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CAUCHY CHARACTERIZATION OF ENRICHED CATEGORIES

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Preface to the reprinted edition

Soon after the appearance of enriched category theory in the sense of Eilenberg-Kelly[1], I wondered whether $\mathcal{V}$-categories could be the same as $\mathcal{W}$-categories for non-equivalent monoidal categories $\mathcal{V}$ and $\mathcal{W}$. It was not until my four-month sabbatical in Milan at the end of 1981 that I made a serious attempt to properly formulate this question and try to solve it.

By this time I was very impressed by the work of Bob Walters [28] showing that sheaves on a site were enriched categories. On sabbatical at Wesleyan University (Middletown) in 1976-77, I had looked at a preprint of Denis Higgs showing that sheaves on a Heyting algebra H could be viewed as some kind of H-valued sets. The latter seemed to be understandable as enriched categories without identities. Walters' deeper explanation was that they were enriched categories (with identities) except that the base was not H but rather a bicategory built from H. A stream of research was initiated in which the base monoidal category for enrichment was replaced, more generally, by a base bicategory.

In analysis, Cauchy complete metric spaces are often studied as completions of more readily defined metric spaces. Bill Lawvere [15] had found that Cauchy completeness could be expressed for general enriched categories with metric spaces as a special case. Cauchy sequences became left adjoint modules[2] and convergence became representability. In Walters' work it was the Cauchy complete enriched categories that were the sheaves.

It was natural then to ask, rather than my original question, whether Cauchy complete $\mathcal{V}$-categories were the same as Cauchy complete $\mathcal{W}$-categories for appropriate base bicategories $\mathcal{V}$ and $\mathcal{W}$. I knew already [20] that the bicategory $\mathcal{V}$-Mod whose morphisms were modules between $\mathcal{V}$-categories could be constructed from the bicategory whose morphisms were $\mathcal{V}$-functors. So the question became: given a base bicategory $\mathcal{V}$, for which

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[1] S. Eilenberg and G.M. Kelly, Closed categories, *Proceedings of the Conference on Categorical Algebra at La Jolla* (Springer, 1966) 421-562.

[2] I like to use 'module from A to B' in place of 'left A-, right B-module'.

base bicategories $\mathcal{W}$ do we have a biequivalence

$$\mathcal{V}\text{-Mod} \sim \mathcal{W}\text{-Mod?}$$

This paper characterizes those bicategories $\mathcal{M}$ biequivalent to ones of the form $\mathcal{W}$-Mod for a base bicategory $\mathcal{W}$ with a small set of objects. The possible $\mathcal{W}$ are those biequivalent to full subbicategories of $\mathcal{M}$ whose objects form a 'small cauchy generator'.

Soon after, it was realized[3] that the bicategory $\mathcal{W}$-Mod of small $\mathcal{W}$-categories and $\mathcal{W}$-modules could be defined whether $\mathcal{W}$ were small or not and, if you omitted the small cauchy generator requirement, that the idempotency property

$$(\mathcal{W}\text{-Mod})\text{-Mod} \sim \mathcal{W}\text{-Mod}$$

could be separated out from my characterization theorem. This idempotency is not only reminiscent of Giraud's theorem

$$\mathrm{Sh}(\mathrm{Sh}(C)) \sim \mathrm{Sh}(C)$$

for a site C but also of the idempotency of Cauchy completion. Traditionally, two $\mathcal{V}$-categories are Morita equivalent if and only if their Cauchy completions are equivalent. So my question about a biequivalence $\mathcal{V}$-Mod $\sim$ $\mathcal{W}$-Mod is a higher-dimensional version of Morita equivalence.

Proposition 1 of the present paper brings out how the existence of colimits in the homs of a bicategory is a structural form of additivity. Not only do coproducts become products but all lax colimits become lax limits. These ideas were developed nicely by Richard Wood.[4]

The 2-category $(\mathcal{W}\text{-Cat})_{cc}$ of Cauchy complete $\mathcal{W}$-categories and $\mathcal{W}$-functors can be obtained up to biequivalence from $\mathcal{W}$-Mod by restricting to left-adjoint morphisms. In further work[5], the question of abstractly characterizing bicategories of the form $(\mathcal{W}\text{-Cat})_{cc}$ was addressed.

I would like to warmly thank Vacation Scholars James Douglas and Rony Kirollos for retyping the paper using modern techniques, and Michael Barr for perfecting their work. In the process some of the original typographical errors were corrected.

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[3]A. Carboni,S. Kasangian and R. Walters, An axiomatics for bicategories of modules, *J.Pure Appl. Algebra* **45** (1987) 127-141.

[4]R.J. Wood, Proarrows. II, *Cahiers Topologie Géom.Differentielle Catég.* **26** (1985) 135-168.

[5]A. Carboni, S. Johnson, R. Street and D. Verity, Modulated bicategories, *J. Pure Appl. Algebra* **94** (1994) 229-282.

Sunto. Si caratterizzano le bicategorie di moduli fra categorie arricchite, a meno di biequivalenze. Tale caratterizzazione permette di introdurre la nozione di cosmos (non elementare), e di mostrare che risulta chiusa rispetto a numerose costruzioni.

