm{ab}} \otimes I(Q/P)) \to Q)$$

where for $m, n \in M,\ x \in I(Q/P)$:

(i) $\zeta(m, [n] \otimes x) = m$;

(ii) the action of Q is given by

$$(m, [n] \otimes x)^q = (m^q, [m^q] \otimes (\overline{q} - 1) + [n^q] \otimes x\overline{q}).$$

The universal map $i : M \to M \times (M^{\mathrm{ab}} \otimes I(Q/P))$ is given by $m \mapsto (m, 0)$.

Proof This could be proved directly (see [63]) but instead, in view of what has already been set up, we will deduce it from Theorem 5.8.6. Specialising this theorem to the current situation, in which $\sigma\mu = 1$ and $i_t(m, t) = (m^t, [m, t])$, yields an isomorphism of crossed Q-modules

$$\iota_*\mathcal{M} \to \mathcal{X} = (\xi = \iota\mu\,\mathrm{pr}_1 : M \times \bigoplus_{t \in T'} (M^{\mathrm{ab}}) \to Q).$$

In $\mathcal{X}$ the action of Q is given as follows, where $m \in M,\ r \in P,\ q = rv$ and $v \in T$:

(i)

$$(m, 0)^q = \begin{cases} (m^q, 0) & \text{if } v = 1, \\ (m^q, [m^r, v]) & \text{if } v \neq 1. \end{cases}$$

(ii) if $tq = pu,\ t \in T',\ p \in P$ and $u \in T$, then

$$(1, [m, t])^q = \begin{cases} (1, [m^p, t]) & \text{if} \quad v = 1, \\ (1, -[m^{pv^{-1}}, v]) & \text{if} \quad v \neq 1,\ u = 1, \\ (1, -[m^{puv^{-1}}, v] + [m^p, u]) & \text{if} \quad v \neq 1,\ u \neq 1. \end{cases}$$

Now we construct an isomorphism

$$\omega : M \times \bigoplus_{t \in T'} (M^{\mathrm{ab}}) \to M \times (M^{\mathrm{ab}} \otimes I(Q/P))$$

where for $m, n \in M,\ t \in T'$,

$$\omega(m, 0) = (m, 0), \quad \omega(m, [n, t]) = (m, [n^t] \otimes (\bar{t} - 1)).$$

Clearly ω is an isomorphism of groups, since it is an isomorphism on the part determined by a fixed $t \in T'$, and $I(Q/P)$ has a basis $\{\bar{t}-1 : t \in T'\}$ when considered as an abelian group. Now we prove that ω preserves the action of Q. Let $m, n \in M,\ t \in T',\ q \in Q$. Let $q = rv,\ tq = pu,\ p, r \in P,\ u, v \in T$. When $v = 1$ we have $tqt^{-1} \in P$ and so $u = t$. Then

$$\begin{aligned}
\omega((m,0)^q) &= \begin{cases} \omega(m^q, 0) & \text{if} \quad v = 1, \\ \omega(m^q, [m^r, v]) & \text{if} \quad v \neq 1. \end{cases} \\
&= \begin{cases} (m^q, 0) & \text{if} \quad v = 1, \\ (m^q, [m^q] \otimes (\overline{v} - 1) & \text{if} \quad v \neq 1, \end{cases} \\
&= (\omega(m,0))^q.
\end{aligned}$$

Further,

$$\begin{aligned}
\omega((1,[m,t])^q) &= \begin{cases} \omega(1, [m^p, t]) & \text{if} \quad v = 1, \\ \omega(1, -[m^{pv^{-1}}, v]) & \text{if} \quad v \neq 1,\ u = 1, \\ \omega(1, -[m^{puv^{-1}}, v] + [m^p, u]) & \text{if} \quad v \neq 1,\ u \neq 1, \end{cases} \\
&= \begin{cases} (1, [m^{pt}] \otimes (\bar{t} - 1)) & \text{if} \quad v = 1, \\ (1, -[m^p] \otimes (\overline{v} - 1)) & \text{if} \quad v \neq 1,\ u = 1, \\ (1, -[m^{pu}] \otimes (\overline{v} - 1) + [m^{pu}] \otimes (\overline{u} - 1)) & \text{if} \quad v \neq 1,\ u \neq 1, \end{cases} \\
&= (1, -[m^{pu}] \otimes (\overline{v} - 1) + [m^{pu}] \otimes (\overline{u} - 1) \qquad \text{in every case,} \\
&= (1, [m^{tq}] \otimes (\bar{t} - 1)\overline{q}), \\
&= (\omega(1, [m,t]))^q
\end{aligned}$$

since, in $I(Q/P)$,

$$(\bar{t} - 1)\overline{q} = \overline{pu} - \overline{rv} = \overline{u} - \overline{v} = (\overline{u} - 1) - (\overline{v} - 1).$$

Finally, we have to compute the universal extension ϕ of β. For this, it is sufficient to determine

$$\begin{aligned}
\phi(1, [n] \otimes (\overline{q} - 1)) &= \phi\omega(1, [n^{v^{-1}}, v]) \\
&= \phi\omega((n^{-1}, 0)\, i_v(n^{v^{-1}}, v)) \\
&= \beta(n^{-1})\, \beta(n^{v^{-1}})^v \\
&= \beta(n^{-1})\, \beta(n^{q^{-1}})^q
\end{aligned}$$

since β is a P-morphism and $\overline{q} = \overline{rv} = \overline{v}$. □

With this description, we can get new results on the fundamental crossed module of a space which is the pushout of classifying spaces. The following corollary is immediate.

Corollary 5.8.13 *Under the assumptions of the theorem, let us consider the space* $X = BQ \cup_{BP} B(P/M)$*. Its fundamental crossed module* $\Pi_2(X, BQ)$ *is isomorphic to the above crossed* Q*-module*

$$(\zeta : M \times (M^{\mathrm{ab}} \otimes I(Q/P)) \to Q).$$

In particular, the second homotopy group $\pi_2(X)$ *is isomorphic to* $M^{\mathrm{ab}} \otimes I(Q/P)$ *as* Q/M*-module.*

Proof The proof is immediate. □

Note one of our major arguments: in order to compute an abelian second homotopy group, we may have to use nonabelian algebraic methods which better reflect the structure of the problem than the usual abelian methods.

Corollary 5.8.14 *In particular, if the index* $[Q:P]$ *is finite, and* $\mathcal{P}$ *is the crossed module* $(1 : P \to P)$, *then* $\iota_*\mathcal{P}$ *is isomorphic to the crossed module* $(\mathrm{pr}_1 : P \times (P^{\mathrm{ab}})^{[Q:P]-1} \to Q)$ *with action as above.*

Remark 5.8.15 In this case, $X = BQ \cup_{BP} B(P/P)$ may be interpreted either as the space obtained from BQ by collapsing BP to a point, or, better, as $X = BQ \cup_{BP} CB(P)$ the space got by attaching a cone. This is a consequence of the gluing theorem for homotopy equivalences proved in [30].

This crossed module is not equivalent to the trivial one. At first sight, it seems that the projection

$$pr_2 : P \times (P^{\mathrm{ab}} \otimes I(Q/P)) \to (P^{\mathrm{ab}} \otimes I(Q/P))$$

determines a morphism of crossed modules to the trivial one $0 : (P^{\mathrm{ab}} \otimes I(Q/P)) \to I(Q/P))$, but this is not so because the map pr_2 is not a Q-morphism.

We are going to show later that this crossed module is not equivalent in a certain sense to the projection crossed module.

We have now completed the applications of the 2-dimensional van Kampen Theorem which we will give in this book. In the next chapter we give the proof of the theorem, using the algebraic concepts of double groupoids. In the next section, we explain how the computer algebra system GAP has been used to give further computations of induced crossed modules, and of course these have topological applications according to the results of this chapter.

5.9 Computational issues for induced crossed modules

The following discusses significant aspects of the computation of induced crossed modules. Let us consider the description of the induced module from a computational point of view. It involves the copower, i.e. a free product of groups. This usually gives infinite groups, but let us consider how to get a finite presentation in the case $M \subseteq P \subseteq Q$.

If $M = \langle X \mid R \rangle$ is a finite presentation of M, there is a finite presentation of M^{*T} with $|X||T|$ generators and $|R||T|$ relations.

Let X^P be the closure of X under the action of P. Then $\iota_*(M) = (M^{*T})/N$ where N is the normal closure in M^{*T} of the elements

(5.9.1) $$\langle (m,t),(n,u) \rangle = (m,t)^{-1}(n,u)^{-1}(m,t)(n,u)^{\delta(m,t)} \quad (m,n \in \Sigma^P,\ t,u \in T).$$

The homomorphism ι_* is induced by the projection $\mathrm{pr}_{()} m = (m, ()\ )$ onto the first factor, and the boundary δ of $\iota_*\mathcal{M}$ is induced from δ' as shown in the following diagram:

$$\begin{array}{ccc} M & \xrightarrow{\iota_*} & (M^{*T})/N \\ {\scriptstyle\mu}\downarrow & & \downarrow{\scriptstyle\delta} \\ P & \xrightarrow[\iota]{} & Q \end{array}$$

When Σ is a set and $\sigma : \Sigma \to Q$ any map, take $M = P = F(\Sigma)$ to be the free group on Σ and let $\mathcal{F}_\Sigma = (\mathrm{id}_{F(\Sigma)} : F(\Sigma) \to F(\Sigma)\ )$. Then σ extends uniquely to a homomorphism $\sigma' : F(\Sigma) \to Q$ and $\sigma'_* \mathcal{F}_\Sigma$ is the free crossed module $\mathcal{F}_\sigma$ described in section 3.4. However, computation in free crossed modules is in general difficult since the groups are usually infinite.

So, in order to compute the induced crossed module $\iota_*\mathcal{M}$ for $\mathcal{M} = (\mu : M \to P)$ a conjugation crossed module and $\iota : P \to Q$ an inclusion, we construct finitely presented groups FM, FP, FQ isomorphic to the permutation groups M, P, Q and monomorphisms $FM \to FP \to FQ$ mimicking the inclusions $M \to P \to Q$.

As well as returning an induced crossed module, the construction should return a morphism of crossed modules $(\iota_*, \iota) : \mathcal{M} \to \iota_*\mathcal{M}$.

A finitely presented form FC for the copower M^{*T} is constructed with $|X||T|$ generators. The relators of FC comprise $|T|$ copies of the relators of FM, suitably renumbered.

The inclusion δ' maps the generators of FM to the first $|X|$ generators of FC. A finitely presented form FI for ι_*M is then obtained by adding to the relators of FC further relators corresponding to the list of elements in equation (5.9.1).

Then we can apply some Tietze transformations to the resulting presentation. During the resulting simplification, some of the first $|X|$ generators may be eliminated, so the projection $\mathrm{pr}_{()}$ may be lost. In order to preserve this projection, and so obtain the morphism ι_*, it is necessary to record for each eliminated generator g a relator gw^{-1} where w is the word in the remaining generators by which g was eliminated.

The Tietze transformation code in GAP was modified so that the resulting presentation **presI** contained an additional field **presI.remember**, namely a list of (at least) $|X||T|$ relators expressing the original generators in terms of the final ones. (In the recent release 3.4.4 of GAP this facility has been made generally available using the **TzInitGeneratorImages** function).

Let us see how this process works in some examples, and notesome of the limitations of the process.

Recall that a *polycyclic group* is a group G with power-conjugate presentation having generators $\{g_1, \ldots, g_n\}$ and relations

$$\{g_i^{o_i} = w_{ii}(g_{i+1}, \ldots, g_n), \ \ g_i^{g_j} = w'_{ij}(g_{j+1}, \ldots, g_n) \quad \forall\, 1 \leqslant j < i \leqslant n\}. \tag{5.9.2}$$

(These are implemented in GAP as **AgGroups** (see [99], Chapters 24, 25)). Since subgroups $M \leqslant P \leqslant G$ have induced power-conjugate presentations, if T is a transversal for the right cosets of P in G, then the relators of M^{*T} are all of the form in (5.9.2).

Furthermore, all the Peiffer relations in equation (5.9.1) are of the form $g_i^{g_j} = g_k^p$, so one might hope that a power conjugate presentation would result. Consideration of the cyclic-by-cyclic case in the following example shows that this does not happen in general.

Example 5.9.1 Let C_n be cyclic of order n and let $\alpha : x \mapsto x^a$ be an automorphism of C_n of order p. Take $G = \langle g, h \mid g^p, h^n, h^g h^{-a} \rangle \cong C_p \ltimes C_n$. It follows from these relators that $h^i g = g h^{ai}$, $0 < i < n$ and that $h^{-1}(gh^{i(1-a)})h = gh^{(i+1)(1-a)}$. So if we put $g_i = gh^{i(1-a)}$, $0 \leqslant i < n$ then $g_i^{g_j} = g_{[j+a(i-j)]}$. When $M = P = C_n \lhd G$ Theorem 5.8.12 apply, and $\iota_* P \cong C_n^m$. Now take $M = P = C_p$, with power-conjugate form $\langle g \mid g^p \rangle$, and $\iota : C_p \to G$. We may choose as transversal $T = \{\lambda, h, h^2, \ldots, h^{n-1}\}$, where λ is the empty word. Then M^{*T} has generators $\{(g, h^i) \mid 0 \leqslant i < n\}$, all of order p, and relators $\{(g, h^i)^p \mid 0 \leqslant i < n\}$. The additional Peiffer relators in equation (5.9.1) have the form

$$(g, h^i)(g, h^j) = (g, h^j)(g^k, h^l) \ \text{ where } \ h^i h^{-j} g h^j = g^k h^l$$

so $k = 1$ and $l = [j + a(i - j)]$. Hence $\theta : \iota_* M \to Q$, $(g, h^i) \mapsto g_i$ is an isomorphism, and $\iota_*\mathcal{M}$ is isomorphic to the identity crossed module on Q. Furthermore, if we take M to be a cyclic subgroup C_m of C_p then $\iota_*\mathcal{M}$ is the conjugation crossed module $(\partial : C_m \ltimes C_n \to C_p \ltimes C_n)$.

Also, we know that many of the induced groups $\iota_* M$ are direct products. However the generating sets in the presentations that arise following the Tietze transformation do not in general split into generating sets for direct summands. This is clearly illustrated by the following simple example.

Example 5.9.2 Let $Q = S_4$, the symmetric group of degree 4, and $M = P = A_4$, the alternating subgroup of Q of index 2. Since the abelianisation of A_4 is cyclic of order 3, Theorem 5.8.12 shows that $\iota_* M \cong A_4 \times C_3$. However a typical presentation for $A_4 \times C_3$ obtained from the program is

$$\langle x, y, z \mid x^3, y^3, z^3, (xy)^2, zy^{-1}z^{-1}x^{-1}, yzyx^{-1}z^{-1}, y^{-1}x^2y^2x^{-1}\rangle,$$

and one generator for the C_3 summand is yzx^2. Converting to an isomorphic permutation group H gives a degree 12 representation with generating set

$$\{(2,9,4)(3,5,6)(8,12,10),\ (1,4,2)(3,5,7)(10,11,12),\ (1,8,3)(2,10,5)(7,9,12)\}.$$

Converting H to an AgGroup produces a 4-generator group with subnormal series $A_4 \times C_3 > A_4 > C_2^2 > C_2 > I$, and $g_1g_2g_4$ is a generator for the normal C_3. After conversion of this AgGroup to a SpecialAgGroup, the corresponding generator is g_1g_2. In all these representations, the cyclic summand remains hidden, and an explicit search among the normal subgroups must be undertaken to find it.

We finish the results obtained in our computation by listing all the induced crossed modules coming from subgroups of groups of order at most 23 (excluding 16) which are not covered by the special cases mentioned earlier. This enables us to exclude abelian and dihedral groups, cases $P \lhd Q$ and $Q \cong C_m \ltimes C_n$.

In the first table, we assume given an inclusion $\iota : P \to Q$ of a subgroup P of a group Q, and a normal subgroup M of P. We list the crossed module $\iota_* M$ induced from $(\mu : M \to P)$ by the inclusion ι. The kernel of $\partial : \iota_* M \to Q$ is written $\nu_2(\iota)$. This kernel is related to the second homotopy group in the topological application (in some cases like Theorems 5.4.4 and 5.4.7 it is exactly the second homotopy group).

In this table the labels I, C_n, D_{2n}, A_n, S_n denote the identity, cyclic, dihedral, alternating and symmetric groups of order $1, n, 2n, n!/2$ and $n!$ respectively. The group H_n is the holomorph of C_n and H_n^+ is its positive subgroup in degree n. $SL(2,3)$ and $GL(2,3)$ are the special and general linear groups of order 24, 48 respectively. Labels of the form **m.n** refer to the nth group of order m according to the GAP numbering.

Table 1

$\lvert Q\rvert$	M	P	Q	$\iota_* M$	$\nu_2(\iota)$
12	C_2	C_2	A_4	H_8^+	C_4
	C_3	C_3	A_4	$SL(2,3)$	C_2
18	C_2	C_2	$C_2 \ltimes C_3^2$	54.10	C_3
	S_3	S_3	$C_2 \ltimes C_3^2$	54.10	C_3
20	C_2	C_2	H_5	D_{10}	C_2
	C_2	C_2^2	D_{20}	D_{10}	I
	C_2^2	C_2^2	D_{20}	D_{20}	I
21	C_3	C_3	H_7^+	H_7+	I

The second table contains the results of calculations with $Q = S_4$, where $C_2 = \langle(1,2)\rangle$, $C_2' = \langle(1,2)(3,4)\rangle$, and $C_2^2 = \langle(1,2),(3,4)\rangle$. The final column contains the automorphism group $\mathsf{Aut}(\iota_* M)$ (where known).

Table 2

M	P	$\iota_* M$	$\nu_2(\iota)$	$\mathsf{Aut}(\iota_* M)$
C_2	C_2	$GL(2,3)$	C_2	$S_4 C_2$
C_3	C_3	$C_3\ SL(2,3)$	C_6	144.?
C_3	S_3	$SL(2,3)$	C_2	S_4
S_3	S_3	$GL(2,3)$	C_2	$S_4 C_2$
C_2'	C_2'	128.?	$C_4 C_2^3$	
C_2'	C_2^2, C_4	H_8^+	C_4	$S_4 C_2$
C_2'	D_8	C_2^3	C_2	$SL(3,2)$
C_2^2	C_2^2	$S_4 C_2$	C_2	$S_4 C_2$
C_2^2	D_8	S_4	I	S_4
C_4	C_4	96.219	C_4	96.227
C_4	D_8	S_4	I	S_4
D_8	D_8	$S_4 C_2$	C_2	$S_4 C_2$

Chapter 6

Double groupoids and the 2-dimensional van Kampen theorem.

In Chapter 2 we saw that an important topological example of crossed module was provided by the fundamental crossed module of a based pair of spaces

$$\Pi_2(X, A)(x) = (\partial : \pi_2(X, A, x) \to \pi_1(A, x)).$$

As in the case of the fundamental group, to prove the van Kampen theorem for crossed modules, it is interesting, even necessary, to include in the same structure all the fundamental crossed modules when varying the base point $x \in A$. In the 1-dimensional case, we generalised the fundamental group to the fundamental groupoid. To prove a van Kampen Theorem in the 2-dimensional case the idea was to use double groupoids but it took some time to find the required 2-dimensional analogue of the fundamental group. After a good deal of trying the structure we need for the van Kampen Theorem in dimension 2 happens to be the double groupoids with connection or, equivalently, a crossed module over a groupoid.

Now the question can be fairly put: Why introduce a new version? The answer is the usual kind of answer, that sometimes the new version is useful for proving theorems. In particular, we are unable to prove directly in terms of crossed modules the version of the 2-dimensional van Kampen theorem which gives a result in terms of the classical crossed modules. One reason for conceiving of the homotopy double groupoid was to find an algebraic gadget more appropriate than groups for giving an

algebraic inverse to subdivision.

This is the slogan underlying the work on higher dimensional van Kampen Theorems. Subdividing a square into little squares has a convenient expression in terms of double groupoids, and much more inconvenient expressions, if they exist at all, in terms of crossed modules. The 2-dimensional van Kampen Theorem was conceived first in terms of double groupoids, and it was only gradually that the link with crossed modules was realised. In the end, the aim of obtaining Whitehead's theorem on free crossed modules (Corollary 5.4.8) as a corollary was an important impetus to forming a definition of a homotopy double groupoid for a pointed *pair* of spaces, since that theorem involved a crossed module defined for such a pair of spaces.

Further, the structure of double groupoids that we use was expressly sought in order to make valid Lemma 6.8.4 in the proof of our 2-dimensional van Kampen Theorem in the last section. This lemma shows that a construction of an element of a double groupoid is independent of all the choices made. This makes use of the notion of commutative cube in a crucial way.

This theory gives also in a sense an *algebraic formulation* of different ways which have been classically used and found necessary in considering properties of second relative homotopy groups. We find that the 2-dimensional double groupoid viewpoint is useful both for understanding the theory and for proving theorems, while the crossed module viewpoint is useful both for specific calculations, and because of its closer relation to chain complexes. The importance of the algebraic formulation of the equivalence between crossed modules and double groupoids is the equivalence between colimits, and in particular pushouts, in the two categories.

Since this is a longish chapter, it seems a good idea to include a more detailed sketch of the way that all this material is presented here.

The first part describes the step up one dimension from groupoids to double groupoids. Since these are double categories where all structures are groupoids and have either a connection pair or a thin structure the first few sections are devoted to defining first double categories and then connections. In parallel another algebraic category is described, that of crossed modules over groupoids, which is equivalent to that of double groupoids. The equivalence is finally proved in Section 6.6

The first Section gives the definition and properties of double categories. Some notions to be used later are also presented here, e.g. the double category of commutative squares or 2-shells in a groupoid.

With this model in mind, we can think of the elements of a double category D as squares. Also, we can restrict our attention to the subspace γD of "squares" having all faces trivial but the top one.

If we restrict ourselves to double categories G that have all three structures groupoids, the space γG is algebraically a crossed module over a groupoid. These algebraic structures are studied in Section 6.2 and they are an easy step away from that of a crossed module over a group.

A direct topological example is the fundamental crossed module of a triple of topological spaces (X, A, C) formed by all the crossed modules $\partial : \pi_2(X, A, x) \to \pi_1(A, x)$ for varying $x \in C$. We denote this crossed module by $\Pi_2(X, A)$ and we shall prove that it is a crossed module in an indirect way by showing in Proposition 6.3.7 that $\Pi_2(X, A)$ is the crossed module associated to the fundamental double groupoid of a triple $\rho(X, A, C)$ defined in Section 6.3.

Both the fundamental crossed module of a triple and the double category of commutative 2-shells on a groupoid have some extra structure that can be defined in two equivalent ways: as a *thin structure* (as in Section 6.4) and as a *connection pair* (in Section 6.5). In this way we define the objects in the category of double groupoids.

Using 2-shells that 'commute up to some element', in Section 6.6 we associate to each crossed module $\mathcal{M}$ a double groupoid $\lambda\mathcal{M}$ in such a way that it is clear that $\gamma\lambda\mathcal{M}$ is naturally isomorphic to $\mathcal{M}$. It is a bit more challenging to prove that for any double groupoid G, $\lambda\gamma G$ is also naturally isomorphic to G. In order to do this we use the folding operation $\Phi : G_2 \to G_2$ which has the effect of folding all faces of an element of G_2 into the top face.

With all the algebra in place, we turn to the topological part. As seen in Chapter 1, the proof of the van Kampen Theorem uses the homotopy commutativity of squares. Thanks to the algebraic machinery developed earlier, we can talk about commutative 3-cubes and prove that any composition of commutative cubes is commutative. This commutativity of the boundary of a cube in $\rho(X, A, C)$ has a homotopy meaning stated in Section 6.7 which is analogous to the 1-dimensional case.

We finish this chapter by giving in Section 6.8 a proof of the 2-dimensional van Kampen theorem for the fundamental double groupoid and the main consequences.

