Title: q-Schur Algebras as Quotients of Quantized Enveloping Algebras

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Abstract We study the properties of the surjective homomorphism, defined by Beilinson, Lusztig and MacPherson, from the quantized enveloping algebra of gl_n to the q-Schur algebra, $S_q(n, r)$. In particular, we find an expression for the preimage of an arbitrary element of $S_q(n, r)$ under this map and a basis for the kernel.

q-Schur Algebras as Quotients of Quantized Enveloping Algebras

By R.M. Green

0. Introduction

Quantized enveloping algebras are quantum analogues of the universal enveloping algebras corresponding to semisimple and reductive Lie algebras over $\mathbf{C}$. They were first discovered, in the simplest case, in 1981, and they have been used in areas of mathematics as diverse as Lie theory, statistical mechanics, knot theory and quantum theory.

The q-Schur algebras, $S_q(n, r)$, are quantum analogues of the Schur algebras $S(n, r)$ which were invented by Schur to classify the homogeneous polynomial representations of GL_n of degree r over $\mathbf{C}$. The algebras $S_q(n, r)$ have applications to the representation theory of GL_n over the finite field $\mathbf{F}_q$ in the nondescribing characteristic.

Beilinson et al. [1, §5.7] define surjective algebra homomorphisms, θ_r, from an integral form, $U_{\mathcal{A}}(gl_n)$, of the quantized enveloping algebra of the Lie algebra gl_n to certain finite-dimensional associative algebras which turn out (see [2]) to be isomorphic to the q-Schur algebras, $S_q(n, r)$. The main aim of this paper is to investigate the properties of the surjective algebra homomorphism, $\theta = \theta_r$. In particular, we would like to be able to find preimages of elements of the q-Schur algebra under this map, and we would like a description of $\ker \theta$. These problems are solved at the end of this paper.

The paper is divided into three sections. In §1, we make some definitions and describe a certain basis for $U_{\mathcal{A}}(gl_n)$. In §2, we study the restriction of θ to certain subalgebras of $U_{\mathcal{A}}(gl_n)$. In §3, we use the properties of quantized codeterminants associated with q-Schur algebras (which were introduced by the author in [6]) to complete our description of the relationship between $U_{\mathcal{A}}(gl_n)$ and $S_q(n, r)$.

1. Quantized enveloping algebras and q-Schur algebras

Let $\mathcal{A} := \mathbf{Z}[v, v^{-1}]$. We will define the quantized enveloping algebra $U(gl_n)$ over $\mathbf{Q}(v)$ (corresponding to the Lie algebra gl_n) and its associated $\mathcal{A}$-form $U_{\mathcal{A}}(gl_n)$. We also introduce the q-Schur algebra $S_q(n, r)$, which is an associative algebra with 1 with base ring $\mathcal{A}$ unless otherwise stated. As in [2], the relationship between q and v is given by $q = v^2$, so, strictly speaking, the q-Schur algebra we use will be $\mathcal{A} \otimes S_q(n, r)$, since $S_q(n, r)$ is usually defined to be over the ring $\mathbf{Z}[q, q^{-1}]$. For this reason, we will often refer to our "extended" $S_q(n, r)$ as the v-Schur algebra, $S_v(n, r)$.

The two algebras $U_{\mathcal{A}}(gl_n)$ and $S_v(n,r)$ specialise (when $v=1$) to an $\mathcal{A}$-form of the universal enveloping algebra $\mathcal{U}(gl_n)$ and the Schur algebra $S(n,r)$, respectively. After specialisation, the base ring becomes $\mathbf{Z}$. The symbols n and r shall be reserved for the integers given in the definitions of these three algebras.

1.1 The Quantized Enveloping Algebra $U(gl_n)$

We now define the algebra $U(gl_n)$ over $\mathbf{Q}(v)$ as in [2]. It is given by algebra generators

$$E_i, F_i, K_j, K_j^{-1},$$

(where $1 \leq i \leq n-1$ and $1 \leq j \leq n$) subject to the following relations:

$$K_i K_j = K_j K_i, \tag{1}$$

$$K_i K_i^{-1} = 1, \tag{2}$$

$$K_i E_j = v^{\epsilon^+(i,j)} E_j K_i, \tag{3}$$

$$K_i F_j = v^{\epsilon^-(i,j)} F_j K_i, \tag{4}$$

$$E_i F_j - F_j E_i = \delta_{ij} \frac{K_i K_{i+1}^{-1} - K_i^{-1} K_{i+1}}{v - v^{-1}}, \tag{5}$$

$$E_i E_j = E_j E_i \quad \text{if } |i-j| > 1, \tag{6}$$

$$F_i F_j = F_j F_i \quad \text{if } |i-j| > 1, \tag{7}$$

$$E_i^2 E_j - (v + v^{-1}) E_i E_j E_i + E_j E_i^2 = 0 \quad \text{if } |i-j| = 1, \tag{8}$$

$$F_j^2 F_i - (v + v^{-1}) F_j F_i F_j + F_i F_j^2 = 0 \quad \text{if } |i-j| = 1. \tag{9}$$

Here,

$$\epsilon^+(i,j) := \begin{cases} 1 & \text{if } j = i; \\ -1 & \text{if } j = i-1; \\ 0 & \text{otherwise;} \end{cases}$$

and

$$\epsilon^-(i,j) := \begin{cases} 1 & \text{if } j = i-1; \\ -1 & \text{if } j = i; \\ 0 & \text{otherwise.} \end{cases}$$

We introduce certain elements of $\mathbf{Q}(v)$, as follows.

We will define the *quantum integer* $[a]_v$, where a is a nonnegative integer, to be

$$\frac{v^a - v^{-a}}{v - v^{-1}.}$$

We also define *quantized factorials* by

$$[a]_v! := \prod_{k=1}^{a} [k]_v,$$

and *quantized binomial coefficients* by

$$\begin{bmatrix} a \\ b \end{bmatrix} := \frac{[a]_v!}{[b]_v![a-b]_v!}.$$

Note that when v is specialised to 1, these become ordinary integers, factorials and binomial coefficients, respectively.

If X is an element of $U(gl_n)$ and c is a nonnegative integer, then the *divided power* $X^{(c)}$ is defined to be

$$\frac{X^c}{[c]_v!}.$$

In this paper, we work with an *integral form* of $U(gl_n)$, which is denoted by $U_{\mathcal{A}}(gl_n)$, or U for short. This is an $\mathcal{A}$-algebra which is generated by the elements of $U(gl_n)$ given by

$$E_i^{(c)} \quad (1 \leq i < n,\ c \in \mathbf{N}) \tag{10}$$

$$F_i^{(c)} \quad (1 \leq i < n,\ c \in \mathbf{N}) \tag{11}$$

$$K_j \quad (1 \leq j \leq n) \tag{12}$$

$$\begin{bmatrix} K_j; 0 \\ t \end{bmatrix} \quad (1 \leq j \leq n,\ t \in \mathbf{N}) \tag{13}$$

Here,

$$\begin{bmatrix} K_i; c \\ t \end{bmatrix} := \prod_{s=1}^{t} \frac{K_i v^{c-s+1} - K_i^{-1} v^{-c+s-1}}{v^s - v^{-s}}.$$

The $\mathcal{A}$-algebra U^- is the subalgebra with 1 generated by the elements in (11), subject to relations of form (7) and (9).

The $\mathcal{A}$-algebra U^0 is the subalgebra with 1 generated by the elements in (12) and (13), subject to relations of form (1) and (2).

The $\mathcal{A}$-algebra U^+ is the subalgebra with 1 generated by the elements in (10), subject to relations of form (6) and (8).

It is known (see [10, §3.2]) that $U \cong U^- \otimes U^0 \otimes U^+$ as $\mathcal{A}$-modules.

1.2 The q-Schur algebra, $S_q(n,r)$

Denote by Θ_r the set of $n \times n$ matrices with nonnegative integer coefficients whose entries sum to r.

Let V be a vector space of dimension r over a field F, and let $\mathcal{F}$ be the set of all n-step flags

$$V_1 \subset V_2 \subset \cdots \subset V_n = V.$$

The group $G = GL(V)$ acts naturally on $\mathcal{F}$, and hence diagonally on $X = \mathcal{F} \times \mathcal{F}$. Choose $(f, f') \in X$. Then

$$f = (V_1 \subset V_2 \subset \cdots \subset V_n), \quad \text{and } f' = (V_1' \subset V_2' \subset \cdots \subset V_n').$$

Set $V_0 = V_0' = \{0\}$ and define

$$a_{ij} = \dim(V_{i-1} + (V_i \cap V_j')) - \dim(V_{i-1} + (V_i \cap V_{j-1}')).$$

The map from (f, f') to (a_{ij}) induces a bijection between the set of G orbits on X and the set Θ_r (see [2]). Define $\mathcal{O}_A$ to be the G-orbit corresponding to $A \in \Theta_r$.