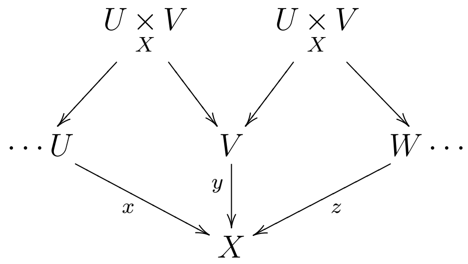
Abstract. A characterization is given of those bicategories which are biequivalent to bicategories of modules for some suitable base. These bicategories are the correct (non elementary) notion of cosmos, which is shown to be closed under several basic constructions.

1. Introduction

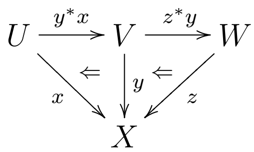
In developing set theory one has a choice as to whether functions or relations should be taken as primitive. Starting with the category Set of sets and functions, one defines a relation from A to B to be a subobject of $B \times A$. Each relation from A to B has a reverse relation from B to A. There is a bicategory Rel whose objects are sets, whose arrows are relations, and whose 2-cells are inclusions. This construction of Rel from Set requires little of the structure available in Set; from any category $\mathcal{E}$ which is finitely complete, has strong epic-monic factorizations, and, has strong epics stable under pullback (that is, a *regular* category in the sense of Barr [2]), one obtains a bicategory Rel $\mathcal{E}$ which has a reversing involution on arrows.

On the other hand, one can start with the bicategory of relations. Axioms have been determined [12], [26], [9] on a bicategory $\mathcal{R}$ with involutions which force it to be biequivalent to Rel $\mathcal{E}$ for a regular category $\mathcal{E}$. Functions from A to B are relations from A to B which have right-adjoint relations; it is not necessary to specify that the involution provides the adjoint as has been done in the literature. It is the author's attitude that the involution should not be taken as an extra datum; the bicategory $\mathcal{R}$ should suffice. When relations are taken as primitive the concepts of set theory have categorical formulations. Monic (respectively, epic) functions are relations with right adjoint for which the unit (respectively, counit) is an identity. Equivalence relations are symmetric monads in the bicategory of relations.

It is of interest to contemplate how extra structure on $\mathcal{E}$ exhibits itself in Rel $\mathcal{E}$. For example, what if $\mathcal{E}$ is a Grothendieck topos? Apart from a generator condition this means that $\mathcal{E}$ satisfies non-elementary extensions of the regularity properties: each family of arrows with the same target factors into a strong epic family and a monic, and, strong epic families are stable under pullback. If $\mathcal{F}$ is a strong epic family of arrows into X in $\mathcal{E}$ and we form the diagram obtained by pulling back pairs of arrows in $\mathcal{F}$ then X is actually the colimit of that diagram.



In Rel$\mathcal{E}$, these pullbacks $U \underset{X}{\times} V$ are arrows $y^*x : U \to V$ formed by making use of the right adjoints y^* for the members y of F. The relevant diagram in Rel $\mathcal{E}$, is as follows, where the 2-cells are induced by the counits of the adjunctions.



In the present paper our interest is in category theory rather than set theory. Previous works [19], [25], [20] have taken the 2-category Cat of categories and functors as primitive and have considered the problem of constructing the bicategory Mod of categories and modules. For categories A, B, a module r from A to B consists of sets $r(b, a)$ indexed by the objects a, b of A, B together with actions:

$$\varrho : B(b', b) \times r(b, a) \to r(b', a), \qquad \lambda : r(b, a) \times A(a, a') \to r(b, a'),$$

which are compatible with each other and with the compositions and identities in A, B.

A little more is lost in the passage from Cat to Mod than in the passage from Set to Rel. An arrow $A \to B$ with a right adjoint in Mod amounts to a functor from A into the idempotent completion of B [11; Chapter 2, Exercise B.2]. Our approach in the present paper is to start with the bicategory Mod so that all we can hope to describe about categories are properties invariant under idempotent completion.

Walters [28] has shown that sheaves on a site are closely related to hom-enriched categories. The hom-enrichment takes place in a bicategory determined by the site. This has led to an extension of the classical theory [15] of categories enriched in a monoidal category to categories enriched in a bicategory $\mathcal{W}$ [8], [23].

Let $\mathcal{W}$ denote a bicategory whose hom-categories have small colimits preserved by composition (that is, $\mathcal{W}$ is locally small-cocomplete) and whose objects form a small set. The 2-category $\mathcal{W}$-Cat of small $\mathcal{W}$-categories and $\mathcal{W}$-functors, and the bicategory $\mathcal{W}$-Mod of small $\mathcal{W}$-categories and $\mathcal{W}$-modules, were both studied in [8].

The purpose of the present paper is to characterize those bicategories $\mathcal{W}$ which are biequivalent to $\mathcal{W}$-Mod for some $\mathcal{W}$ as above.

The remarkable aspect of this characterization is its resemblance to Giraud's characterization [1] of Grothendieck topos. The primitive global constructions are (bicategorical) coproducts and the kleisli constructions on a monad [17]. (The latter is analogous to the quotient of an equivalence relation in a topos.) There is a local cocompleteness condition and a generator requirement. This suggests that bicategories $\mathcal{M}$ satisfying the conditions are the correct (non-elementary) notion of *cosmos*.

Note that $\mathcal{W}$-categories with equivalent cauchy completions determine equivalent objects of $\mathcal{W}$-Mod, so our theorem does not characterize $\mathcal{W}$-Cat up to biequivalence. The work of Walters on the geometric and logical interpretation of enriched category theory, and the author's work on cohomology, indicate that many important properties *are* cauchy invariant.