The whole chapter can be seen as an introduction to the generalisation to all dimensions which is carried out in Part II. Chapter 15 generalises the algebraic part by giving an equivalence between crossed complexes and cubical ω-groupoids with connections, while Chapter 16 covers the topological part, including the statement, proof and applications of the GVKT.

6.1 Double categories

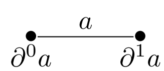
Let us start by pointing out that there are several candidates for the name "double groupoids". We are going to keep that name for the structures which are defined in Section 6.4 and are then used to prove the 2-dimensional van Kampen theorems. We start by investigating what a double category should be.

It is interesting to think of a category in a different way that lends itself better to the generalisation to higher dimensions. As seen in the Appendix, a category C is given by two sets: the set of objects that we denote C_0 and the set of morphisms that we call C_1; three maps among them: the source $\partial^0 : C_1 \to C_0$, target $\partial^1 : C_1 \to C_0$ and identity $1 = \varepsilon : C_0 \to C_1$, satisfying

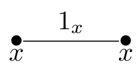
$$\partial^i \varepsilon = \mathrm{Id} \ ;$$

and a partial composition $C_1 \times_{C_0} C_1 \to C_1$ that is associative and has $1_x = \varepsilon(x)$ as right and left identity.

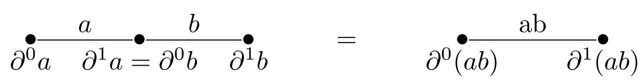
Thus we can think of the elements of C_0 as 0-dimensional, called points, and the elements of C_1 as 1-dimensional and oriented, called arrows. An element $a \in C_1$ is represented by



and for any $x \in C_0$ its identity $1_x = \varepsilon(x)$ is



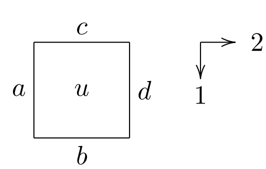
The composition ab of two elements $a, b \in C_1$ is described by juxtaposition:



This gives a 1-dimensional pictorial description of a category.

For a 2-dimensional generalisation, namely a double category D, apart from the sets of "points", D_0 and of "arrows", D_1, we need a set of "squares", D_2. We shall also have two categories associated to the "horizontal" and "vertical" structures on squares, with their faces and compositions. Also, we should have all the appropriate compatibility conditions dictated by the geometry. In some sense these categories are special since the objects of the horizontal and the vertical category structures on squares are the same; in other words, the horizontal and vertical edges of the squares come from the same category. This is the case we need in this book.

Thus we think of an element $u \in D_2$ as a square

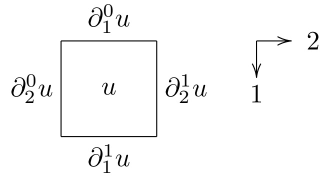


where the directions are labelled as indicated, and we call a, b, c, d the *edges, or faces* of u.

Let us make it formal.

Definition 6.1.1 A *double category* is given by three sets D_0, D_1 and D_2 and three structures of category. The first one on (D_1, D_0) has maps ∂^0, ∂^1 and ε and composition denoted as multiplication. The other two are defined on (D_2, D_1), a "vertical" one with maps $\partial^0_1, \partial^1_1$ and ε_1 and composition denoted by $u +_1 w$ and the "horizontal" one with maps $\partial^0_2, \partial^1_2$ and ε_2 and composition denoted by $u +_2 v$, satisfying some conditions.

Before describing the compatibility conditions it is worth getting used to the diagrammatic expression of the elements in a double category. Thus an element $u \in D_2$ is represented using a matrix like convention

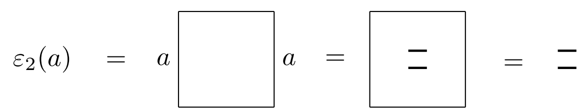


where the labels on the sides are given as indicated.

From this representation it seems indicated, and we assume, that the sources and targets have to satisfy

$$\partial^j \partial_1^i = \partial^i \partial_2^j \quad \text{for} \quad i, j = 0, 1, \tag{DC 1}$$

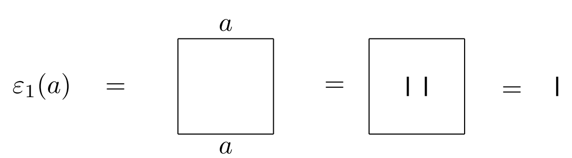
since they represent the same vertex. We shall find it convenient to represent the horizontal identity in several ways, i.e.



In the first representation the unlabeled sides are identities:

$$\partial_1^i \varepsilon_2 = \varepsilon \partial^i \quad \text{for} \quad i = 0, 1. \tag{DC 2.1}$$

In the other two, the sides corresponding to those drawn in the middle are identities. Similarly, the vertical identity is represented by



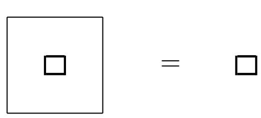
with the same conventions as before. It has also the expected faces in the horizontal direction:

$$\partial_2^i \varepsilon_1 = \varepsilon \partial^i \quad \text{for} \quad i = 0, 1. \tag{DC 2.2}$$

There are also some relations between the identities. The two double degenerate maps are the same and are denoted by 0:

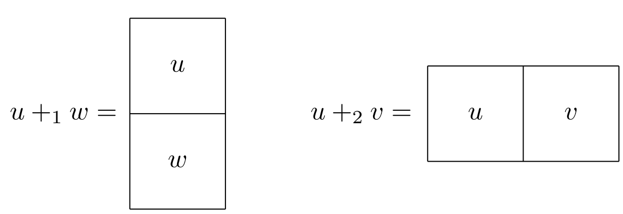
$$\varepsilon_2 \varepsilon = \varepsilon_1 \varepsilon = 0. \tag{DC 3}$$

So $0_x = 0(x)$ is both a horizontal and a vertical identity and is represented as

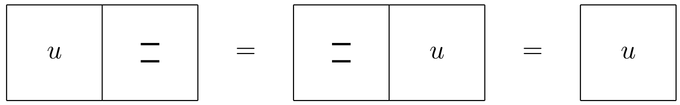


All elements $\varepsilon(x)$, $\varepsilon_1(a)$, $\varepsilon_2(a)$ are called *degeneracies*.

The vertical and horizontal compositions can be represented by "juxtaposition" in the corresponding direction, and are indicated by:



They satisfy all the usual rules of a category, and may be given a diagrammatic representation. For example, the fact that ε_2 is the horizontal identity may be represented as



The composition in one direction satisfies compatibility conditions with respect to the faces and degeneracies in the other direction, i.e. these functions are homomorphisms. This can be read from the representation. Thus the horizontal faces of a vertical composition are

$$\partial_2^i(u +_1 v) = (\partial_2^i u)(\partial_2^i v) \quad \text{for} \quad i = 0, 1. \tag{DC 4.1}$$

and the vertical faces of the horizontal composition are

$$\partial_1^i(u +_2 v) = (\partial_1^i u)(\partial_1^i v) \quad \text{for} \quad i = 0, 1. \tag{DC 4.2}$$

The same applies to the vertical and horizontal identities, i.e.

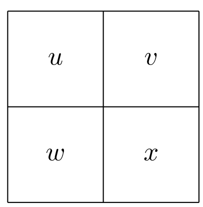
$$\varepsilon_2(ab) = \varepsilon_2(a) +_1 \varepsilon_2(b), \tag{DC 5.1}$$

$$\varepsilon_1(ab) = \varepsilon_1(a) +_2 \varepsilon_1(b). \tag{DC 5.2}$$

Our final compatibility condition is known as the "interchange law" and says that, when composing 4 elements in a square, it is irrelevant if we compose first in the horizontal direction and then in the vertical one, or the other way around, i.e.

$$(u +_2 v) +_1 (w +_2 x) = (u +_1 w) +_2 (v +_1 x) \tag{DC 6}$$

when both sides are defined. This can be represented as giving only one way of evaluating the double composition



To complete the description of the category of double categories, a *double functor* between two double categories D and D' is given by three maps $F_i : D_i \to D'_i$ for $i = 0, 1, 2$ which commute with all structure maps (faces, degeneracies, composition, etc.). In particular, the pair (F_1, F_0) gives a functor from (D_1, D_0) to (D'_1, D'_0).

With these objects and morphisms, we get the category DCat of double categories.

Remark 6.1.2 Thus a double category has a structure which is called a *2-truncated cubical complex with compositions*. Properties (DC 1-3) give the 2-truncated cube structure and (DC 4-6) the compatibility with compositions.

Remark 6.1.3 On matrix notation. There is also a matrix notation for the compositions which will be useful later on and is as follows:

$$u +_1 w = \begin{bmatrix} u \\ w \end{bmatrix} \qquad u +_2 v = [u, v].$$

With this notation we can represent all the rules in the definition of double categories. For instance, we have

$$\begin{bmatrix} u \\ |\;| \end{bmatrix} = \begin{bmatrix} \; u, & \overline{\underline{\quad}} \; \end{bmatrix} = u.$$

Choosing the matrix description, the 'interchange law' (DC 6) may be written

$$\begin{bmatrix} \begin{bmatrix} u \\ w \end{bmatrix} & \begin{bmatrix} v \\ x \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} u & w \end{bmatrix} \\ \begin{bmatrix} w & x \end{bmatrix} \end{bmatrix}$$

This common value is represented by

$$\begin{bmatrix} u & v \\ w & x \end{bmatrix}.$$

Here is a caution about using this interchange law. Let u, v be squares in a double category such that

$$w = \begin{bmatrix} u & v \end{bmatrix} = u +_2 v$$

is defined. Suppose further that

$$u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = u_1 +_1 v_1 \quad v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = u_2 +_1 v_2.$$

Then we can assert

$$w = \begin{bmatrix} u_1 & v_1 \\ u_2 & v_2 \end{bmatrix}$$

only when $u_1 +_2 v_1$, and $u_2 +_2 v_2$ are defined. Thus care is needed in 2-dimensional rewriting.

This matrix notation has a generalisation that we are going to use in proving several equalities.

Definition 6.1.4 Let D be a double category. A *composable array* (u_{ij}) in D, is given by elements $u_{ij} \in D_2$ $(1 \leqslant i \leqslant m, 1 \leqslant j \leqslant n)$ satisfying

$$\begin{cases} \partial_2^1 u_{i,j-1} = \partial_2^0 u_{i,j} & (1 \leqslant i \leqslant m, 2 \leqslant j \leqslant n), \\ \partial_1^1 u_{i-1,j} = \partial_1^0 u_{i,j} & (2 \leqslant i \leqslant m, 1 \leqslant j \leqslant n). \end{cases}$$

It follows from the interchange law that a composable array (u_{ij}) in D can be composed both ways, getting the same result which is denoted by $[u_{ij}]$.

If $u \in D_2$, and (u_{ij}) is a composable array in D satisfying $[u_{ij}] = u$, we say that the array (u_{ij}) is a *subdivision* of u. We also say that u is the *composite* of the array (u_{ij}).

Remark 6.1.5 Subdivisions and their use. The interchange law implies that if in the composable array (u_{ij}) we partition the rows and columns into blocks which produce another composable array and compute the composite v_{kl} of each block, then $[u_{ij}] = [v_{kl}]$. We call the (u_{ij}) a *refinement* of (v_{kl}) in this case.

This observation is used in several ways to prove equalities. The method consists usually in starting from the definition of one side of the equation, then change the array using this subdivision technique and compose the new array getting the other side of the equation.

Changes in a composable array that are clearly possible using this subdivision technique are

1. Select a block of an array and change it for another block having the same composition and the same boundary (see Proposition 6.6.4)

2. Substitute some adjacent columns by another set of adjacent columns having the same boundary and such that each row has the same horizontal composition in both cases. The same can done with rows (see Proposition 6.4.4 and Theorem 6.4.6)

Example 6.1.6 Let us give a couple of examples of double categories associated to a category C. The first one is the double category of "squares" or, better still, "2-shells" in a category C, denoted by $\Box' C$.

The points and arrows of $\Box' C$ and the category structure on $((\Box' C)_1, (\Box' C)_0)$ are the same as those of C. The set of squares $(\Box' C)_2$ is defined by

$$(\Box' C)_2 = \{(a, d, b, c) \in C_1^4 \ : \ \partial^0 b = \partial^1 a, \partial^0 d = \partial^1 c, \partial^1 b = \partial^1 d, \text{and}\, \partial^0 a = \partial^0 c\}.$$

Its elements may be represented by "brackets"

$$\begin{pmatrix} & c & \\ a & & d \\ & b & \end{pmatrix}$$

and the horizontal and vertical face and degeneracy maps are obvious from the representation. The compositions are defined by

$$\begin{pmatrix} & c & \\ a & & d \\ & b & \end{pmatrix} +_1 \begin{pmatrix} & b & \\ f & & h \\ & g & \end{pmatrix} = \begin{pmatrix} & c & \\ af & & dh \\ & g & \end{pmatrix}$$

and

$$\begin{pmatrix} & c & \\ a & & d \\ & b & \end{pmatrix} +_2 \begin{pmatrix} & u & \\ d & & w \\ & v & \end{pmatrix} = \begin{pmatrix} & cu & \\ a & & w \\ & bv & \end{pmatrix}$$

It is easy to see that $\Box' C$ is a double category and that $\Box'$ is a functorial construction. Moreover this functor is right adjoint to the truncation functor which sends each double category D to the category D_1. We leave the proof of adjointness as an exercise.

There are several sub-double-categories of $\Box' C$ that can be obtained taking the same 0 and 1-dimensional part and restricting the 2-dimensional part by putting some commutativity condition on the 2-shells.

Let us consider $\Box C$, the category of "commutative squares" or "commutative 2-shells". Its squares are

$$\Box C_2 = \{(a, d, b, c) \in C_1^4 \ : \ ab = cd\}.$$

The horizontal and vertical face and degeneracy maps and the compositions are the restriction of those in $\Box' C$.

There are quite a few categories that can be defined in the same fashion. All we need is to ask the compositions ab and cd to differ by the action of an element of some fixed subset of C. It is a good exercise to investigate which conditions has to satisfy this subset of C. We shall come back to this in the Example 6.1.8.

As we have stated before, our main objects of interest are double groupoids. These are double categories where all the categories involved are groupoids and which also have an extra structure. Let us start by studying double categories where all category structures are groupoids.

Definition 6.1.7 The category DCatG is the full subcategory of DCat that has as objects double categories in which all three structures are groupoids.

First, recall that a groupoid is a category G which has a map $(\quad)^{-1} : G_1 \to G_1$ such that

$$aa^{-1} = 1_{\partial^0 a} \text{ and } a^{-1}a = 1_{\partial^1 a}.$$

Thus in a double category G where all three category structures are groupoids, there are three "inverse" maps

$$(\quad)^{-1} : G_1 \to G_1, \quad -_1 : G_2 \to G_2 \quad \text{and} \quad -_2 : G_2 \to G_2,$$

where

$$(\varepsilon_i a) +_j (\varepsilon_i a^{-1}) = 0_{\partial^0 a}, \quad (\varepsilon_i a^{-1}) +_j (\varepsilon_i a) = 0_{\partial^1 a}, \text{ for } i \neq j.$$

From the compatibility conditions (DC 4.1, 4.2), we see that the boundary maps preserve inverses in the other direction since they are homomorphisms, i.e.

$$\partial^i_1(-_2 u) = (\partial^i_1(u))^{-1}, \quad \partial^i_2(-_1 u) = (\partial^i_2(u))^{-1}. \tag{DCG 4}$$

From the compatibility conditions (DC 5.1, 5.2), we get that the identity maps also preserve inverses, i.e.

$$\varepsilon_1(a^{-1}) = -_2(\varepsilon_1(a)), \quad \varepsilon_2(a^{-1}) = -_1(\varepsilon_2(a)). \tag{DCG 5}$$

We also easily check from the interchange law that for $u \in G_2$

$$-_1 -_2 u = -_2 -_1 u \tag{DCG 6}$$

and we denote the "rotation" $-_1 -_2$ by $-_{12}$.

Example 6.1.8 In the case G is a groupoid, the double categories $\square G$ of commutative 2-shells and $\square' G$ of 2-shells in G defined in Example 6.1.6 have all three structures of groupoid, the inverses being as follows.

$$u = \begin{pmatrix} & c & \\ a & & d \\ & b & \end{pmatrix}, \quad -_1 u = \begin{pmatrix} & b & \\ a^{-1} & & d^{-1} \\ & c & \end{pmatrix}, \quad -_2 u = \begin{pmatrix} & c^{-1} & \\ d & & a \\ & b^{-1} & \end{pmatrix}, \quad -_1 -_2 u = \begin{pmatrix} & b^{-1} & \\ d^{-1} & & a^{-1} \\ & c^{-1} & \end{pmatrix}.$$

There are interesting differences between the category and groupoid cases with regard to commutative 2-shells. If G is a groupoid, the commutativity condition of a 2-shell can also be stated as $c = abd^{-1}$ or even as $abd^{-1}c^{-1} = 0$.

Thus when searching for new examples of double categories an obvious generalisation of $\square C$ comes by considering 2-shells that are commutative up to an element lying in some subcategory $C' \subseteq C$. That is, instead of $ab = cd$ we require $abd^{-1}c^{-1} \in C'$ which works well in the groupoid case.

It is a nice exercise that you should try at this stage, to check that this works if C is a group and C' is a normal subgroup.

This leads to a possible extension of the notion of normal subgroups to 'normal subgroupoids' (It is also a good exercise for you to think how this extension can be made). At a further stage, the concept of normal subgroupoid can be 'externalised' as a crossed module of groupoids, analogously to what has been done for groups. We shall define this concept and prove that it works in Section 6.2.

6.2 The category XMod of crossed modules of groupoids.

We have explained that there was an early hint that crossed modules (of groupoids) were related to double categories where all structures are groupoids. Since crossed modules appear quite naturally in algebraic topology, that was a suggestion of strong links between higher order groupoids and classical objects of algebraic topology.

Crossed modules of groupoids are an easy step away from crossed modules of groups and mimic the structure of the family of fundamental crossed modules $\Pi_2(X, A, x)$ when $x \in A \subseteq X$. Also, for any double category which has all three structures of groupoid, we get an associated crossed module over a groupoid.

It is natural to define a crossed module of groupoids to be a groupoid morphism $(\mu : M \to P)$ with an action of P on M such that axioms equivalent to CM1) and CM2) are satisfied. Thus, we start with a groupoid P where P_0 its set of vertices, ∂^0, ∂^1 its initial and final maps. We write $P_1(p, q)$ for the set of arrows from p to q $(p, q \in P)$ and $P_1(p)$ for the group $P_1(p, p)$.

Definition 6.2.1 A *crossed module* over the groupoid $P = (P_1, P_0)$ is given by a groupoid $M = (M_2, P_0)$ and a morphism of groupoids which is the identity on objects

$$M \xrightarrow{\mu} P$$

satisfying

- M is a totally disconnected groupoid with the same objects as P. Equivalently, it can be seen as a family of groups $\{M_2(p)\}_{p \in P_0}$.

We shall use additive notation for all groups $M_2(p)$ and we shall use the same symbol 0 for all their identity elements.

Also, μ is given by a family of homomorphisms $\{\mu_p : M_2(p) \to P_1(p)\}_{p \in P_0}$.

- The groupoid P operates on the right on M. The action is denoted $(x, a) \mapsto x^a$. If $x \in M_2(p)$ and $a \in P_1(p, q)$ then $x^a \in M_2(q)$. It satisfies the usual two axioms of an action.

i) $(x^{ab}) = (x^a)^b$

ii) $(xy)^a = x^a y^a$.

(Thus $M_2(p) \cong M_2(q)$ if p and q lie in the same component of the groupoid P.)

- These data satisfy two properties

CM1) μ preserves the actions, i.e. $\mu(x^a) = (\mu x)^a$

CM2) For all $c \in M_2(p)$, μc acts on M by conjugation by c, i.e. for any $x \in M_2(p)$, $x^{\mu c} = -c + x + c$.

Notice that $M_2(p)$ is a crossed module over $P_1(p)$ for all $p \in P_0$. In the case when P_0 is a single point we call μ a *crossed module over a group*, or a *reduced* crossed module.

A *morphism* of crossed modules $f : (\mu : M \to P) \to (\nu : N \to Q)$ is a pair of morphisms of groupoids $f_2 : M \to N$, $f_1 : P \to Q$ inducing the same map of vertices and compatible with the boundary maps and the actions of both crossed modules. We denote by XMod the resulting category of crossed modules over groupoids. Notice that the category XMod/Groups studied in the preceding chapters can be seen as the full subcategory of XMod whose objects are reduced crossed modules of groupoids.

Example 6.2.2 As we have pointed out, there is an immediate topological example. For any topological pair (X, A) and $C \subseteq A$, we consider $P = \pi_1(A, C)$, the fundamental groupoid of (A, C). Recall that the objects of $\pi_1(A, C)$ are the points of C and for any $x, y \in C$, the elements of $\pi_1(A, C)(x, y)$ are the homotopy classes rel $\{0, 1\}$ of maps

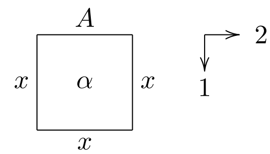
$$\omega : (I, 0, 1) \to (A, x, y).$$

The *fundamental crossed module* $\Pi_2(X, A, C)$ of the triple (X, A, C) is given by the family of groups $\{\pi_2(X, A, x)\}_{x \in C}$. These groups have been defined already in Section 2.1.

Recall that any $[\alpha] \in \pi_2(X, A, x)$ is a homotopy class rel J^1 of maps

$$\alpha : (I^2, \partial I^2, J^1) \to (X, A, x),$$

that can be represented as a square



that is the usual convention for $\mathbb{R}^2$ rotated clockwise through $\pi/2$ to make it equal to the algebraic convention. We shall keep the axes drawn beside the square to make this easier to remember.

The action

$$\pi_2(X, A, x) \times \pi_1(A, C)(x, y) \to \pi_2(X, A, y)$$

was also described in Section 2.1.

The morphism of groupoids $\partial : \pi_2(X, A, C) \to \pi_1(A, C)$ is given, for each $x \in C$, by the restriction to the top face $0 \times I$, so giving

$$\partial(x) : \pi_2(X, A, x) \to \pi_1(A, x).$$

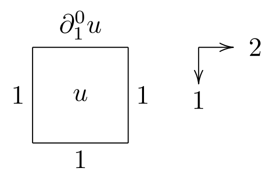
As before, it could be proved directly that these maps satisfy the properties of a crossed module over a groupoid, but we prefer the roundabout way of proving that this crossed module is the one associated to a double groupoid called the *fundamental double groupoid* that shall be defined in Section 6.3.

Let us go back to the general theory and see how to associate to any object $G \in \mathsf{DCatG}$ a crossed module of groupoids which we denote by $\gamma G = (\partial : \gamma G \to P)$. To make it a crossed module we need: groupoid structures on γG and P, a map of groupoids ∂ and an action satisfying CM1) and CM2).

We start by defining P as the groupoid (G_1, G_0). Thus the objects of γG are $(\gamma G)_0 = G_0$ and as morphisms we choose all $u \in G_2$ that have all faces degenerate except $\partial_1^0 u$, i.e.

$$(\gamma G)_2 = \{u \in G_2 : \partial_2^1 u = \partial_2^0 u = \partial_1^1 u = \varepsilon \partial^0 \partial_1^0 u = \varepsilon \partial^1 \partial_1^0 u\}.$$

The reason we chose to use the subindex 2 in the set of morphisms M_2 of M is now apparent: because in this very important example they have "dimension" two. The elements in γG_2, when represented with a matrix like convention, are



With the obvious source, target, and identity, and the composition $u + v$ defined to be $u +_2 v$, we get a totally disconnected groupoid γG.