Now suppose F as above is a finite field with q elements. It is shown in [1] that for $A, A', A'' \in \Theta_r$, there exists a function $g_{A,A',A'',q}$ given by

$$g_{A,A',A'',q} := |\{f \in \mathcal{F} : (f_1, f) \in \mathcal{O}_A, (f, f_2) \in \mathcal{O}_{A'}\}| = c_0 + c_1 q + \cdots + c_m q^m,$$

where the c_i are integers that do not depend on the prime power q, and $(f_1, f_2) \in \mathcal{O}_{A''}$. The $\mathbf{Z}[v^2]$-polynomial $g_{A,A',A''}$ is defined by

$$g_{A,A',A''} := c_0 + c_1 v^2 + \cdots + c_m v^{2m}.$$

We now define (following [2] or [1, Proposition 1.2]) the q-Schur algebra, $S_q(n, r)$, to be the free $\mathbf{Z}[q, q^{-1}]$-module with basis $\{e_A : A \in \Theta_r\}$ and with associative multiplication given by

$$e_A e_{A'} = \sum_{A'' \in \Theta_r} g_{A,A',A''} e_{A''}.$$

Du [2] remarks that this algebra is canonically isomorphic to the q-Schur algebra defined by Dipper and James, by exhibiting the following correspondence between basis elements e_A and basis elements $\phi^d_{\lambda\mu}$ as defined by Dipper and James. Here, the elements λ and μ lie in $\Lambda(n, r)$, which is the set of compositions of r into n parts, and $d \in \mathcal{D}_{\lambda\mu}$, which is the set of distinguished W_λ–W_μ double coset representatives for the Young subgroups W_λ and W_μ of $\mathcal{S}_r$, the symmetric group on r letters. Suppose $\lambda = (\lambda_1, \ldots, \lambda_n)$ and $\mu = (\mu_1, \ldots, \mu_n)$. For each $\alpha \in \mathbf{n}$, we define I_α to be the subset of $\mathbf{n}$ given by

$$I_\alpha := \{\lambda_1 + \cdots + \lambda_{\alpha-1} + 1, \lambda_1 + \cdots + \lambda_{\alpha-1} + 2, \ldots, \lambda_1 + \cdots + \lambda_\alpha\}.$$

Similarly, we define J_α to be the analogous subset of $\mathbf{n}$ corresponding to μ and α. Given a Dipper-James basis element, $\phi^d_{\lambda\mu}$, we define a corresponding matrix A via

$$A_{\alpha\beta} := |d(J_\beta) \cap I_\alpha|.$$

This procedure sets up the required isomorphism by sending $\phi^d_{\lambda\mu}$ to e_A. It should be noted that λ corresponds to the sums of the rows of A, and μ to the sums of the columns of A.

We will also be using certain elements $[A]$ in $S_v(n, r)$. These are closely related to the basis elements e_A via

$$[A] := v^{-\dim \mathcal{O}_A + \dim pr_1(\mathcal{O}_A)} e_A.$$

Here, the map pr_1 is the first projection from X to $\mathcal{F}$. Beilinson et al. [1, 2.3] prove that

$$\dim \mathcal{O}_A - \dim pr_1(\mathcal{O}_A) = \sum_{i,j,k,l} A_{ij} A_{kl},$$

where the indices are required to satisfy $i \geq k$ and $j < l$.

It is convenient to have an alternative description of the basis $\{e_A\}$ of $S_q(n,r)$, which we present below.

Let $I(n,r)$ be the set of all ordered r-tuples of elements from the set $\mathbf{n} := \{1, \ldots, n\}$. The symmetric group $\mathcal{S}_r$ acts on the set $I = I(n,r)$ on the right by place permutation in the obvious way, i.e. via

$$(i_1, \ldots, i_r).\pi := (i_{1.\pi}, \ldots, i_{r.\pi}).$$

It also acts on the set $I \times I$ as $(i,j)\pi = (i\pi, j\pi)$. We write $i \sim j$ if i and j are in the same $\mathcal{S}_r$-orbit of I, and $(i,j) \sim (i',j')$ if (i,j) and (i',j') are in the same $\mathcal{S}_r$-orbit of $I \times I$. We now introduce a set of symbols $\xi_{i,j}$ where $i, j \in I$, and we identify $\xi_{i,j}$ and $\xi_{i',j'}$ if and only if $(i,j) \sim (i',j')$. The set of all $\xi_{i,j}$, as (i,j) ranges over a transversal Ω of all $\mathcal{S}_r$-orbits of $I \times I$ can be shown (see e.g. [4]) to index a basis for $S(n,r)$. We will usually write ξ_l as shorthand for $\xi_{l,l}$.

Let $i = (i_1, \ldots, i_r)$ and $j = (j_1, \ldots, j_r)$ be elements of I. We now identify $\xi_{i,j}$ with e_A, where the (x,y)-entry of the matrix A is given by the number of pairs (i_s, j_s) such that $i_s = x$ and $j_s = y$. It is easily seen that this is well-defined and that the matrix A is an $n \times n$ matrix with nonnegative integer entries summing to r. It is also easy to see that the map is surjective, and hence bijective because both bases contain the same number of elements. From now on, we equate $\xi_{i,j}$ with the element e_A of $S_q(n,r)$ as above.

The following well-known facts about the multiplication in $S_q(n,r)$ are important. (Proofs can be found in [3, §2].)

(i) $\xi_{i,j}\xi_{k,l} = 0$ unless $j \sim k$, in which case it is nonzero.

(ii) $\xi_i\xi_{i,j} = \xi_{i,j}$.

(iii) $\xi_{i,j}\xi_j = \xi_{i,j}$.

1.3 The map $\theta : U_{\mathcal{A}}(gl_n) \to S_v(n,r)$

It is proved in [2] that there exists an algebra homomorphism $\theta : U_{\mathcal{A}}(gl_n) \to S_v(n,r)$. This makes $S_v(n,r)$ into a U-module, and shows that it is a quotient of U. This is given, following [2], as follows:

$$\begin{aligned}
\theta(E_i) &= \sum_{D \in \mathbf{D}, E_{i,i+1}+D \in \Theta_r} [E_{i,i+1} + D], \\
\theta(F_i) &= \sum_{D \in \mathbf{D}, E_{i+1,i}+D \in \Theta_r} [E_{i+1,i} + D], \\
\theta(K_i) &= \sum_{D \in \mathbf{D}_r} v^{d_i}[D], \\
\theta(K_i^{-1}) &= \sum_{D \in \mathbf{D}_r} v^{-d_i}[D].
\end{aligned}$$

Here, $\mathbf{D}$ is the set of diagonal matrices, and $\mathbf{D}_r$ means $\mathbf{D} \cap \Theta_r$. The matrix $E_{a,b}$ has 1 in the (a,b) position and zeros elsewhere, and the matrix D is of the form $\mathrm{diag}(d_1, \ldots, d_n)$.

1.4 Root systems of type A_{n-1}

We now state without proof some properties of root systems. The general theory of these can be found in any good text on Lie algebras.

Associated with the Lie algebra sl_n, or (in our case) gl_n, is a certain collection of vectors in $(n-1)$-dimensional Euclidean space known as a *root system* of type A_{n-1}. It is well-known that this root system contains an independent subset (the *fundamental roots*) $\{\alpha_1, \ldots, \alpha_{n-1}\}$ such that any other root is of form

$$\alpha = \alpha_i + \alpha_{i+1} + \cdots + \alpha_j \quad (1 \leq i \leq j < n)$$

or of form

$$\alpha = -\alpha_i - \alpha_{i+1} - \cdots - \alpha_j \quad (1 \leq i \leq j < n).$$

In the first case, the root α is called *positive*, and in the second case, the root α is called *negative*. Denote these two sets of roots by Φ^+ and Φ^-, respectively.

We will also write $\alpha(i, j+1)$ to denote the positive root $\alpha_i + \alpha_{i+1} + \cdots + \alpha_j$.

We define the *height*, $h(\alpha)$, of $\alpha = \alpha(i,j)$ to be $j - i$.