The characterization allows us to show that the notion of cosmos is closed under several basic constructions: dualization, parametrization and localization. This gives a different approach to the work of R. Betti, A. Carboni, R. Walters and the author which proves that stacks over a site can be interpreted as categories enriched in a bicategory (the case of the chaotic site appears in [8]).

In summary, there is a well-known analogy between set theory and category theory; I suggest that, when set theory is generalized to the study of a Grothendieck topos, the analogous generalization of category theory is the study of a cosmos (and this *means* enriched category theory).

I am grateful to Robert Walters, for his enthusiastic interest in this work, and for suggesting the term “collage” to replace my earlier term “lax bicolimit”.

2. Preliminary Concepts.

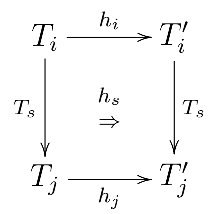
Let $\mathcal{M}$ denote a bicategory. An arrow $f : X \to Y$ in $\mathcal{M}$ is called a *map* when it has a right adjoint $f^* : Y \to X$; the unit and counit are denoted by $\eta_f : 1_X \to f^*f$, $\varepsilon_f : ff^* \to 1_Y$. We write $\mathcal{M}^*$ for the subbicategory of $\mathcal{M}$ with the same objects and with the maps as arrows.

A map $f : X \to Y$ is called *fully faithful* when the unit $\eta_f : 1_X \to f^*f$ is invertible. If f is fully faithful and fm is a map then m is a map.

A family of maps into Y is called *cauchy dense* when the only fully faithful maps into Y, through which all members of the family factor up to isomorphism, are equivalences.

Call $\mathcal{M}$ *locally small-cocomplete* when each hom-category $\mathcal{M}(X, Y)$ has small colimits and composition with arrows in $\mathcal{M}$ preserves these small colimits.

Recall from [3] that the bicategory $\mathrm{Bicat}(\mathcal{J}^{\mathrm{op}}, \mathcal{M}^{\mathrm{op}})^{\mathrm{op}}$ has morphisms of bicategories $T : \mathcal{J} \to \mathcal{M}$ as objects and optransformations $h : T \to T'$ as arrows.



Each object X of $\mathcal{M}$ gives a constant morphism $\triangle X : \mathcal{J} \to \mathcal{M}$ with $(\triangle X)i = X$, $(\triangle X)s = 1_X$. If $h : T \to T'$ is a map in Bicat $(\mathcal{J}^{\mathrm{op}}, \mathcal{M}^{\mathrm{op}})^{\mathrm{op}}$ then $h^* : T' \to T$ is a strong optransformation.

A *collage* for $T : \mathcal{J} \to \mathcal{M}$ is an object L of $\mathcal{M}$ together with an optransformation $l : T \to \triangle L$ which induces an equivalence of categories

$$\mathcal{M}(L, X) \simeq \mathrm{Bicat}(\mathcal{J}^{\mathrm{op}}, \mathcal{M}^{\mathrm{op}})^{\mathrm{op}}(T, \triangle X)$$

for all objects X of $\mathcal{M}$. The components $l_i : T_i \to L$ are called *coprojections.*

In the terminology established in [20], collages are *lax bicolimits.* They are particular kinds of indexed ("weighted" is a better word!) bicolimits. Many interesting constructions are examples of collages and so we believe a special name is in order. Benabou defined the concept in [4] under the name "2-dimensional right limit".

2.1. Examples. Recall that a morphism $T : \mathcal{J} \to \mathcal{M}$ is called *normal* when the structure 2-cells $1_{Ti} \to T1_i$ are all invertible, and is called a *homomorphism* when, furthermore, the structure 2-cells $(Tt)(Ts) \to T(ts)$ are all invertible.

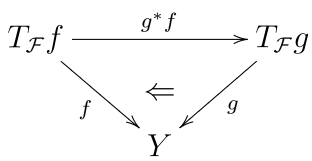
1. Suppose $\mathcal{J}$ is a set (discretely regarded as a bicategory). A family of objects of $\mathcal{M}$ indexed by $\mathcal{J}$ yields a homomorphism $T : \mathcal{J} \to \mathcal{M}$. A collage for T is a *bicoproduct* for the family.

2. When $\mathcal{J}$ is a one-element set, a morphism $T : \mathcal{J} \to \mathcal{M}$ amounts to a *monad in* $\mathcal{M}$. A collage for T is a *kleisli construction* on the monad [17].

3. Let $\mathcal{B}$ denote $\{0,\ 1,\ 2\}$ as an ordered set. A *gamut* in $\mathcal{M}$ is a *normal morphism* $T : \mathcal{B}^{\mathrm{op}} \to \mathcal{M}$. A collage for T with the 1-coprojection omitted gives a cospan in $\mathcal{M}$. When $\mathcal{M}$ is a bicategory of enriched categories and modules, this cospan is the *cofibration* determined by the gamut (see [20, §6]).

We say that $\mathcal{M}$ *admits small collages* when every morphism $T : \mathcal{J} \to \mathcal{M}$, with $\mathcal{J}$ a small bicategory, has a collage.