The next element we need to get a crossed P-module, is a morphism of groupoids. It is defined by

(6.2.1) $$\partial = \partial_1^0 : \gamma G_2 \to P_1.$$

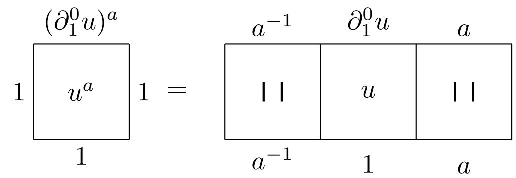
The last ingredient is an action

$$\gamma G_2(x) \times G_1(x, y) \to \gamma G_2(y)$$

for all $x, y \in G_0$. It is given by degeneration and conjugation: i.e. for any $u \in \gamma G_2(x)$ and $a \in G_1(x, y)$,

$$u^a = [-_2\varepsilon_1 a,\ u,\ \varepsilon_1 a], \tag{6.2.2}$$

or, in the usual representation,



Now we have to check that this gives an action that satisfies both properties in the definition of crossed module.

Proposition 6.2.3 *The definition in* (6.2.2) *gives a right action of* G_1 *on* γG_2*.*

Proof From the diagram, it is clear that $u^a \in \gamma G_2$. It is also not difficult to prove all properties of an action:

$$u^{ab} = (u^a)^b, \qquad (u +_2 v)^a = u^a +_2 v^a \qquad \text{and} \qquad u^1 = u.$$

□

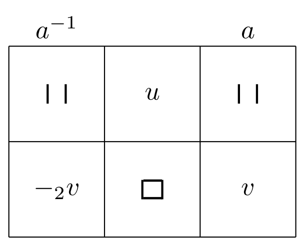
It remains to check the two axioms CM1) and CM2).

Proposition 6.2.4 $\gamma G = (\partial_1^0 : \gamma G_2 \to G_1)$ *is a crossed module with the action defined by* (6.2.2).

Proof For CM1) is clear from the diagram that the top face is the conjugate:

$$\partial(u^a) = \partial_1^0(u^a) = \partial_1^0(-_2\varepsilon_1 a)\partial_1^0 u \partial_1^0(\varepsilon_1 a) = a^{-1}\partial_1^0 u a = (\partial u)^a.$$

Also, for any $a = \partial v, v \in \gamma G_2$, we may construct an array such that when computing both ways gives the equality. In this case the array is



Composing first in the horizontal direction and then in the vertical one, the first row gives u^a and the second one a degenerate square, so we get u^a.

On the other hand, composing first vertically, we get

$$[-_2 v, u, v] = u^v.$$

□

It is important to notice that this construction is functorial, thus giving a functor

$$\gamma : \mathsf{DCatG} \to \mathsf{XMod}.$$

Remark 6.2.5 We finish this section by pointing out that for a double category which has all three structures groupoids we have not only one associated crossed module of groupoids but four, since we may chose any of the sides to be the unique one not equal to the identity. Let us call γG_j^i the crossed module structure on the set of all elements of G_2 having all faces degenerate but the i-face in the j-direction defined by the map ∂_j^i. Then γG_j^0 and γG_j^1 are isomorphic. In general, γG_1^j and γG_2^j are not isomorphic but we shall see that they are isomorphic in the case of interest here, namely Example 6.2.2.

6.3 Fundamental double groupoid

Granted the success of the fundamental groupoid and the known definition of double groupoid, perhaps it was natural in 1966 to attempt to define a fundamental or homotopy double groupoid of a space by considering maps $I^2 \to X$ of a square. Nevertheless, it was not until 1974 that Brown and Higgins realised that a successful theory could be obtained by considering a triple (X, A, C), i.e. a space X and two subspaces $C \subseteq A \subseteq X$.

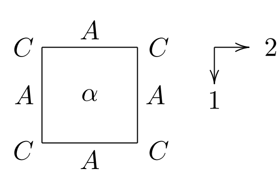
We shall start by describing the space of maps and some structure over it before getting homotopy classes. We consider a triple (X, A, C). We shall use the triple $(I^2, \partial I^2, \partial^2 I^2)$ given by the square, its boundary and the four vertices, respectively. We consider three sets

$$\begin{aligned} R_0(X,A,C) &= C \\ R_1(X,A,C) &= \{\sigma : (I,\{0,1\}) \to (A,C)\} \\ R_2(X,A,C) &= \{\alpha : (I^2,\partial I^2,\partial^2 I^2) \to (X,A,C)\}. \end{aligned}$$

and call the elements of $R_2(X, A, C)$ *filtered maps*

$$\alpha : (I^2, \partial I^2, \partial^2 I^2) \to (X, A, C).$$

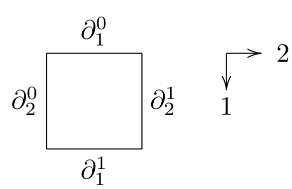
Remark 6.3.1 The elements of R_2 can be represented by squares as follows.



There is an obvious definition of the source and target maps given by restriction to the appropriate faces of I^2. More formally they are composition with the maps

$$\partial_1^i(x) = (i, x) \quad \text{and} \quad \partial_2^i(x) = (x, i) \quad \text{for} \;\; i = 0, 1$$

and they can be seen in the diagram



The identities are given by composing with the projection in the appropriate direction, i.e.

$$p_1(x,y) = x \quad \text{and} \quad p_2(x,y) = y$$

and we use the same notation for degenerate squares as in the previous section.

Also, there are several compositions on R given by juxtaposition. The one in R_1 has been defined when talking about the fundamental groupoid. The set R_2 has two similar compositions given by

$$(\alpha +_1 \beta)(x,y) = \begin{cases} \alpha(2x,y) & \text{if } 0 \leqslant x \leqslant 1/2 \\ \beta(2x-1,y) & \text{if } 1/2 \leqslant x \leqslant 1 \end{cases}$$

and

$$(\alpha +_2 \beta)(x,y) = \begin{cases} \alpha(x,2y) & \text{if } 0 \leqslant y \leqslant 1/2 \\ \beta(x,2y-1) & \text{if } 1/2 \leqslant y \leqslant 1. \end{cases}$$

We leave the reader to check that the interchange law holds for these two compositions. The reverse of an element $\alpha \in R_2$, with respect these two directions are written $-_1\alpha$, $-_2\alpha$ and are defined respectively by $(x,y) \mapsto \alpha(1-x,y)$, $(x,y) \mapsto \alpha(x,1-y)$.

All this structure means in particular that $R(X,A,C)$ is a 2-truncated cubical set with compositions. It is not a double category (no associativity, etc.). Nevertheless, it is useful to fix the meaning of composition of arrays. We study this in the next remark.

Remark 6.3.2 For positive integers m,n let $\varphi_{m,n} : I^2 \to [0,m] \times [0,n]$ be the map $(x,y) \mapsto (mx,ny)$. An $m \times n$ subdivision of a square $\alpha : I^2 \to X$ is a factorisation $\alpha = \alpha' \circ \varphi_{m,n}$; its *parts* are the squares $\alpha_{ij} : I^2 \to X$ defined by

$$\alpha_{ij}(x,y) = \alpha'(x+i-1, y+j-1).$$

We then say that α is the *composite* of the squares α_{ij}, and we write $\alpha = [\alpha_{ij}]$. Similar definitions apply to paths and cubes.

Such a subdivision determines a cell-structure on I^2 as follows. The intervals $[0,m],[0,n]$ have cell-structures with integral points as 0-cells and the intervals $[i,i+1]$ as closed 1-cells. Then $[0,m] \times [0,n]$ has the product cell-structure which is transferred to I^2 by $\varphi_{m,n}^{-1}$. We call the 2-cell $\varphi_{m,n}^{-1}([i-1,i] \times [j-1])$ the *domain* of α_{ij}.

Definition 6.3.3 To define the fundamental double groupoid associated to a triple of spaces (X,A,C) we shall use the three sets

$$\begin{aligned} \rho_0(X,A,C) &= C \\ \rho_1(X,A,C) &= R_1(X,A,C)/\equiv \\ \rho_2(X,A,C) &= R_2(X,A,C)/\equiv . \end{aligned}$$

where $\equiv$ is the relation of homotopy rel vertices on R_1 and of homotopy of pairs rel vertices on R_2. That is, for such a homotopy $H_t : I^2 \to X$, we have $H_t(c) = H_0(c)$ for all $t \in I$ and $c \in \partial^2 I^2$. We call this relation *f-homotopy* (or filter homotopy), to distinguish it from homotopy of maps $I \to A$ or $I^2 \to X$ which we shall write $\simeq$. It is important that f-homotopy is rel vertices, that is that the vertices of I and of I^2 are fixed in the homotopies. This allows us to obtain the groupoid structures on the filtered homotopy classes without adding any condition on the spaces. (In this we diverge from the definition given in [39].)

The *f-homotopy class* of an element α is written $\langle\!\langle \alpha \rangle\!\rangle$.

We expect all the structure maps in $\rho(X, A, C)$ to be those induced by the corresponding structure maps of $R(X, A, C)$. So we have to prove that they are compatible with the homotopies. In the case of the structure maps for (ρ_1, ρ_0) this is clear, since they form the relative fundamental groupoid of the pair (A, C).

Let us try the maps for the horizontal and vertical structure on (ρ_2, ρ_1). There is no problem with the source and target since the homotopies are filtered. Also a homotopy between elements of $R_1(X, A, C)$ gives easily a homotopy between the associated identities. The only problems appear to be with the compositions.

We develop only the horizontal case; the other follows by symmetry. So, let us consider two elements $\langle\langle\alpha\rangle\rangle, \langle\langle\beta\rangle\rangle \in \rho_2(X, A, C)$ such that $\langle\langle\partial_2^1\alpha\rangle\rangle = \langle\langle\partial_2^0\beta\rangle\rangle$, i.e. we have continuous maps

$$\alpha, \beta : (I^2, \partial I^2, \partial^2 I^2) \to (X, A, C)$$

and a homotopy

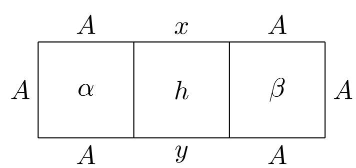
$$h : (I, \partial(I)) \times I \to (A, C)$$

from $\alpha|_{\{1\}\times I}$ to $\beta|_{\{0\}\times I}$ rel vertices, i.e. $h(0 \times I) = y$ and $h(1 \times I) = x$. We define now the composition by

$$\langle\langle\alpha\rangle\rangle +_2 \langle\langle\beta\rangle\rangle = \langle\langle\alpha +_2 h +_2 \beta\rangle\rangle = \langle\langle[\alpha, h, \beta]\rangle\rangle.$$

This is given in a diagram by

(6.3.1)

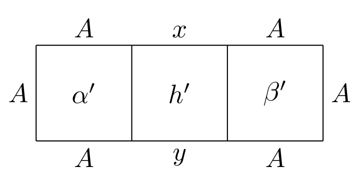


.

Our first important step is that these compositions are well defined.

Proposition 6.3.4 *The compositions are well defined in* $\rho_2(X, A, C) = R_2(X, A, C)/\equiv$.

Proof To prove this we chose two other representatives $\alpha' \in \langle\langle\alpha\rangle\rangle$ and $\beta' \in \langle\langle\beta\rangle\rangle$ and a homotopy h' from $\alpha'|_{\{1\}\times I}$ to $\beta'|_{\{0\}\times I}$. Using them, we get



which should give the same composition in ρ_2 as (6.3.1).

Since $\langle\langle\alpha\rangle\rangle = \langle\langle\alpha'\rangle\rangle, \langle\langle\beta\rangle\rangle = \langle\langle\beta'\rangle\rangle$ there are the f-homotopies $\phi : \alpha \equiv \alpha', \psi : \beta \equiv \beta'$ which can be seen in the next figure, in which the thin lines denote edges on which the maps are constant.

To complete this to an f-homotopy

$$\alpha +_2 h +_2 \beta \equiv \alpha' +_2 h' +_2 \beta'$$

we need to "fill" appropriately the hole in the middle (see Figure 6.1).

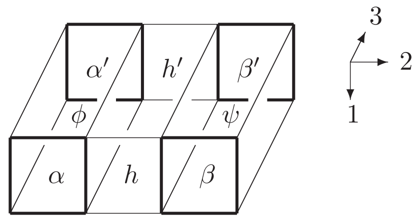


Figure 6.1: Filling the hole in the middle

Let $k : I \times \partial I^2 \to A$ be given by $(r, s, 0) \mapsto h(r, s), (r, s, 1) \mapsto h'(r, s), (r, 0, t) \mapsto \phi_t(r, 1), (r, 1, t) \mapsto \psi_t(r, 0)$. In terms of Figure 6.1, k is the map defined on the four side faces of the central hole. But k is constant on the edges of the bottom face, since all the homotopies are rel vertices. So k extends over $\{1\} \times I^2 \to A$ extending k to five faces of I^3.

Now we can retract I^3 onto any five faces by projecting from a point above the centre of the remaining face. Composing with this retraction, we obtain a further extension $k : I^3 \to A$. The composite cube $\phi +_2 k +_2 \psi$ is an f-homotopy $\gamma \equiv \gamma'$ as required: the key point is that the extension maps the top face of the middle cube into A, since that is true for all the other faces of this middle cube. □

Once we have proved that compositions in ρ_2 are well defined, we can easily prove that they are groupoids, with $\langle\!\langle -_i\alpha \rangle\!\rangle$ being the inverse of $\langle\!\langle \alpha \rangle\!\rangle$ for the composition $+_i, i = 1, 2$. We also need to prove the interchange law.

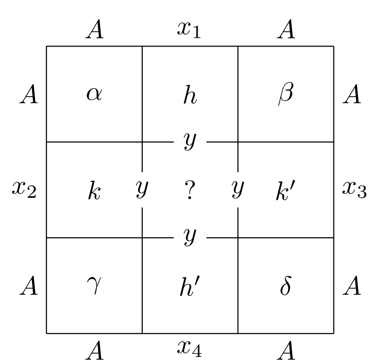
Proposition 6.3.5 *The compositions $+_1, +_2$ in $\rho_2(X, A, C) = R_2(X, A, C)/\equiv$ satisfy the interchange law.*

Proof The argument also involves "filling a hole". So, let us start with four elements $\langle\!\langle \alpha \rangle\!\rangle, \langle\!\langle \beta \rangle\!\rangle, \langle\!\langle \gamma \rangle\!\rangle, \langle\!\langle \delta \rangle\!\rangle \in \rho_2(X, A, C)$ such that $\langle\!\langle \partial_2^1\alpha \rangle\!\rangle = \langle\!\langle \partial_2^0\beta \rangle\!\rangle$, $\langle\!\langle \partial_2^1\gamma \rangle\!\rangle = \langle\!\langle \partial_2^0\delta \rangle\!\rangle$, $\langle\!\langle \partial_1^1\alpha \rangle\!\rangle = \langle\!\langle \partial_1^0\gamma \rangle\!\rangle$ and $\langle\!\langle \partial_1^1\beta \rangle\!\rangle = \langle\!\langle \partial_1^0\delta \rangle\!\rangle$. To prove that

$$(\langle\!\langle \alpha \rangle\!\rangle +_2 \langle\!\langle \beta \rangle\!\rangle) +_1 (\langle\!\langle \gamma \rangle\!\rangle +_2 \langle\!\langle \delta \rangle\!\rangle) = (\langle\!\langle \alpha \rangle\!\rangle +_1 \langle\!\langle \gamma \rangle\!\rangle) +_2 (\langle\!\langle \beta \rangle\!\rangle +_1 \langle\!\langle \delta \rangle\!\rangle)$$

we construct an element of $R_2(X, A, C)$ that represents both compositions.

Using f-homotopies $h : \partial_2^1\alpha \equiv \partial_2^0\beta$, $h' : \partial_2^1\gamma \equiv \partial_2^0\delta$, $k : \partial_1^1\alpha \equiv \partial_1^0\gamma$ and $k' : \partial_1^1\beta \equiv \partial_1^0\delta$ given because the compositions are defined we have a map defined on the whole square except on a hole in the middle:



We only need to fill appropriately the hole. But all homotopies are rel vertices, so the map is constant on the boundary of the hole. So we extend with the constant map, and evaluate the resulting composition in two ways to prove the interchange law. □

Thus we have proved that $\rho(X,A,C)$ is a double category where all three structures are groupoids. We call this *the fundamental double groupoid of the triple* (X,A,C) and leave the study of its extra structure which justifies its name till Section 6.4.

A map $f : (X,A,C) \to (X',A',C')$ of triples clearly defines a morphism $\rho(f) : \rho(X,A,C) \to \rho(X',A',C')$ of double categories.

Proposition 6.3.6 *If $f : (X,A,C) \to (X',A',C')$ is a map of triples such that each of $f : X \to X'$, $f_1 : A \to A'$ are homotopy equivalences, and $f_0 : C \to C'$ is a bijection, then $\rho(f) : \rho(X,A,C) \to \rho(X',A',C')$ is an isomorphism.*

Proof This is an easy consequence of a cogluing theorem for homotopy equivalences. We give the details for the analogous result for filtered spaces in an Appendix. □

Now let us check that the not quite so straightforward fact that the crossed module associated to the fundamental double groupoid $\rho(X,A,C)$ is the fundamental crossed module $\Pi_2(X,A,C)$, i.e. $\gamma(\rho(X,A,C))_2 = \Pi_2(X,A,C)$. Recall that $\gamma(\rho(X,A,C))_2(x)$ is formed by f-homotopy classes of filtered maps $\alpha : (I^2, \partial I^2, \partial^2 I^2) \to (X,A,x)$ such that the restriction to all sides but the last vertical one are homotopically trivial. On the other hand, $\pi_2(X,A,x)$ consists of homotopy classes of maps $\alpha : (I^2, \partial I^2, J^1) \to (X,A,x)$. Let us check that they are the same.

Proposition 6.3.7 *If $x \in C$, then the group $\gamma(\rho(X,A,C))_2(x)$ may be identified with the group $\pi_2(X,A,x)$.*

Proof Recall from the definitions that in both cases the elements are homotopy classes of maps

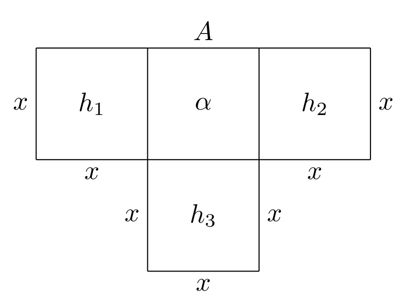
$$\alpha : (I^2, \partial I^2) \to (X,A).$$

For α to define an element in $\pi_2(X,A,x)$, which we are going to denote also by $\langle\!\langle \alpha \rangle\!\rangle$, the maps send all J^1 to x and the same is true for homotopies in this case. In the case $\langle\!\langle \alpha \rangle\!\rangle \in \rho(X,A,C)_2(x)$ the map sends only the vertices to x and the homotopy is rel vertices. Clearly the map

$$\phi : \pi_2(X,A,x) \to \gamma(\rho(X,A,C))_2(x)$$

defined by $\phi(\langle\!\langle \alpha \rangle\!\rangle) = \langle\!\langle \alpha \rangle\!\rangle$ is well defined, is a group homomorphism and preserves action. We only have to prove that ϕ is bijective. We shall use a couple of filling arguments.

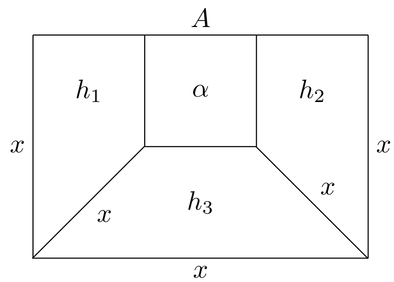
To see that ϕ is onto, let $\langle\!\langle \alpha \rangle\!\rangle \in \gamma(\rho(X,A,C))_2(x)$, i.e. we have a map $\alpha : (I^2, \partial I^2, \partial^2 I^2) \to (X,A,C)$ such that its restrictions to all faces of the square but the top one are homotopic rel vertices to the constant map. Putting all these three homotopies in one diagram we get



We want to get a map $\beta : (I^2, \partial I^2, J^1) \to (X, A, x)$ such that $\phi\langle\langle\beta\rangle\rangle = \langle\langle\alpha\rangle\rangle$ i.e filter homotopic rel vertices to α.

We can fold the above diagram, getting a map defined on four of the six faces of a cube I^3. Thus, composing with the retraction of I^3 onto such four faces, as seen in Figure 2.3, we get both the desired β (the restriction to the top face) and the homotopy (the cube).

Intuitively, the map β is



and the homotopy is got by shrinking the bigger square into the smaller one.

It remains to prove that ϕ is injective, i.e. that Ker ϕ contains only the homotopy class of the constant map.

Thus we start with a map $\alpha : (I^2, \{0\} \times I, J^1) \to (X, A, x)$ so that there is an f-homotopy $h : (I^2, \partial I^2, \partial^2 I^2) \times I \to (X, A, x)$ from α to 0. This h can be represented by a cube that is the constant x both in the back face

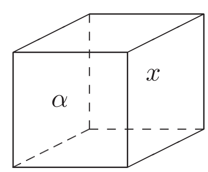
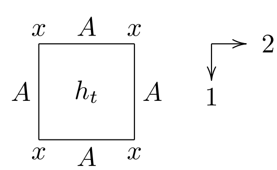


Figure 6.2: f-homotopy from α to the constant map

and in the four slanted lines.

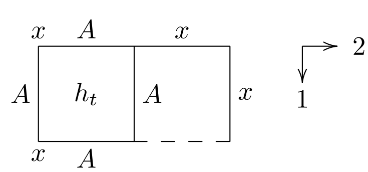
We have to get a homotopy of maps of triples $h' : \alpha \simeq 0$ rel J^1. This $h' : (I^2, \{0\} \times I, J^1) \times I \to (X, A, x)$ is α on the front 2-face and has to be constant not only on $\partial^2 I^2 \times I$ as was h, but also on $(\{1\} \times I \cup I \times \partial I) \times I$.

We will do that by changing h to h' in a similar way to the one used in the first part of this proof. Instead of working in four dimensions, we are going to explain what to do in each section for a fixed third coordinate with the 3-cube given by h. We have the following situation



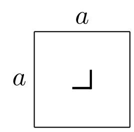
and we want to change this h_t to an h'_t sending all J^1 to x.

So, using a filling argument like the one in 1.3 we extend h_t

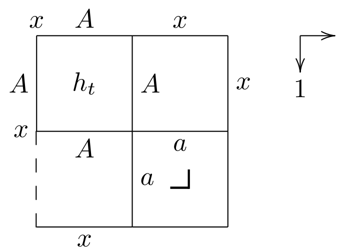


to an square sending one side (the right one) to x. Also, the edge represented by the discontinuous line goes in A, let us call it a.

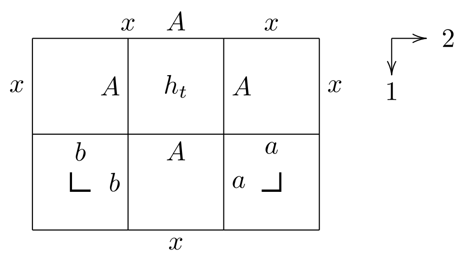
To change another side, we need some way of 'turning right'. This is produced by a degenerate square got by composing a with the map $\sigma : I^2 \to I$ given by $\sigma(s,t) = \max(s,t)$ that is represented by



where the unlabeled sides are constant. Adding this square, we can use a similar filling argument and extend to

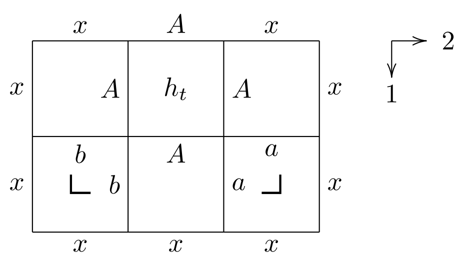


let us call by b the edge without label and repeat the filling argument to get



where the square 'turning left' in the bottom left corner is defined in a similar way as 'turning right'. It is clear that the edges without label goes in A.

