GEOMETRY OF QUANTUM HOMOGENEOUS SUPERVECTOR BUNDLES AND REPRESENTATION THEORY OF QUANTUM GENERAL LINEAR SUPERGROUP

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The quantum general linear supergroup $GL_q(m|n)$ is defined and its structure is studied systematically. Quantum homogeneous supervector bundles are introduced following Connes' theory, and applied to develop the representation theory of $GL_q(m|n)$. Quantum Frobenius reciprocity is proven, and a Borel - Weil theorem is established for the covariant and contravariant tensor irreps.

Quantum supergroup $GL_q(m|n)$

We will work on the complex field $\mathbf{C}$. For conveninece, the Lie superalgebra $gl(m|n)$ will be denoted by $\mathbf{g}$, and its quantized universal enveloping algebra by $U_q(\mathbf{g})$. Here $U_q(\mathbf{g})$ is of Jimbo type with q being specialized to a generic complex parameter. We will largely follow the notations of [1] and [2]. Now the generators of $U_q(\mathbf{g})$ are $K_a,\ K_a^{-1},\ E_{b\ b+1},\ \ E_{b+1,b},\ a\in\mathbf{I},\ b\in\mathbf{I}'$, where $\mathbf{I}=\{1,2,...,m+n\}$, $\mathbf{I}'=\mathbf{I}\backslash\{m+n\}$, and the defining relations are given in [1]. Corresponding to any subset $\mathbf{\Theta}$ of $\mathbf{I}'$, we introduce $U_q(\mathbf{k})$ which is generated by the elements of $\mathcal{S}_k=\{K_a^{\pm1},\ E_{c\,c+1},\ E_{c+1\,c},\ a\in\mathbf{I},\ c\in\mathbf{\Theta}\}$, and $U_q(\mathbf{p}_\pm)$ respectively generated by those of $\mathcal{S}_k\cup\{E_{c\ c+1},\ c\in\mathbf{I}'\backslash\mathbf{\Theta}\}$ and $\mathcal{S}_k\cup\{E_{c+1\ c},\ c\in\mathbf{I}'\backslash\mathbf{\Theta}\}$. Clearly, $U_q(\mathbf{p}_\pm)$ are $\mathbf{Z}_2$ - graded Hopf subalgebras of $U_q(\mathbf{g})$, which will be called parabolic. Also, $U_q(\mathbf{k})$ is a $\mathbf{Z}_2$ - graded Hopf subalgebra of $U_q(\mathbf{p}_\pm)$.

The representation theory of $U_q(\mathbf{g})$ has been systematically developed [1]. Let π be the contravariant vector representation of $U_q(\mathbf{g})$ afforded by the irreducible module E in the standard basis $\{v_a\}$ such that $E_{a\,a\pm1}v_b=\delta_{b\,a\pm1}v_a$, $K_a v_b=q_a^{\delta_{ab}}v_b$. Although finite dimensional representations of $U_q(\mathbf{g})$ are not completely reducible in general, repeated tensor products of π can be reduced into direct sums of contravariant tensor irreps. Let $\Lambda^{(1)}$ be the set of the highest weights of all such irreps. Let $\overline{E}$ be the dual module of E, and $\overline{\pi}$ the covariant vector representation (dual to π). Repeated tensor products of $\overline{\pi}$ are again completely reducible. We denote by $\Lambda^{(2)}$ the highest weights of all the covariant tensor irreps. (The trivial representation is tensorial by convention.) $\Lambda^{(1)}$ and $\Lambda^{(2)}$ can be explicitly characterized [3]. In particular, it is known that $\Lambda^{(1)}\cap\Lambda^{(2)}=\{0\}$. Also, if $\lambda\in\Lambda^{(1)}$, we denote by $\bar{\lambda}$ the lowest weight of the irrep with highest weight λ, and set $\lambda^\dagger=-\bar{\lambda}$. Then $\Lambda^{(2)}=\{\lambda^\dagger|\lambda\in\Lambda^{(1)}\}$.