Following [9, §2.2], we define the function $g(\alpha(i,j)) = j - 1$. (The function g finds the index of the highest fundamental root occurring with nonzero coefficient in its argument.)

The bilinear map $(\ ,\ ) : \Phi^+ \times \Phi^+ \to \mathbf{Z}$ is defined to satisfy

$$(\alpha_i, \alpha_j) := \begin{cases} 2 & \text{if } i = j; \\ -1 & \text{if } |i-j| = 1; \\ 0 & \text{otherwise.} \end{cases}$$

1.5 Tableaux and Codeterminants

Following [6, §1.9], we define a *q-codeterminant*, or "codeterminant" for short (when the context is clear), to be a nonzero product $e_A e_{A'}$ of two basis elements of $S_q(n,r)$. (Note that a codeterminant is more than just a product of two basis elements; the given factorisation is also important.)

We introduce the set

$$\Lambda = \Lambda(n,r) := \{\lambda = (\lambda_1, \ldots, \lambda_n) : \lambda_\nu \in \mathbf{N}_0 \text{ for all } \nu \in \mathbf{n}, \sum_\nu \lambda_\nu = r\}$$

and

$$\Lambda^+ = \Lambda^+(n,r) := \{\lambda \in \Lambda : \lambda_1 \geq \cdots \geq \lambda_n\}.$$

Here, $\mathbf{N}_0$ denotes the set of nonnegative integers.

An element of Λ is called a *weight*, and is *dominant* if $\lambda \in \Lambda^+$. There is an obvious correspondence between elements of Λ^+ and partitions of r into not more than n parts.

The weight $\mathrm{wt}(i)$ of an element $i \in I(n,r)$ is the element $\alpha \in \Lambda$ given by $\alpha_\nu = |\{\rho \in \mathbf{r} : i_\rho = \nu\}|$ for all $\nu \in \mathbf{n}$. If $i, j \in I$, it is clear that $i \sim j$ if and only if $\mathrm{wt}(i) = \mathrm{wt}(j)$.

For each $\lambda \in \Lambda^+$ we define a *basic λ-tableau* T^λ by writing the integers $1, \ldots, r$ into a Young diagram in some arbitrary (but henceforth fixed) order. (In practice, the order we pick will always be row by row, starting with the top row, and filling each row from left to right.) To each $i \in I$ we now associate the λ-tableau $T_i^\lambda = iT^\lambda$. For example, let $n = 4$, $r = 7$, $\lambda = (4, 2, 1, 0)$. Using our choice of basic λ-tableau, then

$$T^\lambda = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 4 \\ \hline 5 & 6 & \multicolumn{2}{c}{} \\ \cline{1-2} 7 & \multicolumn{3}{c}{} \\ \cline{1-1} \end{array},$$

and

$$T_i^\lambda = \begin{array}{|c|c|c|c|} \hline i_1 & i_2 & i_3 & i_4 \\ \hline i_5 & i_6 & \multicolumn{2}{c}{} \\ \cline{1-2} i_7 & \multicolumn{3}{c}{} \\ \cline{1-1} \end{array}.$$

If $\lambda \in \Lambda^+$ and $i \in I$, the λ-tableau T_i^λ is said to be *standard* if the elements in each row increase weakly from left to right, and the elements in each column increase strictly from top to bottom. We define

$$I_\lambda := I_\lambda(n,r) = \{i \in I : T_i^\lambda \text{ is standard}\}.$$

We say a λ-tableau T_i^λ is *row-semistandard* if the elements in each row increase weakly from left to right, and define the set

$$I'_\lambda := I'_\lambda(n,r) = \{i \in I : T_i^\lambda \text{ is row-semistandard}\}.$$

It is clear that in a standard tableau, all the entries equal to s must appear in the first s rows, by an easy induction on s. There is exactly one element of I, denoted by $\ell = \ell(\lambda)$, for which T_ℓ^λ is standard and $\mathrm{wt}(\ell) = \lambda$. (The entries in the s-th row of the tableau T_ℓ^λ are all equal to s.)

We find from the correspondence between the elements e_A and the elements $\xi_{i,j}$ that any codeterminant can be written as $\xi_{i,\ell}\xi_{\ell,j}$, where $\ell = \ell(\lambda)$ for some $\lambda \in \Lambda(n,r)$. Following [6], we shall usually express this as $Y_{i,j}^\lambda$.

With any ordered pair $\langle T, T' \rangle$ of standard tableaux of the same shape, each consisting of r boxes and having entries in $\mathbf{n}$, we associate a certain codeterminant $e_A e_{A'}$ in $S_q(n,r)$ as follows.

The entry A_{ij} of A is defined to be the number of occurrences of i in the j-th row of the tableau T, or zero if there is no such entry.

The entry A'_{ij} of A is defined to be the number of occurrences of j in the i-th row of the tableau T', or zero if there is no such entry.

The fact that the tableaux are of the same shape forces the product $e_A e_{A'}$ to be nonzero, because it is of form $\xi_{i,\ell}\xi_{\ell,j}$, and by using the product rule for $S_q(n,r)$, we find that this is nonzero, and hence $e_A e_{A'}$ is a codeterminant. We call such a codeterminant a *standard codeterminant*. This definition agrees with that in [6].

It should be noted that, because the tableaux T and T' are standard, the matrix A must be lower triangular, and the matrix A' must be upper triangular.

Example

Suppose $n = 3$, $r = 6$,

$$T = \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 2 & 3 & \multicolumn{1}{c}{} \\ \cline{1-2} 3 & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array},$$

and

$$T' = \begin{array}{|c|c|c|} \hline 1 & 1 & 2 \\ \hline 2 & 2 & \multicolumn{1}{c}{} \\ \cline{1-2} 3 & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}.$$

Then the matrices A and A' are given by

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

and

$$A' = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

1.6 Bases for U^- and U^+

We now describe Poincaré-Birkhoff-Witt type bases for U^- and for U^+ which are compatible with the properties of θ.

We define certain elements E_α and F_α for each positive root α corresponding to gl_n as follows.

Let $\alpha = \alpha_i + \alpha_{i+1} + \cdots + \alpha_j$ be a (typical) positive root in type A_{n-1}, where the α_i are, as usual, the fundamental roots. Let $\gamma = \alpha - \alpha_j$ in the case where $i \neq j$. Then define, by induction on $j - i$,

$$E_\alpha := \begin{cases} E_\gamma E_j - v^{-1} E_j E_\gamma & \text{if } i \neq j; \\ E_i & \text{if } i = j. \end{cases}$$

Let $\alpha = \alpha_i + \alpha_{i+1} + \cdots + \alpha_j$ and $\beta = \alpha - \alpha_i$ if $i \neq j$. We define, by induction on $j - i$,

$$F_\alpha := \begin{cases} F_\beta F_i - v^{-1} F_i F_\beta & \text{if } i \neq j; \\ F_i & \text{if } i = j. \end{cases}$$

We also order the elements E_α and the elements F_α as follows.

The element $E_{\alpha(i,j)}$ precedes (or appears to the left of) the element $E_{\alpha(k,l)}$ if $i > k$ or ($i = k$ and $j > l$).

The element $F_{\alpha(i,j)}$ precedes (or appears to the left of) the element $F_{\alpha(k,l)}$ if $j < l$ or ($j = l$ and $i < k$).

Note that this definition is essentially the same as that given by Jimbo [8].

Example

Let $n = 4$. In this case, the positive roots are

$$\alpha_1,\ \alpha_1 + \alpha_2,\ \alpha_1 + \alpha_2 + \alpha_3,\ \alpha_2,\ \alpha_2 + \alpha_3,\ \alpha_3.$$

The ordering on the elements E_α corresponds to the ordering

$$\alpha(3,4) < \alpha(2,4) < \alpha(2,3) < \alpha(1,4) < \alpha(1,3) < \alpha(1,2)$$

on the positive roots, and the ordering on the elements F_α corresponds to the ordering

$$\alpha(1,2) < \alpha(1,3) < \alpha(2,3) < \alpha(1,4) < \alpha(2,4) < \alpha(3,4)$$

on the positive roots.

Lemma 1.1

Define $\psi^+ : U^+ \to U^+$ by $\psi^+(E_i) := E_{n-i}$. Then ψ^+ extends naturally to an $\mathcal{A}$-algebra isomorphism $\psi^+ : U^+ \to U^+$.