Therefore, the above constructions all fit together to obtain h'_t as in the diagram below



Since we could do the above construction for any section t and all of them fit together, we get a homotopy

$$h' : (I^2, \{0\} \times I, J^1) \to (X, A, x)$$

from α to the constant map that is clearly continuous. □

The reader will have noticed the widespread use of filling arguments in the above proofs. These arguments become the key to the proof of corresponding results for higher dimensions which are developed in Chapter 16.

6.4 Thin structures on a double category. The category DGpds of double groupoids.

We have examples of double categories coming from two sources: first, the 2-shells commutative up to an element of a crossed modules over groupoids hinted at the end of Section 6.1 and which will be properly developed in Section 6.6, and second, the fundamental double groupoid of a topological pair seen in Section 6.3. In both cases not only are all three structures groupoids but they have also some extra structure. Let us see one way of introducing this structure.

We have already introduced in Example 6.1.6 the double category $\square' C$ of 2-*shells* in the category C and its sub double category $\square C$ of *commuting* 2-*shells*.

For any double category D there is a morphism of double categories $D \to \square' D_1$ which is the identity in dimensions 0,1 and in dimension 2 gives the bounding shell of any element. On the other hand, there is no natural morphism the other way, from either $\square' D_1$ or $\square D_1$, which is the identity on D_1.

In this Section, we are going to study double categories endowed with such a morphism, i.e. for any given commuting shell in D_1, there is a chosen 'filler' in D_2. Next, in Section 6.5, we develop an alternative approach using some extra degeneracies called *connections*.

Definition 6.4.1 We therefore define a *thin structure* on a double category D to be a morphism of double categories

$$\Theta : \square D_1 \to D$$

which is the identity on D_1, D_0. The 2-dimensional elements of the form $\Theta\alpha$ for $\alpha \in (\square D_1)_2$ will be called *thin* squares in (D, Θ) or simply in D if Θ is given.

Equivalently, the axioms for thin squares are:

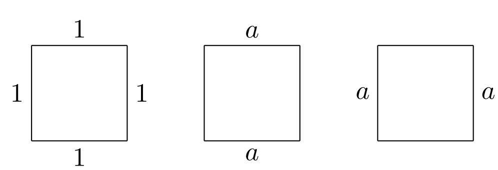
T0) Any identity square in D is thin.

T1) Each commuting shell in D has a unique thin filler.

T2) Any composite of thin squares is thin.

By T0), particular thin squares represent the degenerate squares, namely those of the form

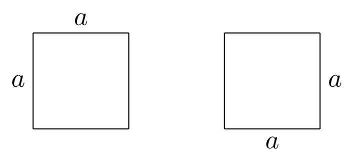
(6.4.1)



which we write in short as

Notice that identity edges are those drawn with a solid line. The notation is ambiguous, since for example the second element is the same as the first if $a = 1$. Also we have not named the vertices. Nevertheless, it is clear that they represent the degenerate squares since Θ is a morphism of double categories.

We also have two new ‘degenerate’ squares

(6.4.2) 

which we write in short as

$$\lrcorner \qquad\qquad \ulcorner \quad .$$

The fact that Θ is a morphism of double categories leads immediately to some equations for compositions of such elements, i.e.

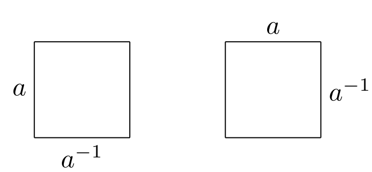
$$(6.4.3) \qquad \begin{bmatrix} \ulcorner & \lrcorner \end{bmatrix} = |\,| \qquad \begin{bmatrix} \ulcorner \\ \lrcorner \end{bmatrix} = \,=\, \quad .$$

In writing such matrix compositions, of course we always assume that the compositions are defined. The reason why these equations hold is that the composites are certainly thin, by T2), and since they are determined by their shell, by T1), they are by T0) of the form given.

Here are some more consequences which are known as “transport laws”:

$$(6.4.4) \qquad \begin{bmatrix} \lrcorner & |\,| \\ = & \lrcorner \end{bmatrix} = \lrcorner \quad , \qquad \begin{bmatrix} \ulcorner & = \\ |\,| & \ulcorner \end{bmatrix} = \ulcorner \quad .$$

If in addition the category D_1 is a groupoid then we have two further thin elements namely

(6.4.5) 

which we write

$$\urcorner \qquad\qquad \llcorner \quad .$$

Those elements give rise to new equations, for example

$$\begin{bmatrix} \ulcorner & \urcorner \\ |\,| & |\,| \end{bmatrix} = \square \quad .$$

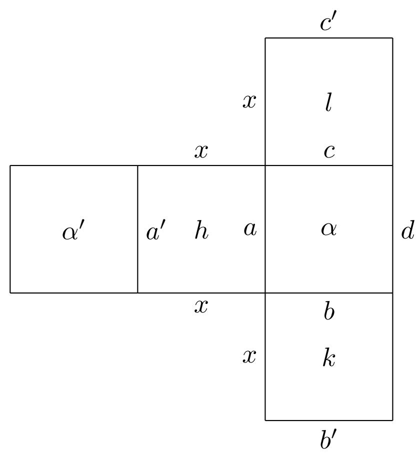
Note here that three of the sides are identities, and hence so also is the fourth, by commutativity.

Now we apply these ideas to the fundamental double groupoid $\rho(X, A, C)$.

Proposition 6.4.2 *The fundamental double groupoid $\rho(X, A, C)$ has a natural thin structure in which a class $\langle\!\langle \alpha \rangle\!\rangle$ is thin if and only it has a representative α such that $\alpha(I^2) \subseteq A$.*

Proof Let $a, b, c, d : I \to A$ be paths in A such that $ab \simeq cd$ in A. It is a standard property of the fundamental groupoid that the given paths can then be represented by the sides of a square $\alpha : I^2 \to A$. We have to prove that such a square is unique in ρ_2.

Let $\alpha' : I^2 \to A$ be another square whose edges a', b', c', d' are equivalent in $\pi_1(A, C)$ to a, b, c, d respectively. If we choose maps $h, k, l : I^2 \to A$ giving homotopies rel end points $a \simeq a', b \simeq b', c \simeq c'$. These homotopies, with α and α' can be represented as



folding the diagram they give a map H from five 2-faces of I^3 to A.

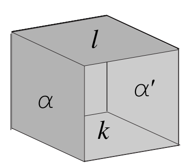


Figure 6.3: Box without a lateral face

Now, using the retraction from I^3, we can extend this to a map $I^3 \to A$. This gives an f-homotopy as required.

Note that this is where we use the fact that an f-homotopy is allowed to move the edges of the square within A. □

Since this important example has this structure, it is reasonable to call them double groupoids (they were called special double groupoids with special connections in [60], since more general connections were discussed there). This leads to:

Definition 6.4.3 A *double groupoid* is a double category such that all three structures are groupoids, together with a thin structure. We write DGpds for the category of double groupoids taking as morphisms the double functors that preserve the given thin structures.

We are interested in the restriction to this category of the functor defined in Section 6.2. It is still denoted

$$\gamma : \mathsf{DGpds} \to \mathsf{Crs}.$$

Notice that the thin elements $\ulcorner$, $\lrcorner$ in $\rho(X, A, C)$ are, like $=$, $\|$, determined by specific maps, namely in the first two cases are composition of a path $I \to A$ with the maps $\max, \min : I^2 \to I$. We will say more on this in the next section.

An important consequence of the existence of a thin structure in a double groupoid is that the vertical and horizontal groupoid structures in dimension 2 are isomorphic. The isomorphism is given by "rotation" maps $\sigma, \tau : G_2 \to G_2$ which correspond to a clockwise and an anticlockwise rotation through $\pi/2$.

Let G be a double groupoid. We define σ, τ for any $u \in G_2$ by

$$\sigma(u) = \begin{bmatrix} \| & \ulcorner & = \\ \llcorner & u & \urcorner \\ = & \lrcorner & \| \end{bmatrix} \quad \text{and} \quad \tau(u) = \begin{bmatrix} = & \urcorner & \| \\ \ulcorner & u & \lrcorner \\ \| & \llcorner & = \end{bmatrix}.$$

To prove the main properties of these operations is a diversion from our main aims, but one which illustrates some points in higher dimensional algebra.

Let us start by proving that σ is a homomorphism from the horizontal to the vertical composition, while τ is a homomorphism from the vertical to the horizontal composition. We next prove that τ is an inverse to σ. It follows that in the case of a double groupoid the horizontal and the vertical groupoid structures in dimension 2 are isomorphic.

Proposition 6.4.4 *For any $u, v, w \in G_2$,*

$$\sigma([u, v]) = \begin{bmatrix} \sigma u \\ \sigma v \end{bmatrix} \quad \text{and} \quad \sigma(\begin{bmatrix} u \\ w \end{bmatrix}) = [\sigma w, \sigma u]$$

$$\tau([u, v]) = \begin{bmatrix} \tau v \\ \tau u \end{bmatrix} \quad \text{and} \quad \tau(\begin{bmatrix} u \\ w \end{bmatrix}) = [\tau u, \tau w]$$

whenever the compositions are defined.

Proof We prove only the first rule and leave the others to the reader.

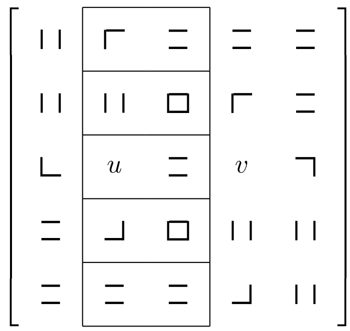
By definition, the element $\sigma([u, v])$ is the composition of the array

$$\begin{bmatrix} \| & \ulcorner & = \\ \llcorner & u +_2 v & \urcorner \\ = & \lrcorner & \| \end{bmatrix}.$$

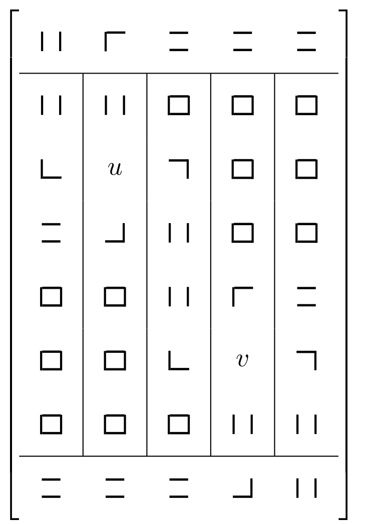
We get a refinement of this array by substituting each element for a box which has the initial element as its composition as follows:

$$\left[\begin{array}{c|cc|c} \| & \ulcorner & = & = \\ \| & \| & \ulcorner & = \\ \hline \llcorner & u & v & \urcorner \\ \hline = & \lrcorner & \| & \| \\ = & = & \lrcorner & \| \end{array}\right].$$

By Remark 6.1.5 this new array has the same composition as the initial one. We now subdivide the second column horizontally in two, getting a new refinement



which still has the same composition. Finally, we expand the three middle rows into six in such a way that we do not change the vertical composition of each column getting

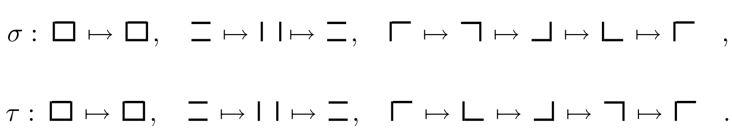


The composition of this array still is $\sigma([u, v])$ by Remark 6.1.5. To get the result, we now see that the composition of the block given by the first four rows is σu and the composition of the other four is σv. □

It is a nice exercise to extend this result to any rectangular array using associativity.

Since thin elements are determined by their boundaries, the next result follows immediately.

Proposition 6.4.5 *The images of thin elements under σ and τ are as follows*



A key fact is that σ is a bijection with inverse τ and that these maps together with the inverse maps $-_1$ and $-_2$, generate all symmetries of a square.

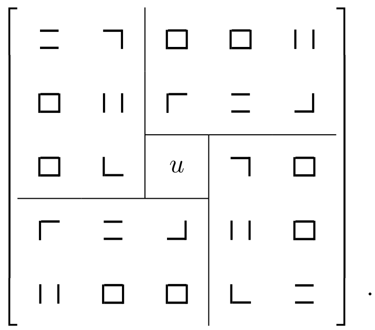
Theorem 6.4.6 *The isomorphisms* $-_1$, $-_2$, σ, τ *and their composites form a group of transformations of* G_2 *which is isomorphic to the group* D_8 *of symmetries of a square.*

Proof We choose a presentation of D_8 and verify that the relations are satisfied:

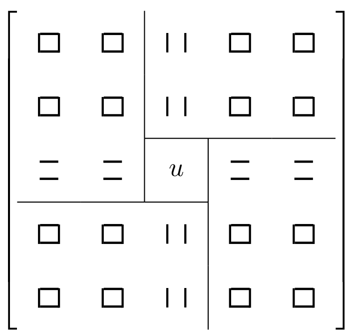
$$D_8 = \langle -_1, -_2, \sigma, \tau : (-_1)^2 = (-_2)^2 = \sigma\tau = (-_{12})^2 = \mathrm{Id}\,,\ -_1\sigma = \tau -_1\,,\ \sigma^2 = -_{12}\rangle.$$

We already know that $\{\mathrm{Id}, -_1, -_2, -_{12}\}$ form a Klein 4-group.

To verify the fourth relation, we show that for any $u \in G_2$, we have $\tau\sigma(u) = u$. It is easily seen that $\tau\sigma(u)$ is the composition of the array

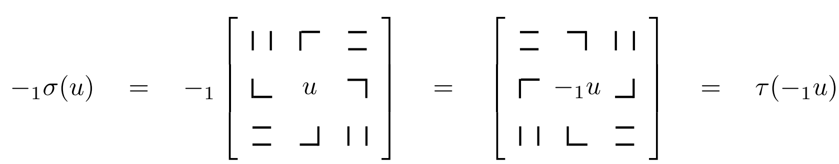


Using Remark 6.1.5 four times, we can change the four blocks one by one and substitute them for another four having the same boundary and composition, getting that $\tau\sigma(u)$ is also the composition of the array



whose composition reduces to u.

We next show that, for any $u \in G_2$, we have



For the final relation we note that

$$\sigma^2 = (\sigma -_1)(-_1\sigma) = (-_1\tau)(\tau -_2) = -_{12}$$

□

Remark 6.4.7 When these results are applied to the fundamental double groupoid $\rho(X, A, C)$, they imply the existence of specific f-homotopies. Indeed one of the aims of higher order groupoid theory is to give an algebraic framework for calculating with homotopies and higher homotopies.

6.5 Connections in a double category: equivalence with thin structure.

The extension of the notion of thin structure to higher dimensions is not straightforward since it would require the notion of commutative n-cube and this notion is not easy even for a 3-cube. We shall return to this at the end of this section.

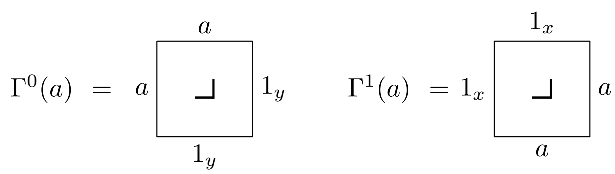
So, we look for an alternative which generalises more easily to higher dimensions. We take as basic the two maps $\Gamma^0, \Gamma^1 : D_1 \to D_2$, that correspond to the thin elements $\lrcorner$, $\ulcorner$, satisfying the properties we have seen in (6.4.3) and (6.4.4). We make this concept clear and develop the equivalence between the two notions in this section.

A *connection pair* on a double category D is a pair of maps

$$\Gamma^0, \Gamma^1 : D_1 \to D_2$$

satisfying the four properties below.

The first one is that the shells are what one expects, i.e., if $a : x \to y$ in D_1 then $\Gamma^0(a), \Gamma^1(a)$ shells are



which can be more formally stated as

$$\partial_2^0 \Gamma^0(a) = \partial_1^0 \Gamma^0(a) = a \quad \text{and} \quad \partial_2^1 \Gamma^0(a) = \partial_1^1 \Gamma^0(a) = \varepsilon \partial^1 a \tag{CON 1}$$

$$\partial_2^1 \Gamma^1(a) = \partial_1^1 \Gamma^1(a) = a \quad \text{and} \quad \partial_2^0 \Gamma^1(a) = \partial_1^0 \Gamma^1(a) = \varepsilon \partial^0 a. \tag{CON$'$ 1}$$

We also assume that the connections associate to a degenerate element a double degenerate one:

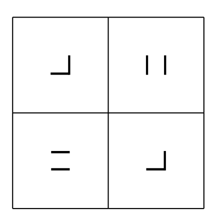
$$\Gamma^0 \varepsilon(x) = 0_x \tag{CON 2}$$

$$\Gamma^1 \varepsilon(x) = 0_x. \tag{CON$'$ 2}$$

The relation with composition is given by the "transport laws" (see (6.4.4)):

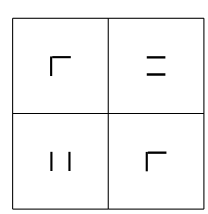
(CON 3)

$$\Gamma^0(ab) = \begin{bmatrix} \Gamma^0 a & | \; | \\ = & \Gamma^0 b \end{bmatrix} =$$



(CON$'$ 3)

$$\Gamma^1(ab) = \begin{bmatrix} \Gamma^1 a & = \\ | \; | & \Gamma^1 b \end{bmatrix} =$$



Intuitively, a feature that 2-dimensional movements can have extra to 1-dimensional movements is the possibility of turning left or right. The transport laws state intuitively that turning left with one's arm outstretched is the same as turning left, and similarly for turning right.

The name 'transport laws' was given because they were initially borrowed from a transport law for path connections in differential geometry, as explained in [61].

A final condition deduced from the same idea is that they are "inverse" to each other in both directions (corresponding to (6.4.3)), i.e.

$$\Gamma^1(a) +_2 \Gamma^0(a) = \varepsilon_1(a) \tag{CON 4}$$

$$\Gamma^1(a) +_1 \Gamma^0(a) = \varepsilon_2(a). \tag{CON$'$ 4}$$

It is interesting to notice that for double categories where all structures are groupoids we need only a map Γ^0 satisfying the conditions CON 1-3 since Γ^1 can be defined using (CON 4).

Proposition 6.5.1 *For a double category in which all structures are groupoids, Γ^0 and Γ^1 may be obtained from each other by the formula*

$$\Gamma^1(a) = -_2 -_1 \Gamma^0(a^{-1}).$$

Proof Let us define $\Gamma''(a) = -_2 -_1 \Gamma^0(a^{-1})$.

Since $\Gamma^0(aa^{-1}) = \Gamma^0(1) = \square$, we obtain from the transport law (CON 3.1) that $\Gamma^0(a^{-1}) = -_1[\Gamma^0 a, \ (\varepsilon_1 a^{-1})]$. Hence $\Gamma''(a) = [(\varepsilon_1 a), \ -_2\Gamma^0 a]$.

This implies that $\Gamma''(a) +_2 \Gamma^0(a) = \varepsilon_1(a)$, and so by (CON 4) $\Gamma''(a) = \Gamma^1(a)$. □

If we use an analogue of our previous notations $\lrcorner$, $\ulcorner$ for Γ^0, Γ^1 respectively then of course we see that all these laws are the ones we have given before for thin elements. So it is not very difficult to see, and it was already done by Brown and Spencer ([60]) in the case that all structures are groupoids, that any thin structure has associated a unique connection, and that the given thin structure is determined by this connection.

Proposition 6.5.2 *If there is a thin structure Θ on D we have an associated connection defined by*

$$\Gamma^0 a = \Theta\begin{pmatrix} & a & \\ a & & 1 \\ & 1 & \end{pmatrix} \quad \textit{and} \quad \Gamma^1 a = \Theta\begin{pmatrix} & 1 & \\ 1 & & a \\ & a & \end{pmatrix}.$$

Moreover, the morphism Θ can be recovered from the connection, since

$$\Theta\begin{pmatrix} & c & \\ a & & d \\ & b & \end{pmatrix} = (\varepsilon_2 a +_1 \Gamma^1 b) +_2 (\Gamma^0 c +_1 \varepsilon_2 d) = (\varepsilon_1 c +_2 \Gamma^1 d) +_1 (\Gamma^0 a +_2 \varepsilon_1 b). \tag{CON 5}$$

Proof The results on the behaviour of Γ^0 and Γ^1 with respect to boundaries and degeneracies are immediate.

Before proving the relation with the compositions, it is worth mentioning that the values of Θ on degenerate elements are determined by the fact that Θ is a morphism of double categories, so, $\Theta\varepsilon_1(b) = \varepsilon_1(b)$ and $\Theta\varepsilon_2(b) = \varepsilon_2(b)$.

Applying Θ to the equation

$$\begin{pmatrix} & ab & \\ ab & & 1 \\ & 1 & \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} & a & \\ a & & 1 \\ & 1 & \end{pmatrix} & \begin{pmatrix} & b & \\ 1 & & 1 \\ & b & \end{pmatrix} \\ \begin{pmatrix} & 1 & \\ b & & b \\ & 1 & \end{pmatrix} & \begin{pmatrix} & b & \\ b & & 1 \\ & 1 & \end{pmatrix} \end{pmatrix}$$

we get the transport law

$$\Gamma^0(ab) = \begin{bmatrix} \Gamma^0 a & \varepsilon_1 b \\ \varepsilon_2 b & \Gamma^0 b \end{bmatrix}.$$

and the one for Γ^1 is obtained along the same lines.

Moreover, it is easy to see that on $\square D$, the element

$$\begin{pmatrix} & c & \\ a & & d \\ & b & \end{pmatrix}$$

may by decomposed as the product of any of the two arrays

$$\begin{pmatrix} \begin{pmatrix} & 1 & \\ a & & a \\ & 1 & \end{pmatrix} & \begin{pmatrix} & c & \\ c & & 1 \\ & 1 & \end{pmatrix} \\ \begin{pmatrix} & 1 & \\ 1 & & b \\ & b & \end{pmatrix} & \begin{pmatrix} & 1 & \\ d & & d \\ & 1 & \end{pmatrix} \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} \begin{pmatrix} & c & \\ 1 & & 1 \\ & c & \end{pmatrix} & \begin{pmatrix} & 1 & \\ 1 & & d \\ & d & \end{pmatrix} \\ \begin{pmatrix} & a & \\ a & & 1 \\ & 1 & \end{pmatrix} & \begin{pmatrix} & b & \\ 1 & & 1 \\ & b & \end{pmatrix} \end{pmatrix}$$

where in the first one we have to compose first columns then rows and in the second one the other way about.

Applying Θ to these expressions, we get both formulae. □

Remark 6.5.3 As we have seen in the proof of the preceding property, the thin elements are composition of degenerate elements and connections. Conversely, all degeneracies and connections lie in the image of Θ, so any composition of such elements is a thin element. Thus we have an easy characterisation of the thin elements.

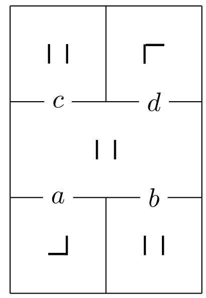
There is more work in obtaining the other implication, i.e. getting the thin structure from the connection maps. We follow the proof given by Brown and Mosa for the case of double categories ([56]). It is easier for double groupoids and in this case the proof may be traced back to Brown-Higgins ([39]). Nevertheless it is interesting to give the proof in the more general case for the possible applications in other situations.