A family $\mathcal{F}$ of maps into Y in $\mathcal{M}$ determines a morphism $T_{\mathcal{F}} : \mathcal{J}_{\mathcal{F}} \to \mathcal{M}$ as follows. The category $\mathcal{J}_{\mathcal{F}}$ is the chaotic category on the set $\mathcal{F}$: that is, its objects are the elements of $\mathcal{F}$ and $\mathcal{J}_{\mathcal{F}}(f, g) = 1$ for all $f,\ g \in \mathcal{F}$. The morphism $T_{\mathcal{F}}$ takes $f \in \mathcal{F}$ to its source object in $\mathcal{M}$, it takes $f \to g$ in $\mathcal{J}_{\mathcal{F}}$ to the arrow $g^* f$ in $\mathcal{M}$, and the structure 2-cells are induced by the units and counits of the adjunctions $f \dashv f^*$. Furthermore, there is an optransformation $T_{\mathcal{F}} \to \triangle Y$ whose component at $f \in \mathcal{J}_{\mathcal{F}}$ is $f : T_{\mathcal{F}} f \to Y$ and whose component at $f \to g$ in $\mathcal{J}_{\mathcal{F}}$ is the 2-cell $\varepsilon_g f : gg^* f \to f$.



2.2. Proposition. *Suppose $\mathcal{M}$ is a locally small-cocomplete bicategory which admits small bicoproducts and the kleisli construction. Then:*

a. *$\mathcal{M}$ admits small collages;*

b. *each coprojection into a collage is a map;*

c. *the coprojections into a bicoproduct are fully faithful and their right adjoints are projections from a biproduct;*

d. *the coprojections into a collage form a cauchy dense family;*

e. *an arrow out of a collage is a map if its composite with all the coprojections is a map;*

f. *each map $f : X \to Y$ factors up to isomorphism as je where j is a fully faithful map and e is the coprojection into the kleisli construction of the monad f^*f generated by $f \dashv f^*$;*

g. *the following conditions on a small family $\mathcal{F}$ of maps into Y are equivalent:*

(i) *$\mathcal{F}$ is cauchy dense;*

(ii) *the diagram*

$$\sum_{f,g\in\mathcal{F}} gg^*ff^* \underset{(\varepsilon_g ff^*)}{\overset{(gg^*\varepsilon_f)}{\rightrightarrows}} \sum_{f\in\mathcal{F}} ff^* \xrightarrow{(\varepsilon_f)} 1_Y$$

is a coequalizer in $\mathcal{M}(Y, Y)$;

(iii) *the optransformation $T_{\mathcal{F}} \to \triangle Y$ exhibits Y as a collage for $T_{\mathcal{F}}$.*

PROOF. Let P be a bicoproduct of a family of objects A_i with coprojections $l_i : A_i \to P$. Define $l_i^* : P \to A_i$ by the conditions:

$$l_i^* l_j \cong \begin{cases} 1_{A_i} & j = i \\ 0 & \text{for } j \neq i \end{cases}$$

where 0 denotes the initial object of $\mathcal{M}(A_j, A_j)$. It follows that $l_i \dashv l_i^*$ with invertible units. The counits determine an isomorphism:

$$\sum_i l_i l_i^* \cong 1_P.$$

Just as in additive category theory, these equations yield that P is also a biproduct of the objects A_i with projections l_i^*. This proves (c). When the collage is a bicoproduct, we also have (b). If $h : P \to X$ is such that each hl_i is a map then h is a map with h^* given by $l_i^* h^* \cong (hl_i)^*$; so we also have (e) when the collage is a bicoproduct.

The following straightforward argument proves (d). Suppose $l : T \to \triangle L$ exhibits L as a collage for T. Suppose $f : X \to L$ is fully faithful and each coprojection l_i factors as $l_i \cong f l'_i$. Since f is fully faithful, each l_s is isomorphic to $f l'_s$ for a unique $l'_s : l'_j \cdot T_s \to l'_i$. This gives an optransformation $l' : T \to \triangle \mathrm{X}$ with $\triangle f \cdot l' \cong l$. So there exists $g : L \to X$ with $\triangle g \cdot l \cong l'$. Thus $\triangle f \cdot \triangle g \cdot l \cong l$. So $fg \cong 1$. Since f is fully faithfully, f is an equivalence.

Each monad m on an object X in $\mathcal{M}$ has a kleisli object X_m with coprojection $e : X \to X_m$. Each arrow $j : X_m \to Y$ determines a triangle:

$$\begin{array}{ccc} \mathcal{M}(Y,A) & \xrightarrow{\mathcal{M}(j,1)} & \mathcal{M}(X_m,A) \\ & {\scriptstyle \mathcal{M}(j_e,1)} \searrow \quad \cong \quad \swarrow {\scriptstyle \mathcal{M}(e,1)} & \\ & \mathcal{M}(X,A) & \end{array}$$

in which the functor $\mathcal{M}(e,1)$ is monadic. By the adjoint triangle theorem [10], since $\mathcal{M}(Y,A)$ has coequalizers preserved by composition, j is a map if and only if je is a map. This gives (b) and (e) in the case where the collage is a kleisli construction.

Now suppose we are given a map $f : X \to Y$ and let m be the monad f^*f on X. Then there exists a comparison arrow j such that $f \cong je$. Since f is a map, the argument of the last paragraph yields that j is a map. Since $\mathcal{M}(f,A)$ preserves coequalizers, $\mathcal{M}(j^*A)$ is fully faithful. So j is fully faithful. This proves (f). Note also that j is an equivalence if and only if $\mathcal{M}(f,A)$ reflects isomorphisms; that is, if and only if

$$ff^*ff^* \underset{\varepsilon_f ff^*}{\overset{ff^*\varepsilon_f}{\rightrightarrows}} ff^* \xrightarrow{\ \varepsilon_f\ } 1_Y$$

is a coequalizer. Using (d), we have the equivalence of (g)(i) and (g)(ii) when $\mathcal{F}$ has only one member.