Define $\psi^- : U^- \to U^-$ by $\psi^-(F_i) := F_{n-i}$. Then ψ^- extends naturally to an $\mathcal{A}$-algebra isomorphism $\psi^- : U^- \to U^-$.

Proof

The map ψ^+ is self-inverse, and preserves the relations (6) and (8).

The map ψ^- is self-inverse, and preserves the relations (7) and (9). ∎

Lemma 1.2

There is an $\mathcal{A}$-algebra isomorphism $\omega^- : U^- \to U^+$ given by $\omega^-(F_i) = E_i$.

There is an $\mathcal{A}$-algebra isomorphism $\omega^+ : U^+ \to U^-$ given by $\omega^+(E_i) = F_i$.

Proof Since ω^- and ω^+ are mutual inverses, and they are clearly surjective, it suffices to check that each one preserves the relations. This is immediate from the nature of the relations (6), (7), (8) and (9). ∎

Lemma 1.3

(i) $\psi^+(\omega^-(F_{\alpha(i,j)})) = E_{\alpha(n+1-j,n+1-i)}$.

(ii) F_α precedes F_β in the ordering on the elements F_γ if and only if $\psi^+(\omega^-(F_\alpha))$ precedes $\psi^+(\omega^-(F_\beta))$ in the ordering on the E_γ.

Proof

We first prove (i), using induction on $h = h(\alpha)$. The case $h = 1$ follows from the definition of ω^-.

For the general case, $F_{\alpha(i,j)} = F_{\alpha(i+1,j)}F_{\alpha(i,i+1)} - v^{-1}F_{\alpha(i,i+1)}F_{\alpha(i+1,j)}$, by definition. By induction, we have

$$\psi^+(\omega^-(F_{\alpha(i,j)})) = E_{\alpha(n-j+1,n-i)}E_{\alpha(n-i,n-i+1)} - v^{-1}E_{\alpha(n-i,n-i+1)}E_{\alpha(n-i,n-j+1)},$$

because $\psi^+\omega^-$ is an algebra isomorphism. The result now follows from the defintion of $E_{\alpha(n-j+1,n-i+1)}$.

The proof of (ii) is immediate from the claim of (i) and the definitions of the two orders. ∎

Definition

Define V^- to be the $\mathcal{A}$-algebra given by generators $\{\widehat{F}_\alpha^{(c)} : \alpha \in \Phi^+, c \in \mathbf{Z}_{\geq 0}\}$ (where $F_\alpha^{(0)} = 1$) and relations

$$\widehat{F}_\alpha^{(c)}\widehat{F}_\alpha^{(b)} = \begin{bmatrix} c+b \\ c \end{bmatrix} \widehat{F}_\alpha^{(c+b)}, \tag{1}$$

$$\widehat{F}_{\alpha_i}^{(c)}\widehat{F}_\alpha^{(b)} = \widehat{F}_\alpha^{(b)}\widehat{F}_{\alpha_i}^{(c)} \qquad \text{if } (\alpha,\alpha_i) = 0 \text{ and } i < g(\alpha), \tag{2}$$

$$\widehat{F}_\alpha^{(b)}\widehat{F}_{\alpha'}^{(c)} = \sum_{j\geq 0; j\leq c; j\leq b} v^{-j-(c-j)(b-j)}\widehat{F}_{\alpha'}^{(c-j)}\widehat{F}_{\alpha+\alpha'}^{(j)}\widehat{F}_\alpha^{(b-j)}, \tag{3}$$

$$v^{cb}\widehat{F}_{\alpha'}^{(c)}\widehat{F}_{\alpha+\alpha'}^{(b)} = \widehat{F}_{\alpha+\alpha'}^{(b)}\widehat{F}_{\alpha'}^{(c)}, \tag{4}$$

$$v^{cb}\widehat{F}_{\alpha+\alpha'}^{(b)}\widehat{F}_\alpha^{(c)} = \widehat{F}_\alpha^{(c)}\widehat{F}_{\alpha+\alpha'}^{(b)}. \tag{5}$$

The relations (3), (4) and (5) are each subject to the restrictions that $(\alpha, \alpha') = -1$ and either ($\alpha' = \alpha_i$ and $i < g(\alpha)$) or ($h(\alpha') = h(\alpha) + 1$ and $g(\alpha') = g(\alpha)$).

Proposition 1.4 (Lusztig)

There is an $\mathcal{A}$-algebra isomorphism $\phi : V^- \to U^-$ satisfying $\phi(\widehat{F}_{\alpha_i}) = F_i$, and the set

$$\left\{ \prod_{\alpha\in\Phi^+} \widehat{F}_\alpha^{(c_\alpha)} : c_\alpha \in \mathbf{Z}_{\geq 0} \right\},$$

is an $\mathcal{A}$-basis for V^-, where the order taken for the product is the same as the order on our elements F_α.

Proof

The required isomorphism is exhibited in [9, Theorem 4.5]. This theorem also shows that with a certain fixed order, the products as shown above form an $\mathcal{A}$-basis for V^-. Fortunately, this fixed order (which is the reverse of the order shown in [9, 2.9 (a)]) is exactly the same as the order we imposed on the elements F_α!

From [9, Corollary 4.3] and [9, Proposition 1.8 (d)], we see that $\phi(\widehat{F}_{\alpha_i}) = F_i$. This completes the proof. ∎

Proposition 1.5

(i) The set

$$B^- := \left\{ \prod_{\alpha \in \Phi^+} F_\alpha^{(c_\alpha)} : c_\alpha \in \mathbf{Z}_{\geq 0} \right\},$$

where the product is taken in the order corresponding to that on the elements F_α, is an $\mathcal{A}$-basis for U^-.

(ii) The set

$$B^+ := \left\{ \prod_{\alpha \in \Phi^+} E_\alpha^{(c_\alpha)} : c_\alpha \in \mathbf{Z}_{\geq 0} \right\},$$

where the product is taken in the order corresponding to that on the elements E_α, is an $\mathcal{A}$-basis for U^+.

Proof

The result (ii) will follow from Lemma 1.3 and (i), so it is enough to prove (i).

To prove (i), notice that the relation (3) before Proposition 1.4 shows that

$$v^{-1}\widehat{F}_{\alpha(i,j)} = \widehat{F}_{\alpha(i+1,j)}\widehat{F}_{\alpha(i,i+1)} - v^{-1}\widehat{F}_{\alpha(i,i+1)}\widehat{F}_{\alpha(i+1,j)}$$

Since we know that $F_{\alpha(i,i+1)} = \widehat{F}_{\alpha(i,i+1)}$, we now see by an induction on $h(\alpha)$ that $F_\alpha = v^{-h(\alpha)+1}\widehat{F}_\alpha$. Since the claim of (i) is true if we replace F by $\widehat{F}$ whenever it appears in the statement, and the element F_α differs from $\widehat{F}_\alpha$ by a unit in $\mathcal{A}$, we see that (i) holds. This completes the proof. ∎

2. The restriction of θ to U^-, U^0 and U^+

The next aim is to obtain precise and general descriptions for $U^+ \cap \ker\theta$ and for $U^- \cap \ker\theta$, concentrating on the first case, because the second case is essentially similar.

Definition Let X be the matrix

$$\begin{pmatrix} 0 & c_1 & c_2 & c_3 & \cdots & c_{n-1} \\ 0 & 0 & c_n & c_{n+1} & \cdots & c_{2n-3} \\ \vdots & & & & \ddots & \\ 0 & 0 & 0 & 0 & \cdots & c_N \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

We define

$$y_{X,r} := \sum_{\lambda_1 + \cdots + \lambda_n + |X| = r} [(X, \lambda_1, \ldots, \lambda_n)],$$

summed over all sets of nonnegative integers $\lambda_1, \ldots, \lambda_n$, where the matrix $(X, \lambda_1, \ldots, \lambda_n)$ is given by $X + \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$, and $|X|$ denotes the sum of the entries in X.

It is known that, for $1 \leq i \leq n-1$, E_i maps under θ to $y_{X,r}$, where X is the matrix $E_{i,i+1}$, having 1 in $(i, i+1)$ place and zeros elsewhere.

In order to prove the main result of this section, we will rely on the the following lemma, which is a simple corollary of a lemma by Beilinson, Lusztig and MacPherson.