Proposition 6.5.4 *If there is a connection on D, we have an associated thin structure Θ defined by the formula* (CON 5) *in Proposition 6.5.2. Moreover, the connection can be recovered from Θ, since*

$$\Gamma^0(a) = \Theta \begin{pmatrix} & a & \\ a & & 1 \\ & 1 & \end{pmatrix} \quad \text{and} \quad \Gamma^1(a) = \Theta \begin{pmatrix} & 1 & \\ 1 & & a \\ & a & \end{pmatrix}.$$

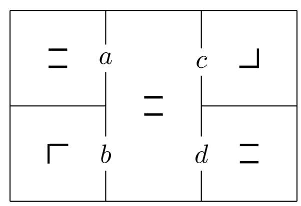
Proof Let us first prove that either formulae gives the same function. This will make it easier to prove the morphism property. We write

$$\Theta_1\begin{pmatrix} & c & \\ a & & d \\ & b & \end{pmatrix} = (\varepsilon_1 c +_2 \Gamma^1 d) +_1 (\Gamma^0 a +_2 \varepsilon_1 b) =$$

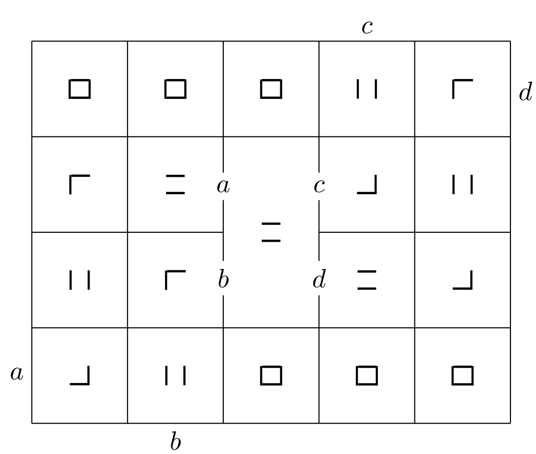


where the last diagram is obtained adding the degenerate middle row, and

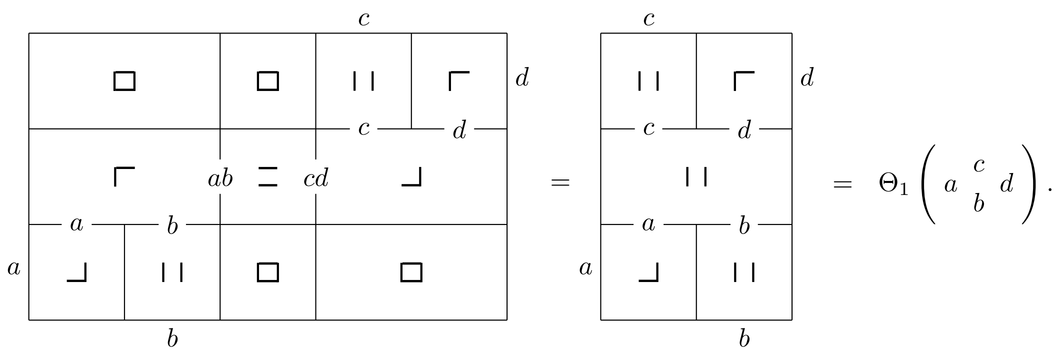
$$\Theta_2\begin{pmatrix} & c & \\ a & & d \\ & b & \end{pmatrix} = (\varepsilon_2 a +_1 \Gamma^1 b) +_2 (\Gamma^0 c +_1 \varepsilon_2 b) =$$



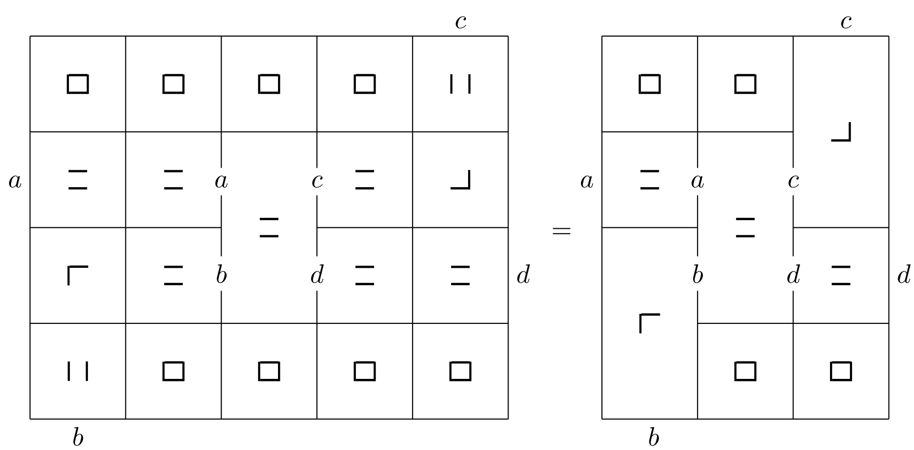
Then we want to prove $\Theta_1 = \Theta_2$. A usual way of proving that two compositions of arrays produce the same result is to construct a common subdivision. One that is appropriate for this case is



From this diagram, we may compose the second and third row using the transport law and then rearrange things, getting Θ_1 as indicated



Similarly, operating in the bottom left and the top right corner, we get



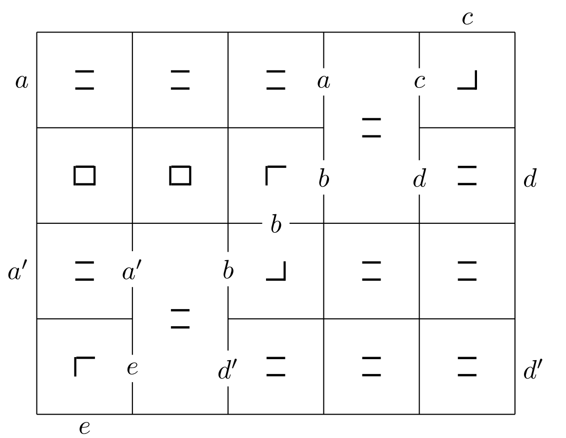
and this last diagram is, quite clearly Θ_2. We write Θ for the common value.

We would like to prove that Θ is a morphism. From any of its representations, it is clear that Θ commutes with faces and degeneracies. The only point we have to prove is that it commutes with both compositions. In this direction, it is good to have two definitions of Θ. First, we use $\Theta = \Theta_2$ to prove that Θ preserves the vertical composition. The use of $\Theta = \Theta_1$ to prove that it preserves the horizontal composition is similar.

So we want to prove

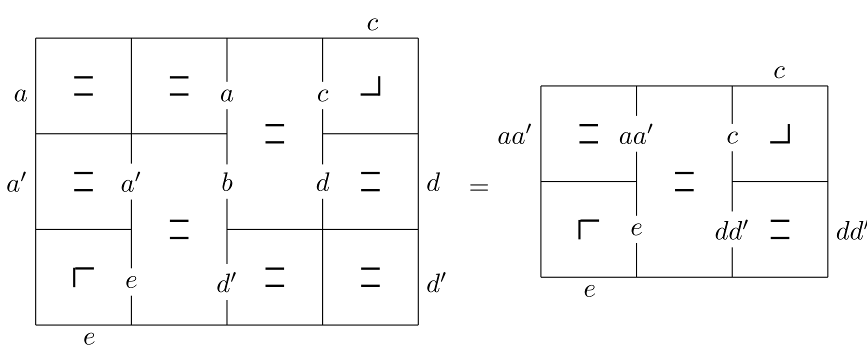
$$\Theta_2 \begin{pmatrix} & c & \\ a & & d \\ & b & \end{pmatrix} +_1 \Theta_2 \begin{pmatrix} & b & \\ a' & & d' \\ & e & \end{pmatrix} = \Theta_2 \begin{pmatrix} & c & \\ aa' & & dd' \\ & e & \end{pmatrix}.$$

As before we compute a common subdivision in two ways. The common subdivision we choose is



If we compose the first two rows, they produce $\Theta_2\begin{pmatrix} & c & \\ a & & d \\ & b & \end{pmatrix}$. Similarly, the two last rows give $\Theta_2\begin{pmatrix} & b & \\ a' & & d' \\ & e & \end{pmatrix}$.

On the other hand, making some easy adjusts on the three middle rows, we get



which clearly is $\Theta_2\begin{pmatrix} & c & \\ aa' & & dd' \\ & e & \end{pmatrix}$. □

6.6 Equivalence between XMod and DGpds: folding

In this section, we prove the equivalence between the category DGpds of double groupoids of Definition 6.4.3 and that of crossed modules of groupoids XMod of Definition 6.2.1.

On the one hand, the crossed module associated to a double groupoid is given by the functor

$$\gamma : \mathsf{DGpds} \to \mathsf{XMod}.$$

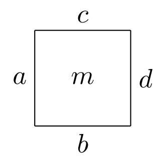
restriction of the one defined in Section 6.2.

On the other hand, there is a double groupoid associated to each crossed module as was already hinted at the end of Section 6.1. We shall develop this idea in this Section. We recall that to generalise the category of shells in a category, we use 2-shells which commute up to some element in the image of the crossed module.

Let $\mathcal{M} = (\mu : M \to P)$ be a crossed module over a groupoid. There is an associated double groupoid $G = \lambda\mathcal{M}$ whose sets are

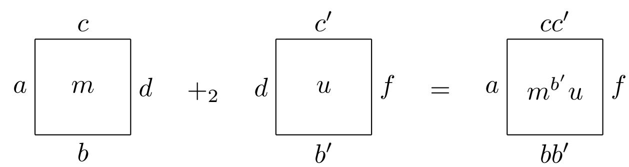
$$G_0 = P_0, \quad G_1 = P_1 \quad \text{and} \quad G_2 = \{(m, (a, b, c, d)) \mid b^{-1}a^{-1}cd = \mu m\}.$$

The elements of G_2 may be represented by

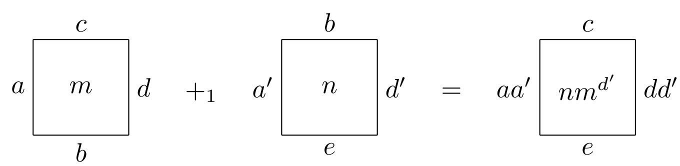


where m measures the lack of commutativity of the boundary, giving the composition of the sides of the boundary in clockwise direction starting from the right bottom corner.

The category structure in (G_1, G_0) is the same as that of (P_1, P_0), so it is a groupoid. The horizontal and vertical structures on (G_2, G_1) have source, target and identities defined as in $\square P$. The compositions deserve some extra care. On the second component they are defined by juxtaposition as in $\square P$. On the first one, they are not just composition since they have to measure the lack of commutativity. After playing a bit with the boundaries, it is not difficult to see that they can be defined by



and



and they are well defined since

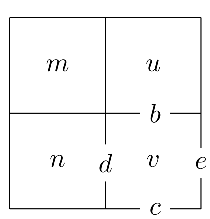
$$\mu(m^{b'}u) = {b'}^{-1}(\mu m)b'(\mu u) = {b'}^{-1}b^{-1}a^{-1}cdb'{b'}^{-1}d^{-1}c'f = {b'}^{-1}b^{-1}a^{-1}cc'f.$$

and

$$\mu(nm^{d'}) = \mu(n){d'}^{-1}\mu(m)d' = e^{-1}{a'}^{-1}bd'{d'}^{-1}b^{-1}a^{-1}cdd' = e^{-1}{a'}^{-1}a^{-1}cdd'.$$

This is where we have used rule CM1) for a crossed module.

It is not difficult to check that with these compositions all three categories are groupoids. We now verify the interchange law, using the following diagram,



Evaluating the rows first gives the first component of the composition, in an abbreviated notation since the edges are omitted, as

$$\begin{bmatrix} m^b u \\ n^c v \end{bmatrix} = (n^c v)(m^b u)^e$$

while evaluating the columns first gives the first component of the composition, in a similar notation, as

$$\begin{bmatrix} nm^d & vu^e \end{bmatrix} = (nm^d)^c vu^e.$$

So to prove the interchange law we have to verify that

$$vm^{be} = m^{dc}v.$$

This follows from CM2) since $\mu v = c^{-1}d^{-1}be$ and then

$$vm^{dc}v^{-1} = (m^{dc})^{\mu v} = m^{dcc^{-1}d^{-1}be} = m^{be}.$$

To finish, we define a thin structure on G by the obvious morphism

$$\Theta : \square P \to G_2$$

given by $\Theta(a,b,c,d) = (1,(a,b,c,d))$.

This gives a functor

$$\lambda : \mathsf{XMod} \to \mathsf{DGpds}$$

and that $\gamma\lambda\mathcal{M}$ is naturally isomorphic to $\mathcal{M}$ is trivial in dimensions 0,1 and in dimension 2 follows from

$$(\gamma\lambda\mathcal{M})_2 = \{(m,(1,1,1,\mu m)) \mid m \in M\} \cong M.$$

It is rather more involved to get a natural isomorphism from G to $\lambda\gamma G$ for any double groupoid G. In order to do this, we shall see first that a double groupoid is "generated" by the thin elements and those that have only one non-degenerate face, which we assume to be the top face. To this end we "fold" all faces to the chosen one.

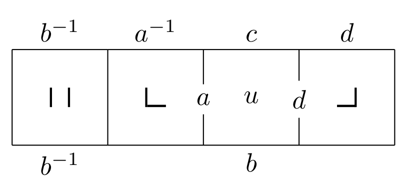
Definition 6.6.1 Let G be a double groupoid. We define the *folding* map

$$\Phi : G_2 \to (\gamma G)_2 \subseteq G_2$$

by the formula $\Phi u = [-_2\varepsilon_1\partial_1^1 u,\ -_2\Gamma^0\partial_2^0 u,\ u,\ \Gamma^0\partial_2^1 u]$. Notice that this can be defined only in the groupoid case because we are using $-_2$.

In the usual description

$\Phi u =$



Now let us see that the boundary of Φu is the one we expect. As a consequence Φ is well defined.

Proposition 6.6.2 *All faces of Φu are identities except the first in the vertical direction, and*

$$\partial_1^0 \Phi u = \partial_1^1 u^{-1} \partial_2^0 u^{-1} \partial_1^0 u \partial_2^1 u.$$

Thus $\Phi u \in \gamma G_2$ and $\mathrm{Im}\ \Phi \subseteq \gamma G_2$.

Proof All are easy calculations which are left as exercises. □

Also from the definition, the following property is clear.

Proposition 6.6.3 *All $u \in \gamma G_2$ satisfy $\Phi u = u$. Thus $\gamma G_2 = \operatorname{Im} \Phi$ and $\Phi\Phi = \Phi$.*

Proof This is immediate since in this case all the elements making up Φu except u itself are identities. □

We are now able to define a map

$$\Psi : G_2 \to (\lambda\gamma)G_2$$

by mapping any element $u \in G_2$ to the element given by the folding map Φu and the shell of u:

$$\begin{array}{ccc} & c & \\ a & \boxed{\;u\;} & d \\ & b & \end{array} \quad \longmapsto \quad \begin{array}{ccc} & c & \\ a & \boxed{\;\Phi u\;} & d \\ & b & \end{array}$$

We shall see that this map is an isomorphism between the two double groupoids.

It is clear that Ψ preserves faces. Also Ψ preserves thin elements since Φ of a thin element is a composition of thin elements and so is thin.

The most delicate part of the proof is the behaviour of the folding map Φ with respect to compositions. We obtain not a homomorphism but a kind of 'derivation', involving conjugacies, or, equivalently, the action in the crossed module γG.

Proposition 6.6.4 *Let $u, v, w \in G_2$ be such that $u +_1 v, u +_2 w$ exist, and let $b = \partial_1^1 u, g = \partial_2^1 v$. Then*

$$\Phi(u +_1 v) = [\Phi v, -_2\varepsilon_1 g, \Phi u, \varepsilon_1 g] = \Phi v +_2 (\Phi u)^g$$

$$\Phi(u +_2 w) = [-_2\varepsilon_1 b, \Phi u, \varepsilon_1 b, \Phi w] = (\Phi u)^b +_2 \Phi w.$$

Proof The proof of the second rule is simple, involving composition and cancelation in direction 2, so we prove in detail only the first rule. As before, this is done by constructing a common subdivision and computing it in two ways. Namely if both u, v are represented by

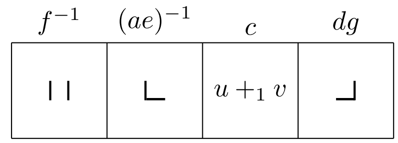
$$\begin{array}{ccc} & c & \\ a & \boxed{\;u\;} & d \\ & b & \end{array} \quad \text{and} \quad \begin{array}{ccc} & b & \\ e & \boxed{\;v\;} & g \\ & f & \end{array}$$

then

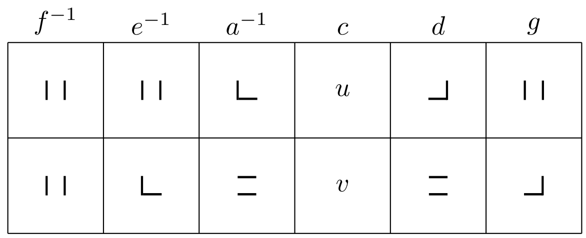
$$u +_1 v \quad = \quad \begin{array}{ccc} & c & \\ ae & \boxed{u +_1 v} & dg \\ & f & \end{array}$$

So we have

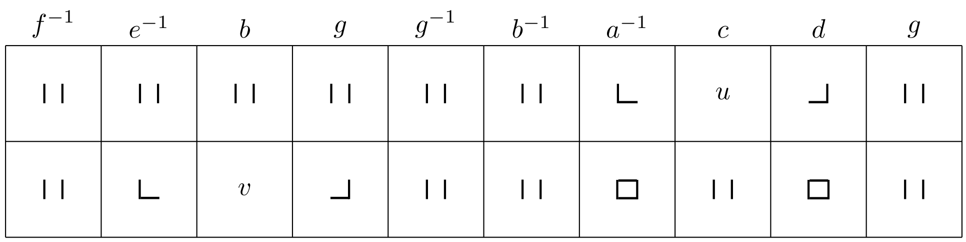
$\Phi(u +_1 v) \quad =$



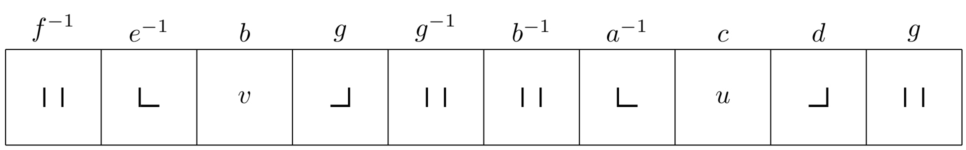
Applying both transport laws to the second and fourth square, we get a refinement



having the same composition by Remark 6.1.5. Next we get another array



having the same composite because each row has same composite in both cases (apply Remark 6.1.5). Now we can compose vertically in this last diagram to get



and this is clearly $\Phi v +_2 (\Phi u)^g$ as stated.

□

The important consequence is that the map

$$\Psi : G_2 \to (\lambda\gamma G)_2$$

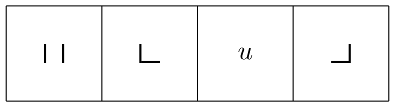
we are studying is a homomorphism with respect to both compositions since the equations proved in the preceding property are part of the definition of the compositions in $(\lambda\gamma G)_2$.

To end our proof of the equivalence between the categories of crossed modules over groupoids and double groupoids, it just remains to prove that the map Ψ is bijective, and preserves the thin structures. Let us start by characterising the thin elements of G_2 using the folding map.

Proposition 6.6.5 *An element $u \in G_2$ is thin if and only if $\Phi u = 1$.*

Proof As we pointed out in the Remark 6.5.3 an element $u \in G_2$ is thin if and only if it is a composition of identities and connections. By the preceding properties, it is clear that both identities and connections go to 1 under the folding map, so the same remains true for their compositions.

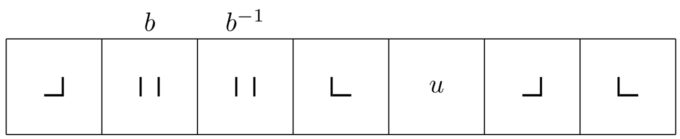
Conversely, if $u \in G_2$ is such that $\Phi u = 1$, by the definition of Φ, we have the following diagram



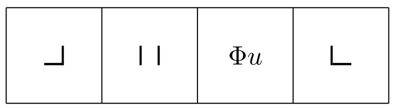
$= 1$

Solving this equation for u, we get that it is a product of identities and connections:

$u =$



$=$



□

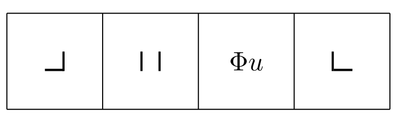
Corollary 6.6.6 *The map Ψ preserves the thin structures.*

Thus we can conclude that an element $u \in G_2$ is uniquely determined by its boundary and its image under the folding map.

Proposition 6.6.7 *Given elements $(a, b, c, d) \in \square G_2$ and $m \in \gamma G_2$, there is an element $u \in G_2$ with boundary (a, b, c, d) and $\Phi u = m$ if and only if $\partial_1^0 m = b^{-1}a^{-1}cd$. Moreover, in this case u is unique.*

Proof As before, we can solve the equation for u getting

$u =$



thus giving the construction of such element u. Uniqueness follows from the result before. □

Corollary 6.6.8 *The map $\Psi : G_2 \to (\lambda\gamma G)_2$ is bijective and determines a natural equivalence of functors $1 \simeq \lambda\gamma$.*

Thus we have completed the proof that the functors γ and λ give an equivalence of categories.

Corollary 6.6.9 *The functor γ preserves pushouts and, more generally, colimits.*

This allows us to prove first a van Kampen Theorem for the fundamental double groupoid and then deduce a corresponding theorem for the fundamental crossed module.

Remark 6.6.10 This equivalence also gives another way of checking some equalities on double groupoids. To see that two elements are equal we just need to know that they have the same boundary and that they fold to the same element. Alternatively, we can just check the equations in a double groupoid of the form $\lambda(M \to P)$. This is how properties of rotations were verified in [61]. The direct proofs are due to Philip Higgins.

Another aspect of the equivalence of categories is that it gives us a large source of double groupoids. Indeed one motivation of the equivalence in the work of [61, 60] was simply to find new examples of double groupoids and these were found since there is a large source of crossed modules.

Work on the double category case, proving the equivalence with 2-categories, was done by Spencer in [165] and additional work by Brown and Mosa [56]. This work has been extended to all dimensions in [4].

6.7 Homotopy commutativity lemma

As we saw in Chapter 1, the desire for the generalisation of the concept of commutative square was one of the motivations behind the search for higher dimensional group theory.

Recall that when proving the van Kampen Theorem 1.6.1, the main idea in the second part was to divide a homotopy into smaller squares and change each one to give a commutative square in π_1. Then we applied the morphisms and got composable commutative 2-shells in K; the fact that in a groupoid any composition of commutative 2-shells is commutative gave the result.

To generalise this to a van Kampen Theorem in dimension 2, we need several points:

- a concept of commutative 3-shell;

- to prove that the composition of 3-shells is commutative; and

- to relate commutative 3-shells with homotopy.

Those are the objectives of this section.

Before getting down to business, let us point out that there is a further generalisation to commutative n-shells for all n which will be explained in Part II (Chapter 15). Nevertheless, in the 3-dimensional case this can be done using connections with some careful handling. In fact, as has already been pointed out, one of the reasons for introducing connections in the paper [61] was to be able to discuss the notion of commutative 3-shell in a double groupoid.

The process generalises the construction of the double categories of 2-shells and commutative 2-shells seen in Example 6.1.8. In the 3-dimensional case we get what could be labelled a "triple category" but we are not formalising this concept at this stage because is not necessary now and can be done in a more natural way in a more general setting (see Chapter 15).

First we consider 3-shells, the definition of which does not use the thin structure. Let us start with the picture of a 3-cube (where we have drawn the directions to make things a bit easier to follow)

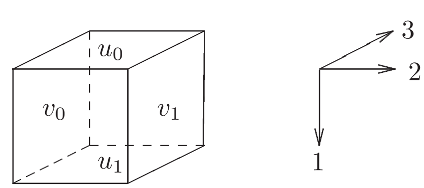


Figure 6.4: cube

Definition 6.7.1 Let D be a double category. A *cube* or (3-*shell)* in D,

$$\alpha = (u_0, u_1, v_0, v_1, w_0, w_1)$$

consists of squares in D_2 which fit together, i.e. such that

$$\partial_1^\sigma u_\tau = \partial_1^\tau v_\sigma;\ \partial_2^\sigma u_\tau = \partial_1^\tau w_\sigma;\ \partial_2^\sigma v_\tau = \partial_2^\tau w_\sigma$$

for $\sigma, \tau = 0, 1$. We also define $\partial_1^\sigma \alpha = u_\sigma, \partial_2^\sigma \alpha = v_\sigma$, and $\partial_3^\sigma \alpha = w_\sigma$ for $\sigma = 0, 1$.