Let $T : \mathcal{J} \to \mathcal{M}$ be a morphism with $\mathcal{J}$ small. Let P be the bicoproduct of the objects $Ti, i \in \mathcal{J}$. Let m: $P \to P$ be the colimit of the functor

$$\prod_{i,j} \mathcal{J}(i,j) \xrightarrow{\prod T_{ij}} \prod_{ij} \mathcal{M}(Ti,Tj) \cong \mathcal{M}(P,P).$$

The 2-cells $1_{T^i} \to T1_i$, $(Tt)(Ts) \to T(ts)$ induce 2-cells $1_p \to m$, $mm \to m$ which enrich m with the structure of monad on P. A kleisli object for m can be shown to be a collage for T. This proves (a). It follows from this construction that (b) and (e) hold since we have already seen that they hold for bicoproducts and the kleisli construction.

Finally, we shall prove (g). Take a small family $\mathcal{F}$ of maps into Y and consider the above construction applied to $T_{\mathcal{F}} : \mathcal{J}_{\mathcal{F}} \to \mathcal{M}$ in place of T. Then P is the bicoproduct of the sources of the $f \in \mathcal{F}$, and m is induced by the arrows g^*f with f, $g \in \mathcal{F}$. Clearly (i) holds precisely when $f : P \to Y$ is cauchy dense. But we have the equivalence of (i) and (ii) for the one-member family consisting of $f : P \to Y$. The diagram of (ii) for this one-member family is isomorphic to the diagram of (ii) for $\mathcal{F}$. So (i), (ii) are equivalent. Using (f) and (d), we see that $f : P \to Y$ is cauchy dense if and only if $P_m \to Y$ is an equivalence. So (i), (iii) are equivalent. ∎

A set $\mathcal{U}$ of objects of $\mathcal{M}$ is called a *cauchy generator* when, for all objects Y of $\mathcal{M}$, there exists a small cauchy-dense family of maps into Y whose sources are all in $\mathcal{U}$.

2.3. Proposition. *Suppose $\mathcal{U}$ is a cauchy generator for a bicategory $\mathcal{M}$. If $\mathcal{F}$ is a family of maps into $Y \in \mathcal{M}$ such that, for each $U \in \mathcal{U}$, each map $U \to Y$ is a colimit in $\mathcal{M}(U, Y)$ of maps $U \to Y$ in $\mathcal{F}$ then $\mathcal{F}$ is cauchy dense.*

Proof. If $f : X \to Y$ is fully faithful then any colimit of maps which lift through f also lifts through f. ∎

3. The Characterization Theorem.

A bicategory $\mathcal{W}$ will be called a *base* when it has a small set of objects and it is locally small-cocomplete. In this case the definition was given in [8] of the bicategory $\mathcal{W}$-Mod whose objects are small $\mathcal{W}$-categories and whose arrows are $\mathcal{W}$-modules.

A bicategory $\mathcal{M}$ will be called a *cosmos* when:

- small bicoproducts exist;
- each monad admits a kleisli construction;
- it is locally small-cocomplete;
- there exists a small cauchy generator.

3.1. Theorem. [Characterization theorem] *A bicategory $\mathcal{M}$ is a cosmos if and only if there exists a base bicategory $\mathcal{W}$ such that $\mathcal{M}$ is biequivalent to* $\mathcal{W}$-Mod*. Indeed, $\mathcal{W}$ can be taken to be any full subbicategory of $\mathcal{M}$ whose objects form a small cauchy generator.*

Proof. For a base bicategory $\mathcal{W}$, we must see that $\mathcal{W}$-Mod is a cosmos. It was observed in [8] that $\mathcal{W}$-Mod is locally small cocomplete. It is also clear from [8] that small coproducts in $\mathcal{W}$-Cat are small bicoproducts in $\mathcal{W}$-Mod. After the work of Thiébaud [27], one expects the kleisli construction to exist in $\mathcal{W}$-Mod. A monad m on a $\mathcal{W}$-category A gives a $\mathcal{W}$-category A_m with the same objects A and $A_m(b, a) = m(b, a)$; the unit and multiplication for m give identities and composition in A_m; the universal property of $A \to A_m$ can be checked. (By Proposition 1, $\mathcal{W}$-Mod has small collages. A direct proof of this result – in the case where $\mathcal{W}$ has one object – was given by Bénabou [4] who pointed out that this generalizes and characterizes the Grothendieck construction of a fibred category from a pseudo-functor.)

A cauchy generator for $\mathcal{W}$-Mod consists of the $\mathcal{W}$-categories U which have only one object over U and none over anything else. The $\mathcal{W}$-functors $U \to A$ provide a small family of maps into A which is cauchy dense.

Conversely, suppose $\mathcal{M}$ is a cosmos. Let $\mathcal{U}$ be a small cauchy generator for $\mathcal{M}$. Let $\mathcal{W}$ be the full subbicategory of $\mathcal{M}$ with these objects. It remains to prove $\mathcal{M} \sim \mathcal{W}$-Mod.