Lemma 2.1 Let X be a strictly upper triangular $n \times n$ matrix, with $|X| \leq r$. Then:

$$\theta(E_h) \times y_{X,r} = (1 - \delta_{|X|,r}) v^{\beta(h+1)} [X_{h,h+1} + 1]_v y_{X+E_{h,h+1},r} + \sum_{p \in [h+2,n]; X_{h+1,p} \geq 1} v^{\beta(p)} [X_{h,p} + 1]_v y_{X+E_{h,p}-E_{h+1,p},r}$$

where, as usual, $E_{i,j}$ is the matrix with 1 in the (i,j) place, and 0 everywhere else.

Here,

$$\beta(p) := \sum_{j>p} (X_{h,j} - X_{h+1,j}).$$

Let X be a strictly lower triangular $n \times n$ matrix. Then:

$$\theta(F_h) \times y_{X,r} = (1 - \delta_{|X|,r}) v^{\beta'(h)} [X_{h+1,h} + 1]_v y_{X+E_{h+1,h},r} + \sum_{p \in [1,h-1]; X_{h,p} \geq 1} v^{\beta'(p)} [X_{h+1,p} + 1]_v y_{X-E_{h,p}+E_{h+1,p},r}.$$

Here,

$$\beta'(p) := \sum_{j<p} (X_{h+1,j} - X_{h,j}).$$

Proof This follows from [1, Lemma 3.2], using the formula for $\theta(E_i)$. ∎

Definition

Let X be an $n \times n$ matrix with entries in $\mathbf{Z}$. We say X has the property $P(i,j)$ (where i and j are integers satisfying $1 \leq i \leq j < n$) if it satisfies one of two conditions. If row $j+1$ of X consists entirely of zeros, then X has the property $P(i,j)$. Otherwise, let p be minimal such that $X_{j+1,p}$ is nonzero. If for each integer s such that $i < s < j+1$, $X_{s,t} = 0$ for $t \geq s+p-j$, then X has the property $P(i,j)$. If neither condition holds, then X does not have the property $P(i,j)$.

Example

Suppose

$$X = \begin{pmatrix} 3 & 5 & 1 & 0 & 3 & 4 \\ 5 & 2 & 3 & 0 & 3 & 1 \\ 0 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 1 & 3 & 0 & 2 & 4 & 4 \end{pmatrix}.$$

Then X has the properties $P(k,k)$ for all $1 \leq k < 6$, and also $P(1,3)$, $P(2,3)$, $P(2,4)$ and $P(3,4)$.

Lemma 2.2 Let $\alpha = \alpha_i + \alpha_{i+1} + \cdots + \alpha_j$ be a positive root.

Assume X is a strictly upper triangular matrix such that if D is a diagonal n by n matrix with nonnegative integer entries, then $X + D$ has the property $P(i,s)$ for all $i < s \leq j$. The action of E_α on $y_{X,r}$ is given by

$$\theta(E_\alpha) \times y_{X,r} = (1 - \delta_{|X|,r}) v^{x(j+1)} [X_{i,j+1} + 1]_v y_{X+E_{i,j+1},r} + \sum_{p \in [j+2,n]; X_{j+1,p} \geq 1} v^{x(p)} [X_{i,p} + 1]_v y_{X+E_{i,p}-E_{j+1,p},r}.$$

Here,

$$x(p) := \sum_{m>p} (X_{i,m} - X_{j+1,m}).$$

Proof The proof is by induction on $n' = j - i$. The case $n' = 0$ is done by Lemma 2.1. It now suffices to check that, when $i < j$ and $\beta = \alpha - \alpha_j$, if the hypothesis works for E_β and for E_j (by induction), then it works for $E_\alpha := E_\beta E_j - v^{-1} E_j E_\beta$.

Using the inductive hypothesis applied to E_β (since we know the matrix X has the property $P(i,s)$ for all $i < s \leq j-1$), we can assume that

$$\theta(E_\beta) \times y_{X,r} = (1 - \delta_{|X|,r}) v^{x'(j)} [X_{i,j} + 1]_v y_{X+E_{i,j},r} + \sum_{p \in [j+1,n]; X_{j,p} \geq 1} v^{x'(p)} [X_{i,p} + 1]_v y_{X+E_{i,p}-E_{j,p},r},$$

where

$$x'(p) := \sum_{m>p} (X_{i,m} - X_{j,m}).$$

Rephrasing this informally, and concentrating on one particular term, $[Y]$, of $y_{X,r}$, we find that a typical term in the action of E_β on $[Y]$ is to decrease an entry in the (j,p)-place of Y by 1, for some suitable p, to increase the entry in the (i,p) place by 1 (resulting in a new matrix Y') and to multiply by

$$v^{y'(p)} [Y'_{i,p}]_v,$$

where

$$y'(p) := \sum_{m>p} (Y_{i,m} - Y_{j,m}).$$

Similarly, considering the action of E_j on $[Z]$ we find that we find that the action of a typical term is, for some suitable p', to decrease an entry in the $(j+1,p')$-place of Y by 1, to increase the entry in the (j,p') place by 1 (resulting in a new matrix Z') and to multiply by

$$v^{z'(p')} [Z'_{i,p'}]_v,$$

where

$$z'(p') := \sum_{m>p'} (Z'_{j,m} - Z'_{j+1,m}).$$

The crucial issue is the extent to which these two actions fail to commute if $p \neq p'$. Let us suppose that $p \neq p'$. We claim that the property P (as above) forces $p \leq p'$. Suppose to the contrary that $p > p'$. This implies that the matrix Y has nonzero entries in the $(j+1, p')$ place and in the (j, p) place. However, the matrix Y is assumed to have property $P(i, j)$, and we are also assuming that row $j+1$ is not empty. Let p'' be minimal such that $Y_{j+1,p''}$ is nonzero. Then we must have $p'' \leq p' < p$. Putting $s = j$ in the definition of the property P (which is valid since $n' = j - i > 0$), we see that $Y_{j,t} = 0$ for $t \geq p''$, so in particular, $Y_{j,p} = 0$. This is a contradiction.

Next, consider a typical term, $[A]$, of $y_{X,r}$. Of course, A is an upper triangular matrix. Acting $E_\beta E_j$ on this matrix, on the left, we obtain the new matrix $[A']$ (with entries (j, p) and $(j+1, p')$ decreased by 1, and entries (i, p) and (j, p') increased by 1) with coefficient

$$[A_{i,p} + 1]_v [A_{j,p'} + 1]_v v^x,$$

where

$$x = \sum_{m>p'} (A_{j,m} - A_{j+1,m}) + \sum_{m>p} (A'_{i,m} - A'_{j,m}),$$

i.e.

$$\sum_{m>p'} (A_{j,m} - A_{j+1,m}) + \sum_{m>p} (A_{i,m} - A_{j,m}) - 1.$$

Similarly, computing the action of $E_j E_\beta$, we find that matrix $[A']$ occurs with coefficient

$$[A_{i,p} + 1]_v [A_{j,p'} + 1]_v v^{x'},$$

where

$$x' = \sum_{m>p'} (A_{j,m} - A_{j+1,m}) + \sum_{m>p} (A_{i,m} - A_{j,m}),$$

which is exactly v times the coefficient with the elements acting the other way round. Thus, in the action of $E_\beta E_j - v^{-1} E_j E_\beta$, the above terms cancel out.

The only remaining terms correspond to the situation $p = p'$. Using similar methods to the above, we find that the coefficient of $[A']$ in the action of $E_\beta E_j - v^{-1} E_j E_\beta$ on $[A]$ is

$$[A_{i,p} + 1]_v [A_{j,p} + 1]_v v^{x''},$$

where

$$x'' = \sum_{m>p} (A_{i,m} - A_{j+1,m}).$$

In fact, $A_{j,p}$ is equal to zero, because the matrix $[A]$ has the property $P(i, j)$ and $A_{j+1,p}$ is nonzero. Hence $[A_{j,p} + 1]_v = 1$.

Notice that this argument is essentially independent of the entries on the diagonal of A. It therefore applies equally well to the action of E_α on $y_{X,r}$, thus establishing the inductive hypothesis for E_α.

This completes the proof. ∎

Recall from §1 the ordering defined on Φ^+ associated with the elements E_α. Denote the last root in this list by β_1, the second from last by β_2, etc.

Proposition 2.3 The element

$$E_{\beta_N}^{(c_N)} \cdots E_{\beta_1}^{(c_1)}$$

maps under θ to $y_{X,r}$. If $|X| > r$, then the given basis element maps to zero.