Now we make these 3-shells into a triple category by defining three partial compositions of 3-shells as follows.

Definition 6.7.2 Let $\alpha = (u_0, u_1, v_0, v_1, w_0, w_1)$ and $\beta = (x_0, x_1, y_0, y_1, z_0, z_1)$ be cubes in D.

(i) If $u_1 = x_0$ we define

$$(u_0, u_1, v_0, v_1, w_0, w_1) +_1 (u_1, u_2, y_0, y_1, z_0, z_1) = (u_0, u_2, v_0 +_1 y_0, v_1 +_1 y_1, w_0 +_1 z_0, w_1 +_1 z_1).$$

(ii) If $v_1 = y_0$ we define

$$(u_0, u_1, v_0, v_1, w_0, w_1) +_2 (x_0, x_1, v_1, v_2, z_0, z_1) = (u_0 +_1 x_0, u_0 +_1 x_0, v_0, v_2, w_0 +_2 z_0, w_1 +_2 z_1).$$

(iii) If $w_1 = z_0$ we define

$$(u_0, u_1, v_0, v_1, w_0, w_1) +_3 (x_0, x_1, y_0, y_1, w_1, w_2) = (u_0 +_1 x_0, u_0 +_1 x_0, v_0 +_2 y_0, v_1 +_2 y_1, w_0, w_2).$$

It is easy to check that this yields a triple category, in the obvious sense. (This construction will be extended to all dimensions in a later chapter using a notation more suitable for the general case.)

Now we have to formulate the notion of *commutative 3-shell*. From the square case it seems that the proper generalisation would be to have that the composition of some faces equals the composition of the remaining faces.

We shall see that this is not so obvious in this dimension. Moreover, we have to work in a double category with connections. In fact, for our purposes, we can restrict to the double groupoid case, only sketching the category case and refer to [36] for more details.

Let us try to give some meaning to one face of a cube being the composition of the remaining five. We can start by thinking in the picture we get by folding flat those five faces of the cube.

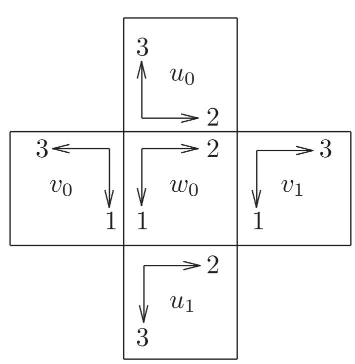


Figure 6.5: Five faces of a cube folded flat

First, notice that already in this figure we need that the double category we are using has all structures groupoids since we are using the inverse of some faces. Also, this is not a composable array in any obvious sense, thus we have to make clear the meaning of its composition. If the double category also has connections, i.e. it is a double groupoid G, we can fill the gaps in the diagram with connections giving a composable array

┌	$-_1u_0$	┐
$-_2v_0$	w_0	v_1
└	u_1	┘

We shall say that *the above* 3*-shell* α *in a double groupoid commutes* if the face $w_1 = \partial_3^1\alpha$ is the composition of the previous array involving the other five faces.

Remark 6.7.3 In the case where D is a double category with thin structure, we cannot get a formula of the above type, because of the lack of inverses. What we can expect as commuting boundary is a formula expressing some of the faces of the cube in terms of the other ones. Let us investigate this case.

If we fold flat the faces of the 3-cube, the six faces look like,

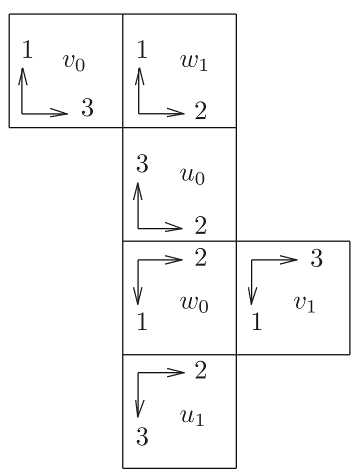


Figure 6.6: Cube boundary folded in a plane

This diagram can be nicely cut in two pieces such that each one can be transformed into a composable array using connections as follows:

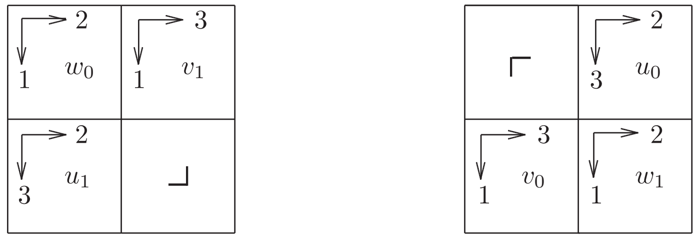
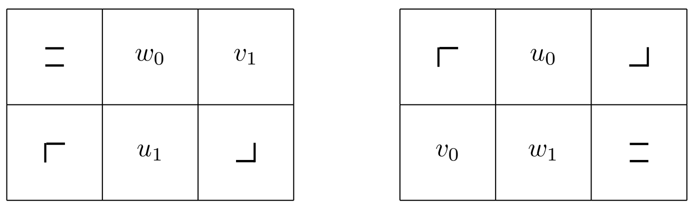


Figure 6.7: Cube boundary decomposed in two

It seems that we could say that a 3-shell is commutative if both compositions are the same, but this does not work because the two squares have different boundary. We expand both squares to get the same boundaries,



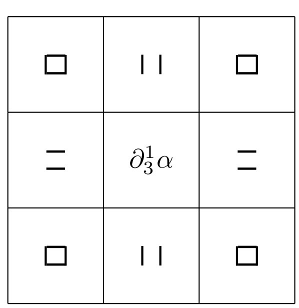
Therefore, we say that *a* 3*-shell* α *in a double category commutes* if both compositions are equal.

Remark 6.7.4 For a corresponding theory in higher dimensions one has at present to take the connections as basic, since their properties in all dimensions are easily expressed in terms of a finite number of axioms, each of which expresses simple geometric features of mappings of cubes. This was done in the groupoid case in [44] and it is developed in Chapter 15. It is then a main feature of the algebra to develop the related notion of thin structure. The chief advantage of the latter is that complicated arguments involving multiple compositions of commuting shells of cubes are reduced to simple arguments on the composition of thin elements.

Now we get two results on commuting cubes which are key to the proof of Theorem 6.8.2, in particular in Lemma 6.8.4. The first one, about 'degenerate' commutative 3-shells, shows that the two non-degenerate faces are equal.

Theorem 6.7.5 *Let* α *be a commutative* 3*-shell in a double groupoid* G*. Suppose that all the faces of* α *not involving direction* 3 *are degenerate. Then* $\partial_3^0\alpha = \partial_3^1\alpha$*.*

Proof In this case the array containing the five faces is



whose composition is clearly $\partial_3^1\alpha$.

Thus the commutativity of the 3-shell implies that $\partial_3^0\alpha = \partial_3^1\alpha$. □

Here is a second result about commutativity of 3-shells being preserved by composition.

Theorem 6.7.6 *In a double groupoid with connections, any composition of commutative 3-shells is commutative.*

Proof It is enough to prove that any composition of two commutative 3-shells is commutative.

So, let us consider $\alpha = (u_0, u_1, v_0, v_1, w_0, w_1)$ and $\beta = (x_0, x_1, y_0, y_1, z_0, z_1)$ two commutative 3-shells in a double groupoid G. This means that w_1 and z_1 are respectively given by

$$w_1 = \begin{array}{|c|c|c|}\hline \ulcorner & -_1u_0 & \urcorner \\ \hline -_2v_0 & w_0 & v_1 \\ \hline \llcorner & u_1 & \lrcorner \\ \hline \end{array} \qquad z_1 = \begin{array}{|c|c|c|}\hline \ulcorner & -_1x_0 & \urcorner \\ \hline -_2y_0 & z_0 & y_1 \\ \hline \llcorner & x_1 & \lrcorner \\ \hline \end{array}$$

We are going to check that composing in any of the three possible directions gives a commutative 3-shell.

If $v_1 = y_0$, the face $\partial_3^1(\alpha +_2 \beta) = w_1 +_2 z_1$ of $\alpha +_2 \beta$ is given by

$$w_1 +_2 z_1 = \begin{array}{|c|c|c|c|c|c|}\hline \ulcorner & -_1u_0 & \urcorner & \ulcorner & -_1x_0 & \urcorner \\ \hline -_2v_0 & w_0 & v_1 & -_2v_1 & z_0 & y_1 \\ \hline \llcorner & u_1 & \lrcorner & \llcorner & x_1 & \lrcorner \\ \hline \end{array}$$

Adding first the central two columns of this array and then the central three columns of the resulting array, we get

$$w_1 +_2 z_1 = \begin{array}{|c|c|c|c|c|}\hline \ulcorner & -_1u_0 & \square & -_1x_0 & \urcorner \\ \hline -_2v_0 & w_0 & \boxminus & z_0 & y_1 \\ \hline \llcorner & u_1 & \square & x_1 & \lrcorner \\ \hline \end{array} = \begin{array}{|c|c|c|}\hline \ulcorner & -_1(u_0 +_2 x_0) & \urcorner \\ \hline -_2v_0 & w_0 +_2 z_0 & y_1 \\ \hline \llcorner & u_1 +_2 x_1 & \lrcorner \\ \hline \end{array}$$

Thus $\alpha +_2 \beta$ is a commutative 3-shell.

Working vertically in the same way we can prove that $\alpha +_1 \beta$, when it is defined, is commutative if both α and β are commutative.

The case $\alpha +_3 \beta$ is a bit different. In this case $w_1 = z_0$, thus we have

$w_1 =$

⌜	$-_1 u_0$	⌝
$-_2 v_0$	w_0	v_1
⌞	u_1	⌟

$w_2 =$

⌜	$-_1 x_0$	⌝
$-_2 y_0$	w_1	y_1
⌞	x_1	⌟

Substituting w_1 in the second array for the first array and subdividing the other blocks to get a composable array, we get that

$w_2 =$

⌜	=	$-_1 u_0$	=	⌝
‖	⌜	$-_1 x_0$	⌝	‖
$-_2 v_0$	$-_2 y_0$	z_0	y_1	v_1
‖	⌞	x_1	⌟	‖
⌞	=	u_1	=	⌟

Now, we can compose by blocks and, using the transport law, we get

$w_2 =$

⌜	$-_1(u_0 +_1 x_0)$	⌝
$-_2(v_0 +_2 y_0)$	w_0	$v_1 +_2 y_1$
⌞	$u_1 +_1 x_1$	⌟

Thus $\alpha +_3 \beta$ is also a commutative 3-shell. □

Let us go now to the case of the fundamental double groupoid of a triple (X, A, C). In particular, we will see that some 3-cubes $h : I^3 \to X$ produce a commutative 3-shell in $\rho(X, A, C)$. This we call a 'homotopy commutativity lemma' reserving the term homotopy addition lemma which we give later for a result expressing the boundary of a cube or simplex in terms of a 'sum' of the faces.

For the statement of the lemma we introduce some notation that represents the changes of coordinates suggested by Figure 6.5. So, if $h : I^3 \to X$ is a cube in X, then the faces of h are given by restriction to the corresponding faces of the cube, i.e.

$$\partial_i^\alpha h = h \circ \eta_i^\alpha,$$

where $\eta_i^\alpha(x_1, x_2) = (y_1, y_2, y_3)$, the y_j being defined by $y_j = x_j$ for $j < i, y_i = \alpha$, and $y_j = x_{j-1}$ for $j > i$.

Also in some of the cases we are going to need some switching of coordinates, so let us consider $\tilde{\eta}_1^\alpha(x_1, x_2) = (\alpha, x_2, x_1)$.

Proposition 6.7.7 *(the homotopy commutativity lemma). Let $(X, A, C), \rho$ be as in section 6.3. Let h be a cube in X with edges in A and vertices in C, and let the elements $u_\alpha, v_\alpha, w_\alpha$ of ρ_2 represented by its faces be respectively the classes of $h \circ \tilde{\eta}_1^\alpha, h \circ \eta_2^\alpha, h \circ \eta_3^\alpha$ $(\alpha = 0, 1)$. Then*

$$w_1 = \begin{bmatrix} \ulcorner & -_1 u_0 & \urcorner \\ -_2 v_0 & w_0 & v_1 \\ \llcorner & u_1 & \lrcorner \end{bmatrix}$$

in ρ_2 where the corner elements are thin elements as above, i.e. the 3-cube in $\rho(X, A, C)$ given by the boundary of h commutes.

Proof Consider the maps $\varphi_0, \varphi_1 : I^2 \to I^3$ defined by

$$\varphi_0 = \begin{bmatrix} -_2 -_1 \Gamma & -_1(\tilde{\eta}_1^0) & -_1\Gamma \\ -_2\eta_2^0 & \eta_3^0 & \eta_2^1 \\ -_2\Gamma & \tilde{\eta}_1^1 & \Gamma \end{bmatrix}, \qquad \varphi_1 = \begin{bmatrix} -_2 -_1 \Gamma & 1 & -_1\Gamma \\ 0 & \eta_3^1 & 0 \\ -_2\Gamma & 1 & \Gamma \end{bmatrix}.$$

where Γ is the map induced by $\gamma : I^2 \to I$ given by $\gamma(x_1, x_2) = \max(x_1, x_2)$.

They are the two compositions given in the next Figure.

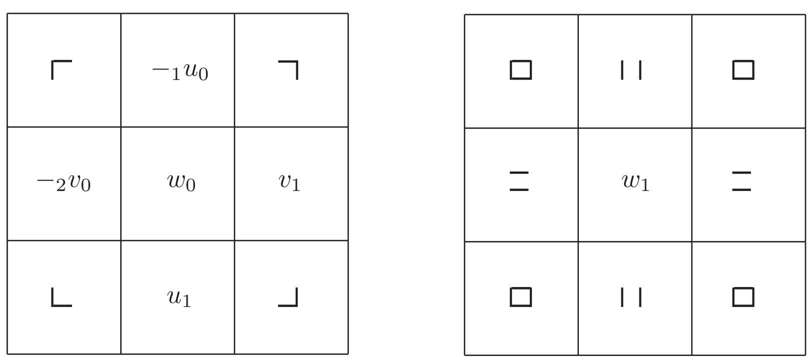


Figure 6.8: Two arrays with the same boundary

Notice that φ_0, φ_1 agree on ∂I^2 and so, since I^3 is convex, the linear homotopy

$$\begin{aligned} F : I^2 \times I &\to I^3 \\ (x_1, x_2), t &\mapsto t\varphi_0(x_1, x_2) + (1 - t)\varphi_1(x_1, x_2) \end{aligned}$$

gives an homotopy rel ∂I^2 between φ_0 and φ_1.

Hence $\langle\langle h\varphi_0 \rangle\rangle = \langle\langle h\varphi_1 \rangle\rangle$ in ρ_2. But $\langle\langle h\varphi_0 \rangle\rangle$ is the composite matrix given in the proposition, and $\langle\langle h\varphi_1 \rangle\rangle = w_1$.

□

6.8 Proof of the 2-dimensional van Kampen Theorem

In this last section of Part I we shall prove a generalised van Kampen Theorem (6.8.2) which includes as a particular case Theorem 2.3.1 some of whose algebraic consequences have been studied in Chapters 4 and 5.

Theorem 6.8.2 is true for triples of spaces (X, A, C) satisfying some connectivity conditions which can be expressed as algebraic conditions on the π_0 and π_1 functors.

Definition 6.8.1 We say that the triple (X, A, C) is *connected* if the following conditions hold:

$(\ddagger)_0$. The maps $\pi_0(C) \to \pi_0(A)$ and $\pi_0(C) \to \pi_0(X)$ are surjective.

$(\ddagger)_1$. The morphism of groupoids $\pi_1(A, C) \to \pi_1(X, C)$ is piecewise surjective.

Notice that condition $(\ddagger)_0$ is equivalent to saying that C intersects all path components of X and all of A. Also condition $(\ddagger)_1$ just says that the function $\pi_1(A)(x, y) \to \pi_1(X)(x, y)$ induced by inclusion is surjective for all $x, y \in C$. It may be shown that given $(\ddagger)_0$, condition $(\ddagger)_1$ may be replaced by

$(\ddagger')_1$. For each $x \in C$, the homotopy fibre over x of the inclusion $A \to X$ is path connected.

That both conditions can be stated in terms of connectivity, explains the origin of the term 'connected' (see [53]).

Let us introduce some notation which will be helpful in both the statement and the proof of Theorem 6.8.2. Suppose we are given a cover $\mathcal{U} = \{U^\lambda\}_{\lambda\in\Lambda}$ of X such that the interiors of the sets of $\mathcal{U}$ cover X. For each $\nu = (\lambda_1, \cdots, \lambda_n) \in \Lambda^n$ we write

$$U^\nu = U^{\lambda_1} \cap \ldots \cap U^{\lambda_n}.$$

An important property of this situation is that a continuous function f on X is entirely determined by a family of continuous functions $f^\lambda : U^\lambda \to X$ which agree on all pairwise intersections $U^{\lambda_1} \cap U^{\lambda_2}$. This is expressed by saying that the following diagram

$$\bigsqcup_{\lambda_1,\lambda_2\in\Lambda} U^{\lambda_1} \cap U^{\lambda_2} \underset{i_2}{\overset{i_1}{\rightrightarrows}} \bigsqcup_{\lambda\in\Lambda} U^\lambda \xrightarrow{\ i\ } X$$

is a coequaliser in the category of topological spaces. The functions i_1, i_2 are determined by the inclusions $U^\nu = U^{\lambda_1} \cap U^{\lambda_2} \to U^{\lambda_1}$, and $U^\nu \to U^{\lambda_2}$ for each $\nu = (\lambda_1, \lambda_2) \in \Lambda^2$, and i is determined by the inclusions $U^\lambda \to X$ for each $\lambda \in \Lambda$.

It is not difficult to extend this to the case of a triple (X, A, C). If we define $A^\nu = U^\nu \cap A$, and $C^\nu = U^\nu \cap C$, we get a similar coequaliser diagram in the category of triples of spaces:

$$\bigsqcup_{\nu\in\Lambda^2}(U^\nu, A^\nu, C^\nu) \underset{i_2}{\overset{i_1}{\rightrightarrows}} \bigsqcup_{\lambda\in\Lambda}(U^\lambda, A^\lambda, C^\lambda) \xrightarrow{\ i\ } (X, A, C).$$

Now we move from this to the homotopical situation, by applying ρ to the coequaliser diagram of triples. So the homotopy double groupoids in the following *ρ-sequence of the cover* are well-defined:

$$\bigsqcup_{\nu\in\Lambda^2} \rho(U^\nu, A^\nu, C^\nu) \underset{i_2}{\overset{i_1}{\rightrightarrows}} \bigsqcup_{\lambda\in\Lambda} \rho(U^\lambda, A^\lambda, C^\lambda) \xrightarrow{\ i\ } \rho(X, A, C). \tag{6.8.1}$$

Here $\bigsqcup$ denotes disjoint union, which is the coproduct in the category of double groupoids. It is an advantage of the approach using a set of base points that the coproduct in this category is so simple to describe. The morphisms i_1, i_2 are determined by the inclusions $U^\nu = U^{\lambda_1} \cap U^{\lambda_2} \to U^{\lambda_1}$, and $U^\nu \to U^{\lambda_2}$ for each $\nu = (\lambda_1, \lambda_2) \in \Lambda^2$, and i is determined by the inclusions $U^\lambda \to X$ for each $\lambda \in \Lambda$.

Theorem 6.8.2 *[39, Theorem B] Assume that for every finite intersection U^ν of elements of $\mathcal{U}$ the triple (U^ν, A^ν, C^ν) is connected. Then*

(Con) *the triple (X, A, C) is connected, and*

(Iso) *in the above ρ-sequence of the cover, i is the coequaliser of i_1, i_2 in the category of double groupoids.*

Proof The proof follows the pattern of the 1-dimensional case (Theorem 1.6.1) and it will take several stages.

We shall be aiming for the coequaliser result (Iso) because the connectivity part (Con) is obtained along the way. So we start with a double groupoid G and a morphism of double groupoids

$$f' : \bigsqcup_{\lambda\in\Lambda} \rho(U^\lambda, A^\lambda, C^\lambda) \to G$$

such that $f'i_1 = f'i_2$. We have to show that there is a unique morphism of double groupoids

$$f : \rho(X, A, C) \to G$$

such that $fi = f'$.

Recall that by the structure of coproduct in the category of double groupoids, the map f' is just the disjoint union of maps $f^\lambda : \rho(U^\lambda, A^\lambda, C^\lambda) \to G$ and the condition $f'i_1 = f'i_2$ translates to f^{λ_1} and f^{λ_2} being the same when restricted to $\rho(U^\nu, A^\nu, C^\nu)$ for $\nu = (\lambda_1, \lambda_2)$.

To define f on $\rho(X, A, C)$ we shall describe how to construct an $F(\alpha) \in G_2$ for all $\alpha \in R_2(X, A, C)$. Then we define $f(\langle\langle\alpha\rangle\rangle) = F(\alpha)$ and prove independence of all choices.

Stage 1.- Define $F(\alpha) \in G_2$ when $\alpha = [\alpha_{ij}]$ such that each α_{ij} lies in some $R_2(U^\lambda, A^\lambda, C^\lambda)$.

The easiest case is when the image of α lies in some U^λ of $\mathcal{U}$. Then α determines uniquely an element $\alpha^\lambda \in R_2(U^\lambda, A^\lambda, C^\lambda)$. The only way to have $fi = f'$ is by defining

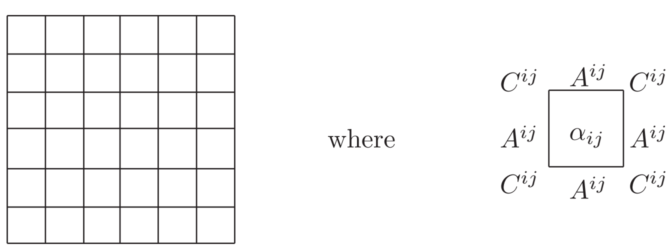
$$F(\alpha) = f^\lambda(\langle\langle\alpha^\lambda\rangle\rangle).$$

This definition does not depend on the choice of λ, because of the condition $f'i_1 = f'i_2$.

Next, suppose that the element $\alpha \in R_2(X, A, C)$ may be expressed as the composition of an array

$$\alpha = [\alpha_{ij}]$$

such that each α_{ij} belongs to $R_2(X, A, C)$, and also the image of α_{ij} lies in some U^λ of $\mathcal{U}$ which we shall denote by U^{ij}.



$\alpha = [\alpha_{ij}]$

Figure 6.9: Case $\alpha = [\alpha_{ij}]$ with $\alpha_{ij} \in R_2(U^{ij}, A^{ij}, C^{ij})$

We can define $F(\alpha_{ij})$ for each ij as before. Since the composite $[\alpha_{ij}]$ is defined, it is easy to check using $f'i_1 = f'i_2$, that the elements $F(\alpha_{ij})$ compose in G_2. We define $F(\alpha)$ to be the composite of these elements of G_2, i.e.

$$F(\alpha) = F([\alpha_{ij}]) = [F(\alpha_{ij})],$$

although *a priori* this definition could depend on the subdivision chosen.

Stage 2.- Define $F(\alpha) \in G_2$ by changing α by an f-homotopy to a map of the type used in Stage 1.