There is a homomorphism £: $\mathcal{M} \to \mathcal{W}$-Mod described as follows (the basic idea was present in [18] and [19]). For each $x \in \mathcal{M}$, choose a small cauchy-dense family $\mathcal{F}_X$ of maps into X with sources in U. An object of $\pounds X$ over U is an element of $\mathcal{F}_X$ with source U; also, put $(\pounds X)(y,x) = y^*x : U \to V$. This gives a $\mathcal{W}$-category $\pounds X$ whose composition is induced by the counits of the adjunctions $y \dashv y^*,\ y \in \mathcal{F}_X$.

For each arrow $h : X \to Y$ in $\mathcal{M}$, define a $\mathcal{W}$-module $\pounds h : \pounds X \to \pounds Y$ as follows:

$(\pounds h)(y,x) = y^*hx : U \to V$ for $x \in (\pounds X)_U,\ y \in (\pounds Y)_V$;

$\lambda : (\pounds Y)(y',y)(\pounds h)(y,x) \to (\pounds h)(y',x)$ is induced by the counit $yy^* \to 1$;

$\varrho : (\pounds h)(y,x)(\pounds X)(x,x') \to (\pounds h)(y,x')$ is induced by the counit $xx^* \to 1$.

In order to see that $\pounds$ is a homomorphism, take $h : X \to Y,\ k : Y \to Z$. Since $\mathcal{F}_Y$ is cauchy dense, we have the coequalizer

$$\sum_{y,y' \in \mathcal{F}_Y} y'y'^*yy^* \rightrightarrows \sum_{y \in \mathcal{F}_Y} yy^* \longrightarrow 1_Y$$

using Proposition 1 (g). Since composition in $\mathcal{M}$ preserves colimits in the hom-categories, we obtain $\pounds(kh)(z,x)$ as a coequalizer of the pair:

$$\sum_{y,y' \in \pounds Y} (\pounds k)(z,y')(\pounds Y)(y',y)(\pounds h)(y,x) \rightrightarrows \sum_{y \in \pounds Y} (\pounds k)(z,y)(\pounds k)(y,x).$$

¿From the definition of composition of $\mathcal{W}$-modules, it follows that $(\pounds k)(\pounds h) \cong \pounds(kh)$.

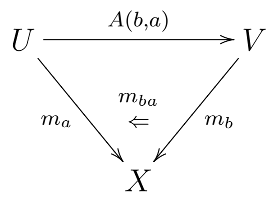
Since homomorphisms preserve equivalences, it follows that $\pounds X$ is independent, up to equivalence, of the choice of $\mathcal{F}_X$. In particular, for $U \in \mathcal{U}$, we can take $\mathcal{F}_U$ to have the one member $1_u : U \to U$; hence $\pounds U \cong U$. Furthermore, since a composite of cauchy dense families is cauchy dense, Proposition 1 (d) yields that $\pounds$ preserves small collages.

There is a homomorphism $\$: \mathcal{W}\text{-Mod} \to \mathcal{M}$ defined as follows. For each $\mathcal{W}$-category A, let J_A denote the category whose objects are objects of A and for which there is precisely one arrow between every two objects. The remaining data for A provide a morphism $T_A : J_A \to \mathcal{W}$ which can also be regarded as landing in $\mathcal{M}$. Define $\$A$ to be the collage for $T_A : J_A \to \mathcal{M}$.

We shall indicate an equivalence of categories

$$\mathcal{M}(\$A, X) \simeq \mathcal{W}\text{-Mod}(A, \pounds X)$$

which is a strong transformation in X; hence \$ becomes a homomorphism such that the equivalence is a strong transformation in A. The category on the left-hand side of the proposed equivalence is equivalent to the category of optransformations $m : T_A \to \triangle X$.



Such an optransformation determines a $\mathcal{W}$-module $\Phi : A \to \pounds X$ given by:

$\Phi(x, a) = x^* m_a : U \to V$ for $a \in A_U, x \in (\pounds X)_V$;

λ is induced by the counits $xx^* \to 1$,

ϱ is induced by the m_{ba}.

Conversely, given a $\mathcal{W}$-module $\Phi : A \longrightarrow \pounds X$, define $m_a : U \longrightarrow X$ to be the coequalizer:

$$\sum_{x,y \in \pounds X} yy^* x\Phi(x, a) \rightrightarrows \sum_{x \in \pounds X} x\Phi(x, a) \longrightarrow m_a,$$

and let $m_{ba} : m_b A(b, a) \to m_a$ be induced by the right actions ϱ of Φ.

Thus we obtain a left biadjoint $ for $\pounds$.

Proposition 1 (g) gives that the counit of this biadjunction is an equivalence $\$\pounds X \simeq X$.

Since $\pounds U \simeq U$, the collage $\pounds\$A$ for the morphism

$$J_a \xrightarrow{T_A} \mathcal{M} \xrightarrow{\pounds} \mathcal{W}\text{-Mod}$$

is equivalent to the collage for

$$J_A \xrightarrow{T_A} \mathcal{W} \longrightarrow \mathcal{W}\text{-Mod}$$

But this last morphism has collage A, by Proposition 1 (g). So the unit for the biadjunction is an equivalence $A \simeq \pounds\$A$. ■

4. Constructions on a Cosmos

4.1. Proposition. *If $\mathcal{M}$ is a cosmos then $\mathcal{M}^{\rm op}$ is a cosmos.*

Proof. The dual of a $\mathcal{W}$-category is a $\mathcal{W}^{\rm op}$-category. This gives an isomorphism:

$$()^{\rm op} : (\mathcal{W}\text{-Mod})^{\rm op} \cong \mathcal{W}^{\rm op}\text{-Mod}.$$ ■