Proof The proof will be by induction on $n' = \sum_{k=1}^N c_k$. The case $n' = 1$ is done by Lemma 2.2. For the general case, we will prove the equivalent statement that

$$\theta(E_{\beta_N}^{c_N} \cdots E_{\beta_1}^{c_1}) = \left(\prod_{k=1}^N [c_k]_v! \right) y_{X,r}$$

Let p be maximal such that c_p is nonzero, and let X' be the matrix obtained by decreasing the entry in the position corresponding to c_p in X by 1. We now need to check that E_{β_p} acts on $y_{X',r}$ to give $[c_p]_v y_{X,r}$, as expected. If $n' > r+1$ then there is nothing to prove, because $y_{X',r} = 0$, by the inductive hypothesis. Otherwise, express β_p in the form $\alpha_i + \alpha_{i+1} + \cdots + \alpha_j$. The ordering chosen for the positive roots guarantees that if D is a diagonal n by n matrix with nonnegative integer entries, then the matrix $X' + D$ has no nonzero entries between rows $i+1$ and $j+1$ inclusive, except possibly on the diagonal. This means that $X' + D$ has the properties $P(i, s)$ for $i < s \leq j$, so we can apply Lemma 2.2. Since $X_{j+1,t} = 0$ for $t > j+1$, we find that only the first term occuring in Lemma 2.2 can appear. If $n' = r$, then we find that this term too is zero, making our element map under θ to zero, as expected. Further scrutiny of the matrices X' and X reveals that $X_{i,m} = 0$ for $m > j+1$; again this is by properties of the ordering chosen on the positive roots. This means that the quantity $x(j+1)$ occurring in Lemma 2.2 is equal to 0, and the inductive step immediately follows. ∎

Corollary 2.4 $U^+ \cap \ker\theta$ is generated by those basis elements which map to zero under θ, i.e.

$$U^+ \cap \ker\theta = \left\langle \left\{ E_{\beta_N}^{(c_N)} \cdots E_{\beta_1}^{(c_1)} : \sum_{i=1}^N c_i \geq r+1 \right\} \right\rangle.$$

Proof The basis elements which do not map to zero all map to elements with different associated matrices X, and hence their images are linearly independent. The corollary now follows. ∎

Corollary 2.5 $\theta(U^+)$ is the subalgebra of $S_q(n,r)$ generated by $y_{X,r}$ for all possible X of zero triangular form.

Proof Clearly $\theta(U^+)$ is the subspace generated by the images of the PBW-type basis elements of U^+. These elements either map to zero, or to a multiple of one of the above elements $y_{X,r}$. ∎

Corollary 2.6 The dimension of $\theta(U^+)$ is given by

$$\binom{\binom{n}{2}+r}{r}.$$

Proof Firstly, observe that by elementary properties of Pascal's triangle,

$$\sum_{i=0}^{r}\binom{\binom{n}{2}+i-1}{i}=\binom{\binom{n}{2}+r}{r}.$$

Secondly, recall that the number of compositions of r into m pieces is known to be

$$\binom{m+r-1}{r}.$$

To count the dimension of $\theta(U^+)$, it suffices to enumerate the possible matrices X, because the corresponding $y_{X,r}$ are linearly independent. The possibilities for X correspond to compositions of integers between 0 and r into $n(n-1)/2$ pieces. The result now follows. ∎

Remarks on the behaviour of U^- The corresponding results for U^- are similar in spirit, and correspond to "rotating the basis matrices by a half turn".

Recall the ordering on Φ^+ associated with the elements F_α. Denote the last root in this list by γ_1, the second from last by γ_2, etc. We have the following result.

Proposition 2.7

Let X be the matrix

$$\begin{pmatrix} 0 & 0 & 0 & 0 & \cdots & 0 \\ c_N & 0 & 0 & 0 & \cdots & 0 \\ \vdots & & & & \ddots & \\ c_{2n-3} & \cdots & c_{n+1} & c_n & 0 & 0 \\ c_{n-1} & \cdots & c_3 & c_2 & c_1 & 0 \end{pmatrix}.$$

We define

$$z_{X,r} := \sum_{\lambda_1+\cdots+\lambda_n+|X|=r} [(X,\lambda_1,\ldots,\lambda_n)],$$

summed over all sets of nonnegative integers $\lambda_1,\ldots,\lambda_n$, where the matrix $(X,\lambda_1,\ldots,\lambda_n)$ is given by $X+\mathrm{diag}(\lambda_1,\ldots,\lambda_n)$, and $|X|$ denotes the sum of the entries in X.

Then the element

$$F_{\gamma_N}^{(c_N)} \cdots F_{\gamma_1}^{(c_1)}$$

maps under θ to $z_{X,r}$. If $|X| > r$, then the given basis element maps to zero.

The intersection of U^- with the kernel of θ is given by

$$U^- \cap \ker\theta = \left\langle \left\{ F_{\gamma_N}^{(a_N)} \cdots F_{\gamma_1}^{(a_1)} : \sum_{i=1}^N a_i \geq r+1 \right\} \right\rangle.$$

Proof This is the same as the proof of the corresponding result for U^+, with trivial changes. ∎

We now investigate the case of $U^0 \cap \ker\theta$. To do this, we require the following result.

Proposition 2.8

A basis for U^0 is given by the set

$$K_1^{\delta_1} K_2^{\delta_2} \cdots K_n^{\delta_n} \begin{bmatrix} K_1; 0 \\ t_1 \end{bmatrix} \begin{bmatrix} K_2; 0 \\ t_2 \end{bmatrix} \cdots \begin{bmatrix} K_n; 0 \\ t_n \end{bmatrix},$$

where $\delta_i \in \{0,1\}$ and $t_i \in \mathbf{N}$, and where

$$\begin{bmatrix} K_1; c \\ t \end{bmatrix} := \prod_{s=1}^t \frac{K_i v^{c-s+1} - K_i^{-1} v^{-c+s-1}}{v^s - v^{-s}}.$$

Proof This follows from [9, Proposition 2.14, Theorem 4.5]. ∎

We wish to investigate the images under θ of these basis elements. We know that

$$\theta(K_i) = \sum_{D \in \mathbf{D}_r} v^{d_i}[D],$$

where $D = \mathrm{diag}(d_1, \ldots, d_n)$, $\mathbf{D}_r$ is the set of $n \times n$ diagonal matrices with nonnegative integer entries summing to r, and as usual

$$[D] = v^{-\dim \mathcal{O}_D + \dim pr_1(\mathcal{O}_D)} e_D,$$

but since D is diagonal, it is not hard to see that $[D] = e_D$. Since e_D corresponds to $\phi^1_{\lambda\lambda}$ in the notation of Dipper and James [3], we see that the e_D form a set of mutually orthogonal idempotents. Armed with this information, it is easy to work out the images of products of the K_i, because the multiplication in the v-Schur algebra is componentwise.

Lemma 2.9

$$\theta\left(K_1^{\delta_1} K_2^{\delta_2} \cdots K_n^{\delta_n} \begin{bmatrix} K_1; 0 \\ t_1 \end{bmatrix} \begin{bmatrix} K_2; 0 \\ t_2 \end{bmatrix} \cdots \begin{bmatrix} K_n; 0 \\ t_n \end{bmatrix} \right) = \sum_{D \in \mathbf{D}_r} v^{\delta_1 d_1 + \cdots + \delta_n d_n} \begin{bmatrix} d_1 \\ t_1 \end{bmatrix} \begin{bmatrix} d_2 \\ t_2 \end{bmatrix} \cdots \begin{bmatrix} d_n \\ t_n \end{bmatrix} [D].$$

Proof This was proved in [2, 3.4 (a)]. ∎

Corollary 2.10

a) If $\sum_{i=1}^n t_i = r$, then

$$\theta\left(K_1^{\delta_1}K_2^{\delta_2}\cdots K_n^{\delta_n}\begin{bmatrix}K_1;0\\t_1\end{bmatrix}\begin{bmatrix}K_2;0\\t_2\end{bmatrix}\cdots\begin{bmatrix}K_n;0\\t_n\end{bmatrix}\right) = v^{\delta_1 t_1+\cdots+\delta_n t_n}[T],$$

where $T \in \mathbf{D}_r$ and the diagonal entries of T are $t_1, \ldots, t_n$.

b) If $\sum_{i=1}^n t_i > r$, then

$$\theta\left(K_1^{\delta_1}K_2^{\delta_2}\cdots K_n^{\delta_n}\begin{bmatrix}K_1;0\\t_1\end{bmatrix}\begin{bmatrix}K_2;0\\t_2\end{bmatrix}\cdots\begin{bmatrix}K_n;0\\t_n\end{bmatrix}\right) = 0.$$