This is done analogously to the 1-dimensional case (Theorem 1.6.1). So, first we apply the Lebesgue covering lemma to get a subdivision $\alpha = [\alpha_{ij}]$ such that for each i, j, α_{ij} lies in some element U^{ij} of the covering. In general, we will not have $\alpha_{ij} \in R_2(U^{ij}, A^{ij}, C^{ij})$, so we have to deform α to another β satisfying this condition. The homotopy for this is given by the next lemma. In this we use the cell-structure on I^2 determined by a subdivision of α as in Remark 6.3.2, and also refer to the 'domain' of α_{ij} as defined there.

Lemma 6.8.3 *Let $\alpha \in R_2(X, A, C)$ and let $\alpha = [\alpha_{ij}]$ be a subdivision of α such that each α_{ij} lies in some U^{ij} of $\mathcal{U}$. Then there is an f-homotopy $h : \alpha \equiv \alpha'$, with $\alpha' \in R_2(X, A, C)$, such that, in the subdivision $h = [h_{ij}]$ determined by that of α, each homotopy $h_{ij} : \alpha_{ij} \simeq \alpha'_{ij}$ satisfies:*

(i) *h_{ij} lies in U^{ij};*

(ii) *α'_{ij} belongs to $R_2(X, A, C)$, and so can be considered an element of $R_2(U^{ij}, A^{ij}, C^{ij})$;*

(iii) *if a vertex v of the domain of α_{ij} is mapped into C, then h is constant on v;*

(iv) *if a cell e of the domain of α_{ij} is mapped by α into A(resp. C), then $e \times I$ is mapped by h into A(resp. C), and hence $\alpha'(e)$ is contained in A(resp. C).*

Proof Let K be the cell-structure on I^2 determined by the subdivision $\alpha = [\alpha_{ij}]$, as in Remark 6.3.2. We define h inductively on $K^n \times I \cup K \times \{0\} \subseteq K \times I$ using the connectivity conditions of the statement, where K^n is the n-skeleton of K for $n = 0, 1, 2$.

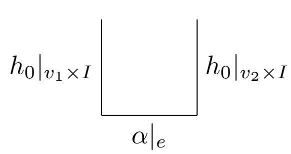
Step 1.- Extend $\alpha|_{K^0 \times \{0\}}$ to $h_0 : K^0 \times I \to C$.

Since the triples (U^ν, A^ν, C^ν) are connected for all finite sets $\nu \subseteq \Lambda$, the map $\pi_0(C^\nu) \to \pi_0(U^\nu)$ is surjective. For each vertex $v \in K$ we can choose a path lying in the intersection of all the U^λ corresponding to all the 2-cells of K containing v (one to four according to the situation of v) and going from $\alpha(v)$ to a point of C.

In particular, when $\alpha(v) \in C$ we choose the constant path and if $\alpha(v) \in A$, using that $\pi_0(C^\nu) \to \pi_0(A^\nu)$ is also surjective, we choose the path lying in A . These paths give a map $h_0 : K^0 \times I \to C$.

Step 2.- Extend $\alpha|_{K^1 \times \{0\}} \cup h_0$ to $h_1 : K^1 \times I \to A$.

For each 1-cell $e \in K$ with vertices v_1 and v_2, we have the following diagram



where on the three sides of $e \times I$ the definition of h_1 is given as indicated. We proceed to extend to $e \times I$ with some care.

If $\alpha(e) \subseteq A$ we consider two cases. When v_1, v_2 are mapped into C, we extend to $e \times I$ using α at each level $e \times \{t\}$. If $\alpha(e) \subseteq A$, and v_1, v_2 are not both mapped into C, since all edges go to A, then we can use a retraction to extend the homotopy.

Otherwise, the product of these three paths defines an element of $\pi_1(U^\nu, C^\nu)$ where U^ν is the intersection of the U^λ corresponding to all the 2-cells containing e (1 or 2 according to the situation of e). Using the condition on the surjectivity of the π_1, we have a homotopy rel $\{0,1\}$ to a path in (A^ν, C^ν). This homotopy gives $h_1|_{e\times\{1\}}$.

Step 3.- Extend $\alpha|_{K\times\{0\}} \cup h_1$ to $h : K \times I \to X$.

This is done using for each 2-cell e the retraction of $e \times I$ to $\partial e \times I \cup e \times 0$

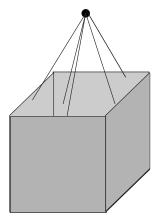


Figure 6.10: Projecting from above in a 3-cube

given by projecting from a point above the centre of the top face. □

The three steps in the construction of h in this Lemma are indicated in Figure 6.11 where the third and fourth diagrams look the same from this direction but from the back the third one looks like a hive with square cells while the fourth diagram is solid.

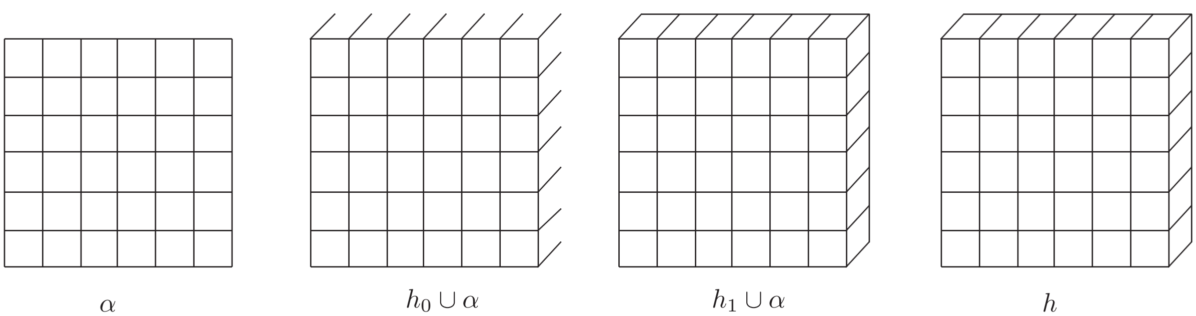


Figure 6.11: Steps in constructing h in Lemma 6.8.3

Notice that the connectivity result (Con) follows immediately from this lemma, particularly (iv), applied to doubly degenerate or to degenerate squares representing elements of an appropriate π_0 or π_1.

We can now define F for an arbitrary element $\alpha \in R_2(X, A, C)$ as follows. First we choose a subdivision $[\alpha_{ij}]$ of α such that for each i, j, α_{ij} lies in some U^{ij}. Then we apply Lemma 6.8.3 to get an element $\alpha' = [\alpha'_{ij}]$ and an f-homotopy $h : \alpha \equiv \alpha'$ decomposing as $h = [h_{ij}]$, the image of each h_{ij} lying in some U^{ij}.

We define

$$F(\alpha) = F(\alpha') = [F({\alpha'}_{ij})],$$

i.e the composition of the array in G got by applying the appropriate f^{λ} to the decomposition resulting on the back face of the last diagram in Figure 6.11. Since this in principle depends on the subdivision and the homotopy h we will sometimes write this element as $F(\alpha, (h_{ij}))$.

Stage 3.- Key lemmas

The tools for our independence of choices are going to be proved at this stage. They are two lemmas considering a homotopy H of maps in $\alpha, \beta \in R_2(X, A, C)$ with a given subdivision $H = [H_{ijk}]$. They are represented in the Figure 6.12.

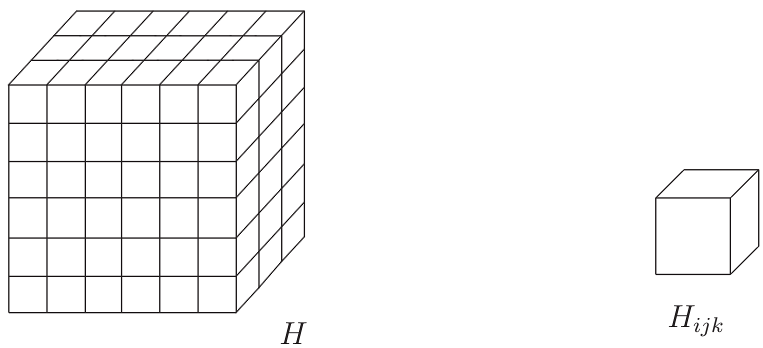


Figure 6.12: Decomposition of a homotopy $H = [H_{ijk}]$

The first lemma is a rather short application of previous results on commutative cubes and states that $F(\alpha) = F(\beta)$ gives particular conditions on α, β and on an f-homotopy $H : \alpha \equiv \beta$.

Lemma 6.8.4 *Let $H : I^3 \to X$ be an f-homotopy of maps $\alpha, \beta : (I^2, \partial I^2, \partial^2 I^2) \to (X, A, C)$. Suppose given a subdivision $H = [H_{ijk}]$ of H such that each H_{ijk} maps its domain D^{ijk} of I^3 into a set U^{ijk} of the cover and maps the vertices and edges of D^{ijk} into C and A respectively, i.e. all its faces lie in $R_2(U^{ijk}, A^{ijk}, C^{ijk})$. Then for the induced subdivisions $\alpha = [\alpha_{ij}], \beta = [\beta_{ij}]$ we have in G that*

(*) $$F(\alpha) = F(\beta).$$

Proof The assumptions imply that each H_{ijk} satisfy the conditions of the homotopy commutativity lemma (6.7.7) and thus defines a commutative 3-shell in $\rho(U^{ijk}, A^{ijk}, C^{ijk}))$. This is mapped by f^{ijk} to give a commutative 3-shell in G. The condition $f' i_1 = f' i_2$ implies that these 3-shells are composable in G, and so, by Theorem 6.7.6, their composition is a commutative cube in G. The assumption that H is an f-homotopy allows us to apply Theorem 6.7.5, and to deduce (*), as required. □

Now we have to prove that we can always obtain from a general f-homotopy between two maps an f-homotopy between associated maps that satisfies the conditions of the previous Lemma. This is where our connectivity assumptions are used again.

Lemma 6.8.5 *Let $H : I^3 \to X$ be an f-homotopy of maps $\alpha, \beta : (I^2, \partial I^2, \partial^2 I^2) \to (X, A, C)$. Suppose given a subdivision $H = [H_{ijk}]$ of H such that each H_{ijk} maps its domain D^{ijk} of I^3 into a set U^{ijk} of the cover. Then there is a homotopy Φ of H to a homotopy H' such that such that in the cell structure K determined by the subdivision of H,*

(i) *H' maps the 0-cells of K into C and the 1-cells into A;*

(ii) *if a 0-cell v of K is mapped by H into C, then Φ is constant on v, and if v is mapped into A by H, then so also is $v \times I$ by Φ;*

(iii) *if a 1-cell e of K is mapped by H into C, then Φ is constant on e, and if e is mapped into A by H, then so also is $e \times I$ by Φ.*

Proof As in Remark 6.3.2, but now in dimension 3, there is a cell structure K on I^3 appropriate to the subdivision of H. We define a homotopy $\Phi : K \times I \to X$ of H by induction on $K^n \times I \cup k \times \{0\} \subseteq K$. The first two steps are as in Lemma 6.8.3. This takes us up to $K^1 \times I \cup K \times \{0\}$. Finally, we extend Φ over the 2- and 3-skeleta of K by using retractions, i.e. by a careful use of the Homotopy Extension Property. □

Remark 6.8.6 The map H' constructed in the Lemma gives an f-homotopy from $\alpha' = H'_0$ to $\beta' = H'_1$. Also there is a decomposition of $\alpha' = [\alpha'_{ij}]$ and $\beta' = [\beta'_{ij}]$ which has each element lying in some $R_2(U^\lambda, A^\lambda, C^\lambda)$. Moreover, the homotopy Φ induces homotopies $h : \alpha \equiv \alpha'$ and $h' : \beta \equiv \beta'$ of the type described in Lemma 6.8.3 and later used to define $F(\langle\langle\alpha\rangle\rangle)$.

In particular, if all the maps in the induced subdivisions $\alpha = [\alpha_{ij}]$ and $\beta = [\beta_{ij}]$ lie in some $R_2(U^\lambda, A^\lambda, C^\lambda)$, the map H' constructed in the lemma gives an f-homotopy $H' : \alpha \equiv \beta$.

<u>Stage 4</u>.- Independence of choices inside the same f-homotopy class.

Now we can prove that f is well defined, proving independence of two choices.

1.- Independence of the subdivision and the homotopy h of Lemma 6.8.3.

Let us consider two subdivisions of the same map $\alpha \in R_2(X, A, C)$. As there is a common refinement we can assume that one is a refinement of the other. We shall write them $\alpha = [\alpha_{ij}]$ and $\alpha = [\alpha^{ij}_{kl}]$ where for a fixed ij we have $\alpha_{ij} = [\alpha^{ij}_{kl}]$.

Using Lemma 6.8.3, we get f-homotopies $h : \alpha \equiv \alpha'$, with $\alpha' \in R_2(X, A, C)$, such that, in the subdivision $h = [h_{ij}]$ determined by that of α, each homotopy $h_{ij} : \alpha_{ij} \simeq \alpha'_{ij}$ and $h' : \alpha \equiv \alpha''$, with $\alpha'' \in R_2(X, A, C)$, such that, in the subdivision $h' = [h'^{ij}_{kl}]$ determined by that of α, each homotopy $h'^{ij}_{kl} : \alpha^{ij}_{kl} \simeq \alpha''^{ij}_{kl}$. We want to prove that

$$[F(\alpha'_{ij})] = [F(\alpha''^{ij}_{kl})].$$

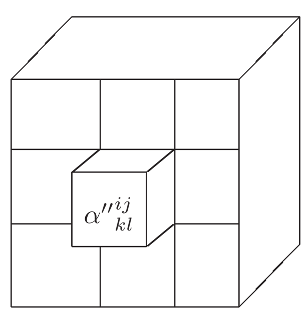


Figure 6.13: Independence of subdivision

The situation for a fixed ij is described in Figure 6.13 where the smaller cube at the front represents h'^{ij}_{kl} and the larger cube at the back is h_{ij}.

If we denote by h'_{ij} the composition of the array $h'_{ij} = [h'^{ij}_{kl}]$ and by α'_{ij} the composition of the array $\alpha'_{ij} = [\alpha'^{ij}_{kl}]$, we have $h'_{ij} : \alpha_{ij} \simeq \alpha''_{ij}$.

Now $\overline{h}h'$ gives an f-homotopy satisfying the conditions of Lemma 6.8.5 if we denote by $\overline{h}$ the homotopy given by $\overline{h}(x,y,t) = h(x,y,1-t)$. First, we change this homotopy using Lemma 6.8.5 and we then apply Lemma 6.8.4, to get

$$[F(\alpha'_{ij})]) = [F(\alpha''_{ij})].$$

On the other hand since the second is a refinement of the first, we have

$$[F(\alpha''_{ij})]) = [F(\alpha''^{ij}_{kl})].$$

As a consequence to define the element $F(\alpha)$ we can choose whatever subdivision and homotopy we want insofar as the conditions of Lemma 6.8.3 are met.

2.- Independence of the choice inside the same f-homotopy class.

Let $H : \alpha \equiv \beta$ be an f-homotopy of elements of $R_2(X, A, C)$. We choose a subdivision $H = [H_{ijk}]$ of H so that each H_{ijk} maps into a set of $\mathcal{U}$. On both extremes there are induced subdivisions $\alpha = [\alpha_{ij}], \beta = [\beta_{ij}]$. We apply Lemma 6.8.3 to H, getting $H' : \alpha' \equiv \beta'$.

As indicated in the Remark 6.8.6, these α', β' satisfy the conditions to be used when defining $F(\alpha)$ and $F(\beta)$. Also H' satisfies the conditions of Lemma 6.8.4. Thus

$$F(\alpha) = [F(\alpha'_{ij})] = [F(\beta'_{ij})] = F(\beta).$$

Stage 5.- End of proof

Now we have proved that there is a well-defined map $f : \rho(X, A, C)_2 \to G_2$, given by $f(\langle\langle\alpha\rangle\rangle) = F(\alpha, (h_{ij}))$, which satisfies $fi = f'$ at least on the 2-dimensional elements of ρ.

The remainder of the proof of (Iso), that is the verification that f is a morphism, and is the only such morphism, is straightforward. It is easy to check that f preserves addition and composition of squares, and it follows from (iii) of Lemma 6.8.3 that f preserves thin elements.

It is now easy to extend f to a morphism $f : \rho(X, A, C) \to G$ of double groupoids, since the 1- and 0-dimensional parts of a double groupoid determine degenerate 2-dimensional parts. Clearly this f satisfies $fi = f'$ and is the only such morphism.

This completes the proof of Theorem 6.8.2. □

Of especial interest (but not essentially easier to prove) is the case of the Theorem in which the cover $\mathcal{U}$ has only two elements; in this case Theorem 6.8.2 gives a push-out of double groupoids. In the applications in previous chapters we have considered only path-connected spaces and assumed that $C = \{x\}$ is a singleton. Taking x as base point, the double groupoids can then be interpreted as crossed modules of groups to give the 2-dimensional analogue of the Seifert-van Kampen theorem given as Theorem 2.3.1 earlier. We do not know how to prove that theorem without using groupoids in some form. A higher dimensional form of this proof and theorem is given in the second part of this book.

Proof of Theorem 2.3.1 In the case where (X, A) is a based pair with base point x, $\rho(X, A, x)$ is abbreviated to $\rho(X, A)$. That we obtain a pushout of crossed modules under the hypothesis of Theorem 2.3.1 is simply a special case of Theorem 6.8.2, together with Proposition 6.3.7, which gives the equivalence between double groupoids and crossed modules.

The corresponding result of Theorem 2.3.3 follows from Theorem 2.3.1 by standard techniques using mapping cylinders. For analogues of these techniques for the fundamental groupoid, see Chapter 8 of [30]. □

Remark 6.8.7 An examination of the proof of Theorem 6.8.2 shows that conditions $(\ddagger)_0$ and $(\ddagger)_1$ are required only for 8-fold intersections of elements of $\mathcal{U}$. However, it has been shown by Razak-Salleh [161] that in fact one need only assume $(\ddagger)_0$ for 4-fold intersections and $(\ddagger)_1$ for 3-fold intersections. Further, these conditions are best possible. The reader may like to try to recover these results using using the tool of Lebesgue covering dimension as in the paper [58].

Remark 6.8.8 Theorem 6.8.2 contains 1-dimensional information which includes most known results expressing the fundamental group of a space in terms of an open cover, but it does not assume that the spaces of the cover or their intersections are path-connected. That is, it contains the van Kampen theorem on $\pi_1(X, A)$ given in Chapter 1.

Thus we have completed the aims of Part I, to give a reasonably full and we hope comprehensible account of what we understand as 2-dimensional nonabelian algebraic topology, which is essentially the theory and application to algebraic topology of crossed modules, double groupoids and related structures.

Now in Part II we move on to the higher dimensional theory. The situation is more complicated because there are several generalisations of crossed modules and double groupoids, with applications to algebraic topology, basically in terms of crossed complexes, or in terms of crossed n-cubes of groups. The theory of crossed complexes is limited in its applications, because it starts as being a purely 'linear' theory. However, even this theory has advantages, in the range of applications, its relation to well known theories, such as chain complexes with a group of operators, its use of groupoids, and its intuitive basis as a development of the methods of Part I. So this is the account we give, in the space we have here.

Bibliography

[1] J. F. Adams, *Four applications of the self-obstruction invariants*, J. London Math. Soc. **31** (1956), 148–159.

[2] ______, *Infinite loop spaces*, Annals of Mathematics Studies, vol. 90, Princeton University Press, Princeton, N.J., 1978.

[3] ______, *Maps from a surface to the projective plane*, Bull. London Math. Soc. **14** (1982), no. 6, 533–534.

[4] F. A. Al-Agl, R. Brown, and R. Steiner, *Multiple categories: the equivalence of a globular and a cubical approach*, Adv. Math. **170** (2002), no. 1, 71–118.

[5] P. S. Aleksandrov and S. P. Finikov, *Eduard Čech: Obituary*, Uspehi Mat. Nauk **16** (1961), no. 1 (97), 119–126. (1 plate).

[6] M. Alp and C. D. Wensley, *Enumeration of* cat^1*-groups of low order*, Internat. J. Algebra Comput. **10** (2000), no. 4, 407–424.

[7] N. Ashley, *Simplicial* T*-complexes and crossed complexes: a nonabelian version of a theorem of Dold and Kan*, Dissertationes Math. (Rozprawy Mat.) **265** (1988), 1–61, With a preface by R. Brown.

[8] M. Barr and C. Wells, *Toposes, triples and theories*, Grundlehren der Mathematischen Wissenschaften, vol. 278, Springer-Verlag, New York, 1985.

[9] A. Bastiani and C. Ehresmann, *Categories of sketched structures*, Cahiers Topologie Géom. Différentielle **13** (1972), 105–214.

[10] H. J. Baues, *The homotopy type of 4-dimensional CW-complexes*, Preprint Max Plank Inst., 1986.

[11] ______, *Algebraic homotopy*, Cambridge Studies in Advanced Mathematics, vol. 15, Cambridge University Press, Cambridge, 1989.

[12] H. J. Baues and R. Brown, *On relative homotopy groups of the product filtration, the James construction, and a formula of Hopf*, J. Pure Appl. Algebra **89** (1993), no. 1-2, 49–61.

[13] H. J. Baues and D. Conduché, *The central series for Peiffer commutators in groups with operators*, J. Algebra **133** (1990), no. 1, 1–34.

[14] ______, *On the tensor algebra of a nonabelian group and applications*, K-Theory **5** (1991/92), no. 6, 531–554.

[15] A. J. Berrick, *Group extensions and their trivialisation*, Enseign. Math. (2) **31** (1985), no. 1-2, 151–172.

[16] A. L. Blakers, *Some relations between homology and homotopy groups*, Ann. of Math. (2) **49** (1948), 428–461.

[17] W. A. Bogley and M. A. Gutiérrez, *Mayer-Vietoris sequences in homotopy of* 2*-complexes and in homology of groups*, J. Pure Appl. Algebra **77** (1992), no. 1, 39–65.

[18] N. Bourbaki, *Éléments de mathématique. Théorie des ensembles*, Hermann, Paris, 1970.

[19] H. Brandt, *Uber eine verallgemeinerung des gruppenbegriffes*, Math. Ann. **96** (1926), no. 4, 360–366.

[20] K. S. Brown, *Cohomology of groups*, Graduate Texts in Mathematics, vol. 87, Springer-Verlag, New York, 1994, Corrected reprint of the 1982 original.

[21] R. Brown, *On Künneth suspensions*, Proc. Cambridge Philos. Soc. **60** (1964), 713–720.

[22] ———, *On a method of P. Olum*, J. London Math. Soc. **40** (1965), 303–304.

[23] ———, *Groupoids and van Kampen's theorem*, Proc. London Math. Soc. (3) **17** (1967), 385–401.

[24] ———, *Fibrations of groupoids*, J. Algebra **15** (1970), 103–132.

[25] ———, *On the second relative homotopy group of an adjunction space: an exposition of a theorem of J. H. C. Whitehead*, J. London Math. Soc. (2) **22** (1980), no. 1, 146–152.

[26] ———, *The homotopy classification of maps from a surface to the projective plane*, The Pure Math. Preprint, vol. March, UOW, Bangor, 1982.

[27] ———, *Coproducts of crossed P-modules: applications to second homotopy groups and to the homology of groups*, Topology **23** (1984), no. 3, 337–345.

[28] ———, *Nonabelian cohomology and the homotopy classification of maps*, Algebraic homotopy and local algebra (Luminy, 1982), Astérisque, vol. 113, Soc. Math. France, Paris, 1984, pp. 167–172.

[29] ———, *From groups to groupoids: a brief survey*, Bull. London Math. Soc. **19** (1987), no. 2, 113–134.

[30] ———, *Topology*, second ed., Ellis Horwood Series: Mathematics and its Applications, Ellis Horwood Ltd., Chichester, 1988, A geometric account of general topology, homotopy types and the fundamental groupoid.

[31] ———, *Higher order symmetry of graphs*, Irish Math. Soc. Bull. (1994), no. 32, 46–59.

[32] R. Brown and G. J. Ellis, *Hopf formulae for the higher homology of a group*, Bull. London Math. Soc. **20** (1988), no. 2, 124–128.