4.2. Proposition. *In any cosmos $\mathcal{M}$, bitensoring with small categories exists. If $f : X \to Y$ is a functor between small categories and A is an object of $\mathcal{M}$ then the arrow $f * A : X * A \to Y * A$ induced between the bitensor products is a map.*

Proof. Take $\mathcal{M} = \mathcal{W}$-Mod. Let $T_A : J_A \to \mathcal{W}$ be the morphism determined by A, where J_A is the chaotic category with the same objects as $A \in \mathcal{W}$-Cat. Let $X * A$ denote the collage of the composite morphism:

$$X \times J_A \xrightarrow{\text{proj}_2} J_A \xrightarrow{T_A} \mathcal{W} \longrightarrow \mathcal{W}\text{-Mod}$$

The following desired universal property can be verified:

$$\mathcal{M}(X^* A, C) \cong [X, \mathcal{M}(A, C)].$$

The second sentence of the Proposition is a consequence of Proposition 1 (b), (e). ■

4.3. Theorem. *Let $\mathcal{C}$ be a small bicategory and $\mathcal{M}$ be a cosmos. The full subbicategory $\mathcal{N}$ of* $\mathrm{Hom}(\mathcal{C}^{\mathrm{op}}, \mathcal{M})$*, consisting of these homomorphisms which take arrows in $\mathcal{C}$ to maps in $\mathcal{M}$, is a cosmos.*

Proof. Since $\mathcal{M}$ is locally small-cocomplete, local small colimits in $\mathrm{Hom}(\mathcal{C}^{\mathrm{op}}, \mathcal{M})$ can be calculated evaluationwise. Since $\mathcal{N}$ is a full subbicategory of $\mathrm{Hom}(\mathcal{C}^{\mathrm{op}}, \mathcal{M})$, it is also locally small-cocomplete.

Small collages are formed pointwise in $\mathrm{Hom}(\mathcal{C}^{\mathrm{op}}, \mathcal{M})$. If follows from Proposition 1 (b), (e) that $\mathcal{N}$ is closed under these constructions.

We claim that, if $\mathcal{U}$ is a small cauchy generator for $\mathcal{M}$, then the objects $\mathcal{C}(-, u) * U$, where $U \in \mathcal{U}$, $u \in \mathcal{C}$, form a small cauchy generator for $\mathcal{N}$. That these objects exist and are in $\mathcal{N}$ follows from Proposition 4.2. Using [22; Lemma 7.4, p. 327], we see that, if A is in $\mathcal{N}$, then the arrow $\mathcal{C}(-, u) * U \to A$, corresponding to a map $U \to Au$ in $\mathcal{M}$, is a map. Thus the small cauchy-dense families of maps into all Au with sources in $\mathcal{U}$ correspond to a small cauchy-dense family of maps into A with sources of the form $\mathcal{C}(-, u) * U$. ■

Writing $(\mathcal{W}\text{-Cat})_{\mathrm{cc}}$ for $(\mathcal{W}\text{-Mod})^*$, we obtain the following generalization of [8; Theorem 18].

4.4. Corollary. *For each small bicategory $\mathcal{C}$, there exists a base bicategory $\mathcal{W}$ with the same objects as $\mathcal{C}$ such that there is a biequivalence:*

$$\mathrm{Hom}(\mathcal{C}^{\mathrm{op}}, \mathrm{Cat}_{\mathrm{cc}}) \sim (\mathcal{W}\text{-Cat})_{\mathrm{cc}}$$

Proof. Take $\mathcal{M}$ to be the bicategory Mod of small categories and modules. Apply Theorem 5 and the Characterization Theorem. ■

A *localization* of a cosmos $\mathcal{M}$ is a homomorphism of bicategories $\Gamma : \mathcal{M} \to \mathcal{N}$ satisfying the following properties:

(i) for each object X of $\mathcal{N}$ there is an object A of $\mathcal{M}$ and an equivalence $\Gamma A \simeq X$;

(ii) for all objects A, B of $\mathcal{M}$ the functor

$$\Gamma_{AB} : \mathcal{N}(A, B) \to \mathcal{M}(\Gamma A, \Gamma B)$$

has a fully faithful right adjoint.

4.5. Theorem. *If* $\Gamma\colon \mathcal{M} \to \mathcal{N}$ *is a localization of a cosmos* $\mathcal{M}$ *then* $\mathcal{N}$ *is a cosmos and* Γ *preserves the cosmos structure.*

Proof. The category $\mathcal{N}(\Gamma A, \Gamma B)$ is equivalent to a full reflective subcategory of $\mathcal{M}(A, B)$ by (ii); so $\mathcal{N}(\Gamma A, \Gamma B)$ has, and Γ_{AB} preserves, small colimits. By (i), each hom category in $\mathcal{N}$ has small colimits. Since Γ preserves composition up to isomorphism, it follows that composition in $\mathcal{N}$ preserves small colimits. So $\mathcal{N}$ is locally small cocomplete.

Let (X_i) be a small family of objects in $\mathcal{N}$. By (i), there is a small family of objects in (A_i) in $\mathcal{M}$ with $\Gamma A_i \simeq X_i$. Let P be a bicoproduct for (A_i) in $\mathcal{M}$; this property is expressible in terms of "direct sum" equations in the homs of $\mathcal{M}$ (Proposition 1). Since Γ preserves small coproducts in the homs, ΓP satisfies the "direct sum" equations for the family (ΓA_i). So ΓP is a bicoproduct for (X_i) (Proposition 1).