Proof Note that if $d_i < t_i$ then

$$\begin{bmatrix}d_i\\t_i\end{bmatrix} = 0.$$

It now follows, in case a), that there is only one term in the sum given in Lemma 2.9, namely the one given. By a similar argument, we find that in case b) there are no surviving terms in the sum of Lemma 2.9. ∎

Corollary 2.10 provides us with a large part of $U^0 \cap \ker\theta$. The rest comes from considering basis elements H of U^0 satisfying $\sum_{i=1}^n t_i \leq r$, and subtracting off other elements corresponding, via Corollary 2.10, to the terms appearing in $\theta(H)$, yielding:

Corollary 2.11

$$\theta\left(K_1^{\delta_1}\cdots K_n^{\delta_n}\begin{bmatrix}K_1;0\\t_1\end{bmatrix}\cdots\begin{bmatrix}K_n;0\\t_n\end{bmatrix} - \sum_{D\in\mathbf{D}_r} v^{\delta_1 d_1+\cdots+\delta_n d_n}\begin{bmatrix}d_1\\t_1\end{bmatrix}\cdots\begin{bmatrix}d_n\\t_n\end{bmatrix}\begin{bmatrix}K_1;0\\d_1\end{bmatrix}\cdots\begin{bmatrix}K_n;0\\d_n\end{bmatrix}\right) = 0$$

Proof Use Lemma 2.9 and Corollary 2.10. ∎

Using the basis of U^0 described earlier, it is now possible to describe $U^0 \cap \ker\theta$ exactly.

Proposition 2.12 Let $\kappa_{n,r}$ be the function on the basis of U^0 given in Proposition 2.8 sending

$$K_1^{\delta_1}K_2^{\delta_2}\cdots K_n^{\delta_n}\begin{bmatrix}K_1;0\\t_1\end{bmatrix}\begin{bmatrix}K_2;0\\t_2\end{bmatrix}\cdots\begin{bmatrix}K_n;0\\t_n\end{bmatrix}$$

to

$$K_1^{\delta_1}\cdots K_n^{\delta_n}\begin{bmatrix}K_1;0\\t_1\end{bmatrix}\cdots\begin{bmatrix}K_n;0\\t_n\end{bmatrix} - \sum_{D\in\mathbf{D}_r} v^{\delta_1 d_1+\cdots+\delta_n d_n}\begin{bmatrix}d_1\\t_1\end{bmatrix}\cdots\begin{bmatrix}d_n\\t_n\end{bmatrix}\begin{bmatrix}K_1;0\\d_1\end{bmatrix}\cdots\begin{bmatrix}K_n;0\\d_n\end{bmatrix}.$$

Then a basis for $U^0 \cap \ker\theta$ is given by $\{\kappa_{n,r}(H)\}$ as H runs through basis elements of the form

$$K_1^{\delta_1}K_2^{\delta_2}\cdots K_n^{\delta_n}\begin{bmatrix}K_1;0\\t_1\end{bmatrix}\begin{bmatrix}K_2;0\\t_2\end{bmatrix}\cdots\begin{bmatrix}K_n;0\\t_n\end{bmatrix}$$

except those where all the δ_i are equal to zero and $\sum_{i=1}^{n} t_i = r$.

Proof By Corollary 2.11, $\theta(\kappa_{n,r}(H)) = 0$ for all basis elements H. By Corollary 2.10, if $\sum_{i=1}^{n} t_i \neq r$, or $\sum_{i=1}^{n} t_i = r$ and not all the δ_i are zero, then H occurs as a "lowest term" (since $d_i \geq t_i$) of coefficient 1 in $\kappa_{n,r}(H)$. In this case, $\kappa_{n,r}(H)$ is nonzero. This proves that the elements given in the statement of the Proposition are linearly independent in U^0.

Also by Corollary 2.10, if $\sum_{i=1}^{n} t_i = r$, and all the δ_i are zero, then $\kappa_{n,r}(H) = 0$. In this case, $\theta(H)$ is some diagonal basis element $[D]$, and all such basis elements turn up as the image of exactly one such H. This proves that that the elements $\{\kappa_{n,r}(H)\}$ as given in the statement of the proposition span $U_0 \cap \ker\theta$, as required. ∎

Definition Take as a basis B^0 of U^0 the elements

$$K_1^{\delta_1} K_2^{\delta_2} \cdots K_n^{\delta_n} \begin{bmatrix} K_1; 0 \\ t_1 \end{bmatrix} \begin{bmatrix} K_2; 0 \\ t_2 \end{bmatrix} \cdots \begin{bmatrix} K_n; 0 \\ t_n \end{bmatrix},$$

where $t_i \in \mathbf{N}_0$, and $\sum_{i=1}^{n} t_i > r$ or both $\sum_{i=1}^{n} t_i = r$ and all the δ_i are equal to zero, together with the elements

$$K_1^{\delta_1} \cdots K_n^{\delta_n} \begin{bmatrix} K_1; 0 \\ t_1 \end{bmatrix} \cdots \begin{bmatrix} K_n; 0 \\ t_n \end{bmatrix} - \sum_{D \in \mathbf{D}_r} v^{\delta_1 d_1 + \cdots + \delta_n d_n} \begin{bmatrix} d_1 \\ t_1 \end{bmatrix} \cdots \begin{bmatrix} d_n \\ t_n \end{bmatrix} \begin{bmatrix} K_1; 0 \\ d_1 \end{bmatrix} \cdots \begin{bmatrix} K_n; 0 \\ d_n \end{bmatrix},$$

where $t_i \in \mathbf{N}_0$, and $\sum_{i=1}^{n} t_i < r$ or both $\sum_{i=1}^{n} t_i = r$ and at least one δ_i is nonzero.

This basis will be useful in §3.

Note that we now know that $B = B^- \otimes B^0 \otimes B^+$ is a basis for U. This provides a very convenient context in which to study θ.

3. Codeterminants, explicit surjectivity and ker (θ)

We now study some of the properties of $U^{\geq 0}$, the subalgebra of U generated by U^0 and U^+, and $U^{\leq 0}$, the subalgebra of U generated by U^0 and U^-. We concentrate on $U^{\geq 0}$.

Proposition 3.1 Let A be an upper triangular matrix corresponding to a basis element of $S_q(n, r)$. Let t_i be the sum of the entries in row i of A. Then the element of $U^{\geq 0}$ given by

$$\begin{bmatrix} K_1; 0 \\ t_1 \end{bmatrix} \begin{bmatrix} K_2; 0 \\ t_2 \end{bmatrix} \cdots \begin{bmatrix} K_n; 0 \\ t_n \end{bmatrix} E_{\beta_N}^{(c_N)} \cdots E_{\beta_1}^{(c_1)}$$

maps under θ to $[A]$, where the c_i correspond to the entries above the diagonal in A according to the order on the positive roots associated with the elements E_α.

Proof This follows from Proposition 2.3, Corollary 2.10 and the following fact. In the q-Schur algebra as presented by Dipper and James [3, §2], they show that that

$$\phi^d_{ab}\phi^1_{cc} = \delta_{bc}\phi^d_{ac},$$

and that

$$\phi^1_{aa}\phi^d_{bc} = \delta_{ab}\phi^d_{ac}.$$

Recalling the discussion before Lemma 2.9, and the correspondence between $[A]$-basis and $\phi_{\lambda\mu}$-basis, it becomes clear that $[D].[A] = [A]$ if the diagonal entries of $[D]$ are the row sums of $[A]$, and $[D][A] = 0$ otherwise. Therefore, the effect of the U^0 part of the element given in the Proposition is to pick out one term of the $y_{X,r}$ expression corresponding to the U^+-part of the element in the Proposition. The matrix C corresponding to the term $[C]$ picked out in this way has the same row sums as A, and is the same above the diagonal, so it must actually be A.

These remarks suffice to prove the Proposition. ∎

Corollary 3.2 The image of $U^{\geq 0}$ under θ is precisely the subspace of $S_q(n,r)$ spanned by the upper triangular basis elements.

Proof All basis elements in the basis $B^0 \otimes B^+$ of $U^{\geq 0}$ map under θ to sums of upper triangular basis elements. We know from Proposition 3.1 that we can find an element of $U^{\geq 0}$ mapping to any desired upper triangular basis element. This completes the proof. ∎

Remark The image of $U^{\geq 0}$ under θ is, in the classical case, precisely the so-called *Borel subalgebra* of the Schur algebra, often denoted by S^+. Similarly, the image of $U^{\leq 0}$ is the subalgebra denoted by S^- in the literature, and $\theta(U^{\leq 0})$ is the subalgebra spanned by lower triangular basis elements.