It follows from standard Hopf algebra theory that $(U_q(\mathbf{g}))^0$ is a Hopf superalgebra with a structure dualizing that of $U_q(\mathbf{g})$. Consider $T_q\subset(U_q(\mathbf{g}))^0$ generated by the matrix elements of π: $t_{ab}\in(U_q(\mathbf{g}))^0$, $a,b\in\mathbf{I}$. It is a $\mathbf{Z}_2$ - graded bi - subalgebra of $(U_q(\mathbf{g}))^0$ with the generators satisfying an 'RTT' relation [2], but does not admit an antipode. Our earlier discussions on representations of $U_q(\mathbf{g})$ imply that the matrix elements of the contravariant tensor irreps form a Peter - Weyl basis for T_q. Similarly, the matrix elements $\overline{t}_{ab}\in(U_q(\mathbf{g}))^0$ of $\overline{\pi}$ also generate a $\mathbf{Z}_2$ - graded bi - subalgebra $\overline{T}_q$ of $(U_q(\mathbf{g}))^0$, for which the matrix elements of the covariant tensor irreps form a Peter -

Weyl basis.

We define the algebra of functions G_q on $GL_q(m|n)$ to be the subalgebra of $(U_q(\mathbf{g}))^0$ generated by $\{t_{ab},\ \overline{t}_{ab}|a, b \in \mathbf{I}\}$. It inherits a bi - superalgebra structure from T_q and $\overline{T}_q$, and also admits an antipode. Thus G_q is a Hopf superalgebra. By considering the universal R - matrix of $U_q(\mathbf{g})$ we can easily show that the following relation is satisfied in G_q

$$R_{12}^{\overline{\pi}\,\pi}\ \overline{t}_1\ t_2 \ \ = \ \ t_2\ \overline{t}_1\ R_{12}^{\overline{\pi}\,\pi}, \tag{1}$$

where $R^{\overline{\pi}\,\pi} = q^{-\sum_{a\in\mathbf{I}} e_{a\,a}\otimes e_{a\,a}(-1)^{[a]}} - (q-q^{-1})\sum_{a<b} e_{b\,a}\otimes e_{b\,a}(-1)^{[a]+[b]+[a][b]}$. An immediate consequence of (1) is the factorization $G_q = T_q\overline{T}_q$. It can also be shown [2] that G_q separates points of $U_q(\mathbf{g})$, that is, for any nonvanishing $x \in U_q(\mathbf{g})$, there exists $f \in G_q$ such that $f(x) \neq 0$.

Induced representations and quantum supervector bundles

Let us introduce two left actions L and R of $U_q(\mathbf{g})$ on G_q. For any $x \in U_q(\mathbf{g})$ and $f \in G_q$, we define $R_x(f) = \sum_{(f)}(-1)^{[x]([f]+[x])}\ f_{(1)}\langle f_{(2)},\ x\rangle$ and $L_x(f) = \sum_{(f)}\langle f_{(1)},\ S^{-1}(x)\rangle f_{(2)}$, where Sweedler's sigma notation is employed. We *trivially* extend them to $V \otimes_{\mathbf{C}} G_q$, and still denote the resultant actions by the same notations. Clearly, L and R graded - commutate. Now define

$$\mathcal{A}_q \ \ := \ \ \{f \in G_q | R_y(f) = \epsilon(y) f,\ \forall y \in U_q(\mathbf{k})\}, \tag{2}$$

which forms a subalgebra of G_q. It can be regarded as the superalgebra of functions on a quantum homogeneous superspace. Given any finite dimensional $U_q(\mathbf{k})$ - module V, we define

$$\mathcal{E}(V) \ \ := \ \ \{\zeta \in V \otimes_{\mathbf{C}} G_q \,|\, R_y(\zeta) = (\,S(y) \otimes id_{G_q})\zeta,\ \forall\ y \in U_q(\mathbf{k})\}. \tag{3}$$