Let n be a monad on X in $\mathcal{N}$. By (i) there is an A in $\mathcal{M}$ with $\Gamma A \simeq X$. By (ii), there is a monad m on A with $\Gamma m \simeq n$ (note that the right adjoint for Γ_{AA} is a monoidal functor). The kleisli construction $e : A \to A_m$ exists for m in $\mathcal{M}$. Since Γ preserves coequalizers in the homs, $\Gamma e : \Gamma A \to \Gamma A_m$ is a cauchy dense map (Proposition 1(g)). Thus we obtain a kleisli construction $X \simeq \Gamma A \to \Gamma A_m$ for n (Proposition 1 again).

Proposition 1(g)(i), (ii) implies that, if $\mathcal{U}$ is a cauchy generator for $\mathcal{M}$, the objects $\Gamma U, U \in \mathcal{U}$, form a cauchy generator for $\mathcal{N}$. ∎

A *topology* J on a base bicategory $\mathcal{W}$ assigns, to each pair of objects U, V of $\mathcal{W}$, an idempotent monad J_{UV} on the category $\mathcal{W}(U, V)$ such that, for all $u : U \to V$, $v : V \to W$ in $\mathcal{W}$, the units of the monads J_{UV}, J_{VW} induce an isomorphism:

$$J_{UV}(vu) \cong J_{UW}(J_{VW}(v)J_{UV}(u))$$

Let $\mathcal{W}_J$ denote the bicategory with the same objects as $\mathcal{W}$ and with hom category $\mathcal{W}_J(U, V)$ taken to be the full subcategory of $\mathcal{W}(U, V)$ consisting of those u for which the unit $u \to J_{UV}(u)$ is an isomorphism. Using the fact that J_{UV} inverts the canonical 2-cell $\operatorname{colim} u_i \to \operatorname{colim} J_{UV}(u_i)$, one easily checks that $\mathcal{W}_J$ *is a base bicategory.* There is a homomorphism of bicategories:

$$H_J : \mathcal{W} \to \mathcal{W}_J$$

which is the identity on objects and which is given on homs $\mathcal{W}(U, V) \to \mathcal{W}_J(U, V)$ by applying J_{UV}. There is a morphism of bicategories:

$$H_J^* : \mathcal{W}_J \to \mathcal{W}$$

which is the identity on objects and which is given on homs by the inclusions $\mathcal{W}_J(U, V) \to (U, V)$.

Since H_J is a homomorphism which preserves small colimits in the homs, it extends to a homomorphism:

$$\Gamma_J : \mathcal{W}\text{-Mod} \to \mathcal{W}_J\text{-Mod}$$

For each $\mathcal{W}$-category A, the W_J-category $\Gamma_J A$ has objects over U precisely the objects of A over U; and $(\Gamma_J A)(b, a) = H_J A(b, a)$. For each $\mathcal{W}$-module $m : A \to B$, the $\mathcal{W}_J$-module $\Gamma_J A \to \Gamma_J B$ is given by $(\Gamma_J m)(b, a) = H_J m(b, a))$. ∎

4.6. THEOREM. *Each topology J on a base bicategory $\mathcal{W}$ determines a localization*

$$\Gamma_J : \mathcal{W}\text{-Mod} \to \mathcal{W}_J\text{-Mod}$$

as defined above. If $\Gamma : \mathcal{W}\text{-Mod} \to \mathcal{N}$ is a localization then there exists a topology J on $\mathcal{W}$ and a biequivalence $\mathcal{N} \sim \mathcal{W}_J$-Mod whose composite with Γ is equivalent to Γ_J.

PROOF. The category $\mathcal{W}_J$-Mod $(\Gamma_J, A, \Gamma_J, B)$ is equivalent to the full subcategory of $\mathcal{W}$-Mod(A, B) consisting of those modules $m : A \to B$ for which $m(b, a) \to J_{UV} m(b, a)$ is invertible for all a, b in A, B over U, V, respectively. Each $\mathcal{W}_J$-category X can be regarded as a $\mathcal{W}$-category via $H_J{}^*$, and $\Gamma_J X = X$. So Γ_J is a localization.

Given a localization $\Gamma : \mathcal{W}\text{-Mod} \to \mathcal{U}$, the functor:

$$\mathcal{W}(U, V) \simeq \mathcal{W}\text{-Mod}(U, V) \xrightarrow{\Gamma_{UV}} \mathcal{N}(\Gamma U, \Gamma V)$$

has a fully faithful right adjoint. So we obtain an idempotent monad J_{UV} on $\mathcal{W}(U, V)$. Since Γ is a homomorphism, this gives a topology J on $\mathcal{W}$. Furthermore, we have a canonical equivalence $\mathcal{N}(\Gamma U, \Gamma V) \simeq W_J(U, V)$. By Theorem 7 and the Characterization Theorem, the desired biequivalence $\mathcal{N} \sim \mathcal{W}_J$-Mod exists. ∎

4.7. COROLLARY. *For each small bisite $\mathcal{C}$ [24], there exists a base bicategory $\mathcal{W}$ with the same objects as $\mathcal{C}$ such that there is a biequivalence:*

$$\text{Stack}(\mathcal{C}^{\text{op}}, \text{Cat}_{\text{cc}}) \sim (\mathcal{W}\text{-Cat})_{\text{cc}}. \qquad \blacksquare$$

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