[33] R. Brown and M. Golasiński, *A model structure for the homotopy theory of crossed complexes*, Cahiers Topologie Géom. Différentielle Catég. **30** (1989), no. 1, 61–82.

[34] R. Brown, M. Golasiński, T. Porter, and A. Tonks, *Spaces of maps into classifying spaces for equivariant crossed complexes*, Indag. Math. (N.S.) **8** (1997), no. 2, 157–172.

[35] ———, *Spaces of maps into classifying spaces for equivariant crossed complexes. II. The general topological group case*, K-Theory **23** (2001), no. 2, 129–155.

[36] R. Brown, K. A. Hardie, K. H. Kamps, and T. Porter, *A van Kampen theorem for the homotopy double groupoid of a Hausdorff space*, Preprint.

[37] R. Brown and P. R. Heath, *Coglueing homotopy equivalences*, Math. Z. **113** (1970), 313–325.

[38] R. Brown, P. R. Heath, and K. H. Kamps, *Coverings of groupoids and Mayer-Vietoris type sequences*, Categorical topology (Toledo, Ohio, 1983), Sigma Ser. Pure Math., vol. 5, Heldermann, Berlin, 1984, pp. 147–162.

[39] R. Brown and P. J. Higgins, *On the connection between the second relative homotopy groups of some related spaces*, Proc. London Math. Soc. (3) **36** (1978), no. 2, 193–212.

[40] ______, *Sur les complexes croisés d'homotopie associés à quelques espaces filtrés*, C. R. Acad. Sci. Paris Sér. A-B **286** (1978), no. 2, A91–A93.

[41] ______, *Colimit theorems for relative homotopy groups*, J. Pure Appl. Algebra **22** (1981), no. 1, 11–41.

[42] ______, *The equivalence of ∞-groupoids and crossed complexes*, Cahiers Topologie Géom. Différentielle **22** (1981), no. 4, 371–386.

[43] ______, *The equivalence of ω-groupoids and cubical T-complexes*, Cahiers Topologie Géom. Différentielle **22** (1981), no. 4, 349–370.

[44] ______, *On the algebra of cubes*, J. Pure Appl. Algebra **21** (1981), no. 3, 233–260.

[45] ______, *Crossed complexes and nonabelian extensions*, Int. Conf. on Category theory (Gummersbach, 1981), Lecture Notes in Math., vol. 962, Springer, Berlin, 1982, pp. 39–50.

[46] ______, *Tensor products and homotopies for ω-groupoids and crossed complexes*, J. Pure Appl. Algebra **47** (1987), no. 1, 1–33.

[47] ______, *Crossed complexes and chain complexes with operators*, Math. Proc. Cambridge Philos. Soc. **107** (1990), no. 1, 33–57.

[48] ______, *The classifying space of a crossed complex*, Math. Proc. Cambridge Philos. Soc. **110** (1991), no. 1, 95–120.

[49] R. Brown and J. Huebschmann, *Identities among relations*, Low-dimensional topology (Bangor, 1979), London Math. Soc. Lecture Note Ser., vol. 48, Cambridge Univ. Press, Cambridge, 1982, pp. 153–202.

[50] R. Brown and G. Janelidze, *Galois theory and a new homotopy double groupoid of a map of spaces*, App. Cat. Struct. **12** (2004), 63–80.

[51] R. Brown and J.-L. Loday, *Excision homotopique en basse dimension*, C. R. Acad. Sci. Paris Sér. I Math. **298** (1984), no. 15, 353–356.

[52] ______, *Homotopical excision, and Hurewicz theorems for n-cubes of spaces*, Proc. London Math. Soc. (3) **54** (1987), no. 1, 176–192.

[53] ______, *Van Kampen theorems for diagrams of spaces*, Topology **26** (1987), no. 3, 311–335, With an appendix by M. Zisman.

[54] R. Brown and K. C. H. Mackenzie, *Determination of a double Lie groupoid by its core diagram*, J. Pure Appl. Algebra **80** (1992), no. 3, 237–272.

[55] R. Brown, E. J. Moore, and C. D. Wensley, *Free crossed resolutions for the fundamental groupoid of a graph of groups*, In preparation.

[56] R. Brown and G. H. Mosa, *Double categories, R-categories and crossed modules*, University College of North Wales preprint 88.11.

[57] R. Brown and T. Porter, *On the Schreier theory of non-abelian extensions: generalisations and computations*, Proc. Roy. Irish Acad. Sect. A **96** (1996), no. 2, 213–227.

[58] R. Brown and A. Razak Salleh, *A van Kampen theorem for unions on nonconnected spaces*, Arch. Math. (Basel) **42** (1984), no. 1, 85–88.

[59] ______, *Free crossed resolutions of groups and presentations of modules of identities among relations*, LMS J. Comput. Math. **2** (1999), 28–61 (electronic).

[60] R. Brown and C. B. Spencer, *Double groupoids and crossed modules*, Cahiers Topologie Géom. Différentielle **17** (1976), no. 4, 343–362.

[61] ______, *G-groupoids, crossed modules and the fundamental groupoid of a topological group*, Nederl. Akad. Wetensch. Proc. Ser. A **79**=Indag. Math. **38** (1976), no. 4, 296–302.

[62] R. Brown and C. D. Wensley, *On finite induced crossed modules, and the homotopy* 2*-type of mapping cones*, Theory Appl. Categ. **1** (1995), No. 3, 54–70 (electronic).

[63] ______, *Computing crossed modules induced by an inclusion of a normal subgroup, with applications to homotopy* 2*-types*, Theory Appl. Categ. **2** (1996), No. 1, 3–16 (electronic).

[64] ______, *Computation and homotopical applications of induced crossed modules*, J. Symbolic Comput. **35** (2003), no. 1, 59–72.

[65] E. Čech, *Höherdimensionale homotopiegruppen*, Verdandlungen des Internationalen Mathematiker-Kongresses Zurich, Band 2, 1932.

[66] R. B. Coates, *Semantics of generalised algebraic structures*, Ph.D. thesis, University of London, 1974.

[67] D. E. Cohen, *Combinatorial group theory: a topological approach*, London Mathematical Society Student Texts, vol. 14, Cambridge University Press, Cambridge, 1989.

[68] M. M. Cohen, *A course in simple-homotopy theory*, Graduate Texts in Mathematics, vol. 10, Springer-Verlag, New York, 1973.

[69] D. Conduché, *Modules croisés généralisés de longueur* 2, Proceedings of the Luminy conference on algebraic K-theory (Luminy, 1983), vol. 34, 1984, pp. 155–178.

[70] R. H. Crowell, *Corresponding group and module sequences*, Nagoya Math. J. **19** (1961), 27–40.

[71] ______, *The derived module of a homomorphism*, Advances in Math. **6** (1971), 210–238 (1971).

[72] R. H. Crowell and R. H. Fox, *Introduction to knot theory*, Graduate Texts in Mathematic, vol. 57, Springer-Verlag, New York, 1977, Reprint of the 1963 original.

[73] M. K. Dakin, *Kan complexes and multiple groupoid structures*, Esquisses Math. **32** (1983), With a preface by R. Brown.

[74] R. K. Dennis, *In search of new "homology" functors having a close relationship to K-theory*, Preprint Cornell University (1976).

[75] J. Dieudonné, *A history of algebraic and differential topology. 1900–1960*, Birkhäuser Boston Inc., Boston-Basel, MA, 1989.

[76] A. Dold, *Homology of symmetric products and other functors of complexes*, Ann. of Math. (2) **68** (1958), 54–80.

[77] J. Duskin, *Simplicial methods and the interpretation of "triple" cohomology*, Mem. Amer. Math. Soc. **3** (1975), no. issue 2, 163, v+135.

[78] C. Ehresmann, *Esquisses et types des structures algébriques*, Bul. Inst. Politehn. Iaşi (N.S.) **14 (18)** (1968), no. fasc. 1-2, 1–14.

[79] ______, *Œuvres complètes et commentées. I-1,2. Topologie algébrique et géométrie différentielle*, Cahiers Topologie Géom. Différentielle **24** (1983), no. suppl. 1, xxix+601 pp. (1984), With commentary by W. T. van Est, M. Zisman, G. Reeb, P. Libermann, R. Thom, J. Pradines, R. Hermann, A. Kock, A. Haefliger, J. Bénabou, R. Guitart, and A. C. Ehresmann, Edited by A. C. Ehresmann.

[80] S. Eilenberg and S. Mac Lane, *Cohomology theory in abstract groups. I*, Ann. of Math. (2) **48** (1947), 51–78.

[81] S. Eilenberg and S. MacLane, *Relations between homology and homotopy groups of spaces*, Ann. of Math. (2) **46** (1945), 480–509.

[82] G. J. Ellis, *An application of some theorems of J. H. C. Whitehead*, M.Sc. dissertation, University of Wales, 1983.

[83] ______, *Crossed modules and their higher dimensional analogues*, Ph.D. thesis, University of Wales, 1984.

[84] ______, *The group $K_2(\Lambda; I_1, \cdots, I_n)$ and related computations*, J. Algebra **112** (1988), no. 2, 271–289.

[85] ______, *Homotopy classification the J. H. C. Whitehead way*, Exposition. Math. **6** (1988), no. 2, 97–110.

[86] M. Évrard, *Homotopie des complexes simpliciaux et cubiques*, Preprint.

[87] H. Federer, *Lectures in algebraic topology*, Lecture notes, Brown University, Providence, R. I., 1962.

[88] R. H. Fox, *Free differential calculus. I. Derivation in the free group ring*, Ann. of Math. (2) **57** (1953), 547–560.

[89] P. Freyd, *Aspect of topoi*, Bull. Austral. Math. Soc. **7** (1972), 1–76.

[90] P. Gabriel and F. Ulmer, *Lokal präsentierbare Kategorien*, Lecture Notes in Mathematics, vol. 221, Springer-Verlag, Berlin, 1971.

[91] P. Gabriel and M. Zisman, *Calculus of fractions and homotopy theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 35, Springer-Verlag New York, Inc., New York, 1967.

[92] R.B. Gardner, *Review of "Invariant theory", by J.A. Springer*, Bull. Amer. Math. Soc. **2** (1980), 246–256.

[93] N. D. Gilbert, *On the fundamental* cat^n*-group of an n-cube of spaces*, Algebraic Topology, Barcelona, 1986, Lecture Notes in Math., vol. 1298, Springer, Berlin, 1987, pp. 124–139.

[94] N. D. Gilbert and P. J. Higgins, *The nonabelian tensor product of groups and related constructions*, Glasgow Math. J. **31** (1989), no. 1, 17–29.

[95] S. Gitler, *Cohomology operations with local coefficients*, Amer. J. Math. **85** (1963), 156–188.

[96] M. Golasiński, *Homotopy groups of small categories and derived functors*, Cahiers Topologie Géom. Différentielle Catég. **28** (1987), no. 2, 89–97.

[97] D. H. Gottlieb, *Covering transformations and universal fibrations*, Illinois J. Math. **13** (1969), 432–437.

[98] A. Grothendieck, *Catégories cofibrées additives et complexe cotangent relatif*, Lecture Notes in Mathematics, vol. 79, Springer-Verlag, Berlin, 1968.

[99] GAP Group, *Gap – groups, algorithms, and programming, version 4.3, manual*, 2002, (`http://www.gap-system.org`).

[100] K. W. Gruenberg, *Residual properties of infinite soluble groups*, Proc. London Math. Soc. (3) **7** (1957), 29–62.

[101] K. W. Gruenberg and K. W. Roggenkamp, *Extension categories of groups and modules. II. Stem extensions*, J. Algebra **67** (1980), no. 2, 342–368.

[102] D. Guin-Waléry and J.-L. Loday, *Obstruction à l'excision en K-théorie algébrique*, Algebraic K-theory, Evanston 1980 (Proc. Conf., Northwestern Univ., Evanston, Ill., 1980), Lecture Notes in Math., vol. 854, Springer, Berlin, 1981, pp. 179–216.

[103] V. L. Hansen, *Spaces of maps into Eilenberg-Mac Lane spaces*, Canad. J. Math. **33** (1981), no. 4, 782–785.

[104] A. Heyworth and B. Reinert, *Applications of Gröbner bases to group rings and identities among relations*, UOW, Bangor Maths. Preprint 99.09.

[105] A. Heyworth and C.D. Wensley, *Loggged Knut-Bendix procedures and identities among relations*, UOW, Bangor Maths. Preprint 99.07.

[106] P. J. Higgins, *Presentations of groupoids, with applications to groups*, Proc. Cambridge Philos. Soc. **60** (1964), 7–20.

[107] ______, *Notes on categories and groupoids*, Mathematical Studies, vol. 32, Van Nostrand Reinhold Co., London, 1971.

[108] ______, *The fundamental groupoid of a graph of groups*, J. London Math. Soc. (2) **13** (1976), no. 1, 145–149.

[109] S. Hintze, *Polysets, $\square$-sets and semi-cubical sets*, M.Phil thesis, University of Warwick, 1973.

[110] C. Hog-Angeloni and W. Metzler (eds.), *Two-dimensional homotopy and combinatorial group theory*, London Mathematical Society Lecture Note Series, vol. 197, Cambridge University Press, Cambridge, 1993.

[111] D. F. Holt, *An interpretation of the cohomology groups $H^n(G, M)$*, J. Algebra **60** (1979), no. 2, 307–320.

[112] H. Hopf, *Fundamentalgruppe und zweite Bettische Gruppe*, Comment. Math. Helv. **14** (1942), 257–309.

[113] J. Howie, *Pullback functors and crossed complexes*, Cahiers Topologie Géom. Différentielle **20** (1979), no. 3, 281–296.

[114] S.-T. Hu, *Homotopy theory*, Pure and Applied Mathematics, Vol. VIII, Academic Press, New York, 1959.

[115] J. Huebschmann, *Verschränkte n-fache erweiterungen von gruppen und cohomologie*, Ph.D. thesis, Diss. Eidg. Techn. Hochschule, Zurich, 1977.

[116] ______, *Crossed n-fold extensions of groups and cohomology*, Comment. Math. Helv. **55** (1980), no. 2, 302–313.

[117] S. Jajodia, *Homotopy classification of lens spaces for one-relator groups with torsion*, Pacific J. Math. **89** (1980), no. 2, 301–311.

[118] D. L. Johnson, *Presentations of groups*, second ed., London Mathematical Society Student Texts, vol. 15, Cambridge University Press, Cambridge, 1997.

[119] K. H. Kamps, *Kan-Bedingungen und abstrakte Homotopiethoerie*, Math. Z. **124** (1972), 215–236.

[120] K. H. Kamps and T. Porter, *Abstract homotopy and simple homotopy theory*, World Scientific Publishing Co. Inc., River Edge, NJ, 1997.

[121] D. M. Kan, *Abstract homotopy. I*, Proc. Nat. Acad. Sci. U.S.A. **41** (1955), 1092–1096.

[122] ———, *Functors involving c.s.s. complexes*, Trans. Amer. Math. Soc. **87** (1958), 330–346.

[123] G. M. Kelly, *Basic concepts of enriched category theory*, London Mathematical Society Lecture Note Series, vol. 64, Cambridge University Press, Cambridge, 1982.

[124] M. Kneser, M. Ojanguren, M.-A. Knus, R. Parimala, and R. Sridharan, *Composition of quaternary quadratic forms*, Compositio Math. **60** (1986), no. 2, 133–150.

[125] F. J. Korkes and T. Porter, *Continuous derivations, profinite crossed complexes and pseudocompact chain complexes*, U.C.N.W. Pure Math. Preprint **87.18**.

[126] F. W. Levi, *The commutator group of a free product*, J. Indian Math. Soc. **4** (1940), 136–144.

[127] J.-L. Loday, *Cohomologie et groupe de Steinberg relatifs*, J. Algebra **54** (1978), no. 1, 178–202.

[128] ———, *Spaces with finitely many nontrivial homotopy groups*, J. Pure Appl. Algebra **24** (1982), no. 2, 179–202.

[129] S. J. Lomonaco, Jr., *The homotopy groups of knots. I. How to compute the algebraic* 2*-type*, Pacific J. Math. **95** (1981), no. 2, 349–390.

[130] A. S.-T. Lue, *Cohomology of groups relative to a variety*, J. Algebra **69** (1981), no. 1, 155–174.

[131] R. C. Lyndon and P. E. Schupp, *Combinatorial group theory*, Classics in Mathematics, Springer-Verlag, Berlin, 2001, Reprint of the 1977 edition.

[132] S. Mac Lane, *Homology*, Die Grundlehren der mathematischen Wissenschaften, vol. 114, Academic Press Inc., Publishers, New York, 1963.

[133] ———, *Categories for the working mathematician*, Graduate Texts in Mathematics, vol. 5, Springer-Verlag, New York, 1971.

[134] ———, *Origins of the cohomology of groups*, Enseign. Math. (2) **24** (1978), no. 1-2, 1–29.

[135] ———, *Historical note*, J. Algebra **69** (1979), 319–320.

[136] S. Mac Lane and I. Moerdijk, *Topos theory*, Handbook of algebra, Vol. 1, North-Holland, Amsterdam, 1996, pp. 501–528.

[137] S. Mac Lane and J. H. C. Whitehead, *On the* 3*-type of a complex*, Proc. Nat. Acad. Sci. U. S. A. **36** (1950), 41–48.

[138] K. Mackenzie, *Lie groupoids and Lie algebroids in differential geometry*, London Mathematical Society Lecture Note Series, vol. 124, Cambridge University Press, Cambridge, 1987.

[139] ———, *Classification of principal bundles and Lie groupoids with prescribed gauge group bundle*, J. Pure Appl. Algebra **58** (1989), no. 2, 181–208.

[140] W. Magnus, A. Karrass, and D. Solitar, *Combinatorial group theory: Presentations of groups in terms of generators and relations*, Interscience Publishers [John Wiley & Sons, Inc.], New York-London-Sydney, 1966.

[141] W. S. Massey, *Singular homology theory*, Graduate Texts in Mathematics, vol. 70, Springer-Verlag, New York, 1980.

[142] J. P. May, *Simplicial objects in algebraic topology*, Mathematical Studies, vol. 11, D. Van Nostrand Co., Inc., Princeton, N.J.-Toronto, Ont.-London, 1967.

[143] ———, *The geometry of iterated loop spaces*, Lectures Notes in Mathematics, vol. 271, Springer-Verlag, Berlin, 1972.

[144] ———, *E_∞ ring spaces and E_∞ ring spectra*, Lecture Notes in Mathematics, vol. 577, Springer-Verlag, Berlin, 1977, With contributions by F. Quinn, N. Ray, and J. Tornehave.

[145] ———, *Infinite loop space theory*, Bull. Amer. Math. Soc. **83** (1977), no. 4, 456–494.

[146] ———, *Multiplicative infinite loop space theory*, J. Pure Appl. Algebra **26** (1982), no. 1, 1–69.

[147] J. F. McClendon, *Obstruction theory in fiber spaces*, Math. Z. **120** (1971), 1–17.

[148] J. M. McPherson, *On the nullity and enclosure genus of wild knots*, Trans. Amer. Math. Soc. **144** (1969), 545–555.

[149] J. Milnor, *Introduction to algebraic K-theory*, Annals of Mathematics Studies, vol. 72, Princeton University Press, Princeton, N.J., 1971.

[150] E.J. Moore, *Graphs of groups: word computations and free crossed resolutions*, Ph.D. thesis, University of Wales, 2001.

[151] G. Nan Tie, *A Dold-Kan theorem for crossed complexes*, J. Pure Appl. Algebra **56** (1989), no. 2, 177–194.

[152] ———, *Iterated $\overline{W}$ and T-groupoids*, J. Pure Appl. Algebra **56** (1989), no. 2, 195–209.

[153] P. Olum, *Mappings of manifolds and the notion of degree*, Ann. of Math. (2) **58** (1953), 458–480.

[154] R. Peiffer, *Über Identitäten zwischen Relationen*, Math. Ann. **121** (1949), 67–99.

[155] S. P. Plotnick and A. I. Suciu, *k-invariants of knotted 2-spheres*, Comment. Math. Helv. **60** (1985), no. 1, 54–84.

[156] H. Poincaré, *Œuvres. Tome VI*, Les Grands Classiques Gauthier-Villars., Éditions Jacques Gabay, Sceaux, 1996, Géométrie. Analysis situs (topologie), Reprint of the 1953 edition.

[157] T. Porter, *Some categorical results in the theory of crossed modules in commutative algebras*, J. Algebra **109** (1987), no. 2, 415–429.

[158] ———, *n-types of simplicial groups and crossed n-cubes*, Topology **32** (1993), no. 1, 5–24.

[159] D. G. Quillen, *Homotopical algebra*, Lecture Notes in Mathematics, vol. 43, Springer-Verlag, Berlin, 1967.

[160] ______, *The geometric realization of a Kan fibration is a Serre fibration*, Proc. Amer. Math. Soc. **19** (1968), 1499–1500.

[161] A. Razak Salleh, *Union theorems for double groupoids and groupoids; some generalisations and applications*, Ph.D. thesis, University of Wales, 1976.

[162] K. Reidemeister, *Über Identitäten von Relationen*, Abh. Math. Sem. Univ. Hamburg **16** (1949), 114–118.

[163] B. Schellenberg, *The group of homotopy self-equivalences of some compact CW-complexes*, Math. Ann. **200** (1973), 253–266.

[164] J. Siegel, *Higher order cohomology operations in local coefficient theory*, Amer. J. Math. **89** (1967), 909–931.

[165] C. B. Spencer, *An abstract setting for homotopy pushouts and pullbacks*, Cahiers Topologie Géom. Différentielle **18** (1977), no. 4, 409–429.

[166] M. Steinberger and J. West, *Covering homotopy properties of maps between CW-complexes or ANRs*, Proc. Amer. Math. Soc. **92** (1984), no. 4, 573–577.

[167] J. Taylor, *Group actions on ω-groupoids and crossed complexes, and the homotopy groups of orbit spaces.*, Ph.D. thesis, University of Durham, 1982.

[168] R. L. Taylor, *Compound group extensions. I. Continuations of normal homomorphisms*, Trans. Amer. Math. Soc. **75** (1953), 106–135.

[169] ______, *Covering groups of nonconnected topological groups*, Proc. Amer. Math. Soc. **5** (1954), 753–768.

[170] R. Thom, *L'homologie des espaces fonctionnels*, Colloque de topologie algébrique, Louvain, 1956, Georges Thone, Liège, 1957, pp. 29–39.

[171] T. tom Dieck, K. H. Kamps, and D. Puppe, *Homotopietheorie*, Lecture Notes in Mathematics, vol. 157, Springer-Verlag, Berlin, 1970.

[172] A. Tonks, *Theory and applications of crossed complexes*, Ph.D. thesis, University of Wales, 1994.

[173] A. P. Tonks, *On the Eilenberg-Zilber theorem for crossed complexes*, J. Pure Appl. Algebra **179** (2003), no. 1-2, 199–220.

[174] A. Weinstein, *Groupoids: unifying internal and external symmetry. A tour through some examples*, Groupoids in analysis, geometry, and physics (Boulder, CO, 1999), Contemp. Math., vol. 282, Amer. Math. Soc., Providence, RI, 2001, pp. 1–19.

[175] G. W. Whitehead, *Elements of homotopy theory*, Graduate Texts in Mathematics, vol. 61, Springer-Verlag, New York, 1978.

[176] J. H. C. Whitehead, *On adding relations to homotopy groups*, Ann. of Math. (2) **42** (1941), 409–428.

[177] ______, *Note on a previous paper entitled "On adding relations to homotopy groups."*, Ann. of Math. (2) **47** (1946), 806–810.

[178] ______, *Combinatorial homotopy. I*, Bull. Amer. Math. Soc. **55** (1949), 213–245.

[179] ______, *Combinatorial homotopy. II*, Bull. Amer. Math. Soc. **55** (1949), 453–496.

[180] ______, *A certain exact sequence*, Ann. of Math. (2) **52** (1950), 51–110.