Then it immediately follows from the commutativity of R and L that $\mathcal{E}(V)$ furnishes a left $U_q(\mathbf{g})$ - module under L. We will call $\mathcal{E}(V)$ an induced $U_q(\mathbf{g})$ - module. There exists the following quantum Frobenius reciprocity: *if W is a quotient $U_q(\mathbf{g})$ - module of $\bigoplus_{k,l}\left\{E^{\otimes k} \otimes \overline{E}^{\otimes l}\right\}$, then there is the canonical isomorphism*

$$Hom_{U_q(\mathbf{g})}(W,\ \mathcal{E}(V)) \ \ \cong \ \ Hom_{U_q(\mathbf{k})}(W,\ V). \tag{4}$$

To understand the underlying geometry of induced representations, observe that $\mathcal{E}(V)$ forms a two - sided $\mathcal{A}_q$ - module under the multiplication of G_q. If W is a $U_q(\mathbf{g})$ - module, then $\mathcal{E}(W)$ is free. It follows that if V is a $U_q(\mathbf{k})$ - module which is contained as a direct summand in a $U_q(\mathbf{g})$ - module, then $\mathcal{E}(V)$ is project, and in this case, we may regard $\mathcal{E}(V)$ as the space of sections of a quantum supervector bundle [4] associated with $\mathcal{A}_q$. When $m \notin \mathbf{\Theta}$, every finite dimensional $U_q(\mathbf{k})$ - module V has this property. Also, for general $\mathbf{\Theta}$, $\mathcal{E}(V)$ is projective if it yields non - zero $\mathcal{O}(\mu)$ or $\overline{\mathcal{O}}(\mu)$(defined below).

Let V be a finite dimensional irreducible module over $U_q(\mathbf{p}) = U_q(\mathbf{p}_\pm)$ with the $U_q(\mathbf{k})$ highest weight μ and lowest weight $\tilde{\mu}$. Define the subspaces $\mathcal{O}(\mu)$ and $\overline{\mathcal{O}}(\mu)$ of $\mathcal{E}(V)$ by

$$\overline{\mathcal{O}}(\mu) := \{\zeta \in \mathcal{E}(V) \cap (V \otimes T_q) \,|R_y(\zeta) = (S(y) \otimes id_{G_q})\zeta, \ \forall \ y \in U_q(\mathbf{p})\};$$
$$\mathcal{O}(\mu) := \{\zeta \in \mathcal{E}(V) \cap \left(V \otimes \overline{T}_q\right) |R_y(\zeta) = (S(y) \otimes id_{G_q})\zeta, \ \forall \ y \in U_q(\mathbf{p})\}. \quad (5)$$

Then we have the following quantum Borel - Weil theorem for the covariant and contravariant tensor irreps [2]: *As* $U_q(\mathbf{g})$ *- modules,*

$$\mathcal{O}(\mu) \cong \begin{cases} W((-\tilde{\mu})^\dagger), & \text{if } \tilde{\mu} \in -\Lambda^{(2)}, \quad U_q(\mathbf{p}) = U_q(\mathbf{p}_+), \\ W(\mu), & \text{if } \mu \in \Lambda^{(1)}, \quad U_q(\mathbf{p}) = U_q(\mathbf{p}_-), \\ \{0\}, & \text{otherwise}. \end{cases}$$
$$\overline{\mathcal{O}}(\mu) \cong \begin{cases} W((-\tilde{\mu})^\dagger), & \text{if } \tilde{\mu} \in -\Lambda^{(1)}, \quad U_q(\mathbf{p}) = U_q(\mathbf{p}_+), \\ W(\mu), & \text{if } \mu \in \Lambda^{(2)}, \quad U_q(\mathbf{p}) = U_q(\mathbf{p}_-), \\ \{0\}, & \text{otherwise}, \end{cases} \quad (6)$$

where the notation $W(\lambda)$ *signifies the irreducible* $U_q(\mathbf{g})$ *- module with highest weight* λ.

References

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[4] A. Connes, *Noncommutative geometry*, Academic Press (1994).