The situation for $U^{\leq 0}$ is extremely similar. For example, Proposition 3.1 becomes:

Let A be a lower triangular matrix corresponding to a basis element of $S_q(n,r)$. Let t_i be the sum of the entries in column i of A. Then the element of $U^{\geq 0}$ given by

$$F^{(c_N)}_{\gamma_N} \cdots F^{(c_1)}_{\gamma_1} \begin{bmatrix} K_1; 0 \\ t_1 \end{bmatrix} \begin{bmatrix} K_2; 0 \\ t_2 \end{bmatrix} \cdots \begin{bmatrix} K_n; 0 \\ t_n \end{bmatrix}$$

maps under θ to $[A]$, where the c_i correspond to the entries below the diagonal in A according to the usual order on the positive roots associated with the elements F_α.

We are now in a position to describe the relationship between $U(gl_n)$ and $S_v(n,r)$. Recall from §1.1 that

$$U \cong U^- \otimes U^0 \otimes U^+.$$

We will work with the basis $B = B^- \otimes B^0 \otimes B^+$ for the algebra U.

We find from earlier results that all but finitely many elements of B map to zero under θ. We will show that the elements of B which do not map to zero map to elements of $S_v(n,r)$ which are closely related to codeterminants.

Definitions If $Y_{i,j}^\lambda$ is a typical q-codeterminant given by the factorisation $e_A e_{A'}$, then denote by $\widehat{Y}_{i,j}^\lambda$ the element of $S_v(n,r)$ given by $[A][A']$. We will call the elements $\widehat{Y}_{i,j}^\lambda$ *v-codeterminants.* Recall that the triple (λ, i, j) is related to the product $[A][A']$ as follows. The entry $A_{a,b}$ of A is defined to be the number of occurrences of a in the b-th row of the tableau of shape λ corresponding to i. The entry $A'_{a,b}$ of A' is defined to be the number of occurrences of b in the a-th row of the tableau of shape λ corresponding to j.

We now recall the straightening formula for q-codeterminants.

Proposition 3.3

Let $Y_{i,j}^\lambda$ be any q-codeterminant.

Then $Y_{i,j}^\lambda$ is a $\mathbf{Z}[q,q^{-1}]$-linear combination of standard q-codeterminants. The coefficients arising in this expression are unique.

Proof This is done in [6, Corollary 3.7], where it was shown that the standard q-codeterminants form a free basis for $S_q(n,r)$. ∎

We immediately have the following Corollary:

Corollary 3.4

Let $\widehat{Y}_{i,j}^\lambda$ be any v-codeterminant.

Then $\widehat{Y}_{i,j}^\lambda$ is a $\mathbf{Z}[v,v^{-1}]$-linear combination of standard v-codeterminants. The coefficients arising in this expression are unique.

Proof This follows because the element $\widehat{Y}_{i,j}^\lambda$ is a power of v times $Y_{i,j}^\lambda$, and $q = v^2$. ∎

Definitions

Define the elements $C^{\lambda,i,j}_{\mu,a,b} \in \mathcal{A}$ to be such that

$$\widehat{Y}^{\lambda}_{i,j} = \sum_{\mu,a,b} C^{\lambda,i,j}_{\mu,a,b} \widehat{Y}^{\mu}_{a,b},$$

where the sum is taken over all triples (μ, a, b) such that $Y^{\mu}_{a,b}$ is a standard codeterminant.

Let $Y^{\lambda}_{i,j}$ be a codeterminant, equal to $e_A e_{A'}$. If A is lower triangular and A' is upper triangular, then we say $Y^{\lambda}_{i,j}$ and $\widehat{Y}^{\lambda}_{i,j}$ are *distinguished.*

Note that any standard codeterminant is distinguished. This follows from the correspondence between the $\xi_{i,j}$ and the e_A, and the fact that in a standard tableau, all entries in the i-th row are greater than or equal to i. (Refer to §1.5 for more details.)

Theorem 3.5 Any element of the basis B of U which does not map to zero maps to a distinguished v-codeterminant. Every distinguished v-codeterminant is the image of one element of B in a natural way.

Proof From our earlier analysis of $\ker\theta \cap U$, we find that the only elements of B which do not map to zero under θ are of form

$$b = \prod_{\alpha\in\Phi^+} F_{\alpha}^{(c_\alpha)} \prod_{i=1}^{n} \begin{bmatrix} K_i;0 \\ t_i \end{bmatrix} \prod_{\alpha\in\Phi^+} E_{\alpha}^{(b_\alpha)},$$

where $\sum_\alpha c_\alpha \leq r$, $\sum_\alpha b_\alpha \leq r$ and $\sum_i t_i = r$. We know that the element $\prod_{i=1}^{n} \begin{bmatrix} K_i;0 \\ t_i \end{bmatrix}$ maps under θ to $[D]$, where $D = \mathrm{diag}(t_1, \ldots, t_n)$, and that $[D][D] = [D]$. Thus $\theta(b) = \theta(b^-)\theta(b^+)$, where b^- is given by

$$\prod_{\alpha\in\Phi^+} F_{\alpha}^{(c_\alpha)} \prod_{i=1}^{n} \begin{bmatrix} K_i;0 \\ t_i \end{bmatrix},$$

and b^+ is given by

$$\prod_{i=1}^{n} \begin{bmatrix} K_i;0 \\ t_i \end{bmatrix} \prod_{\alpha\in\Phi^+} E_{\alpha}^{(b_\alpha)},$$

so in particular $\theta(b^-) \neq 0 \neq \theta(b^+)$. We know from our study of $U^{\leq 0}$ and $U^{\geq 0}$ that if $\theta(b^-) \neq 0$ then $\theta(b^-) = [L]$ for some lower triangular matrix L. Similarly $\theta(b^+) = [U]$ for some upper triangular matrix U. The number t_i is thus the sum of the entries in row i of U, and also the sum of the entries in column i of L. This means that in this case $e_L e_U = \xi_{i,j}\xi_{j,k}$ for suitable $i, j, k \in I(n, r)$, and we see that this quantity is nonzero because Schur's product rule shows that it is nonzero in the classical case. Therefore $\theta(b) = [L][U]$ is a distinguished v-codeterminant.

Conversely, any distinguished v-codeterminant $[L][U]$ such that $[L][U] \neq 0$ has a unique element of B which maps to it, by reversing the previous argument. (The integers in the expression of b determine the coefficients of the matrices L and U, and vice versa.) ∎

Definition For each distinguished v-codeterminant $\widehat{Y}^{\lambda}_{i,j} = [L][U]$, we define the element $\Upsilon^{\lambda}_{i,j}$ to be that element of B (as above) which maps under θ to $[L][U]$.

Corollary 3.6 (kernel of θ) The kernel of $\theta : U(gl_n) \to S_v(n,r)$ has as a basis all elements of B which map to zero under θ together with all elements of form

$$\Upsilon^{\lambda}_{i,j} - \sum_{\mu,a,b} C^{\lambda,i,j}_{\mu,a,b} \Upsilon^{\mu}_{a,b},$$

where $Y^{\lambda}_{i,j}$ is a distinguished but non-standard codeterminant and $Y^{\mu}_{a,b}$ is a standard codeterminant.

Proof This follows from the straightening formula for v-codeterminants applied to the set of distinguished v-codeterminants and the the fact that each element of B maps to zero or to a distinguished v-codeterminant.

∎

Corollary 3.7 (explicit surjectivity) Let $[A]$ be a basis element of $S_v(n,r)$, where e_A corresponds to $\xi_{i,j}$. (We assume, as we may, that $j_a \leq j_b$ whenever $a < b$.) Then the element of U given by

$$\sum_{\mu,a,b} C^{\mathrm{wt}(j),i,j}_{\mu,a,b} \Upsilon^{\mu}_{a,b},$$

where the sum is taken over all (μ, a, b) such that $Y^{\mu}_{a,b}$ is standard, maps under θ to $[A]$.

Proof The element $[A]$ is equal to the codeterminant $\widehat{Y}^{\mathrm{wt}(j)}_{i,j}$, because $e_A = \xi_{i,j} = \xi_{i,j}\xi_j$ and $j = \ell(\mathrm{wt}(j))$. This is equal, by the straightening formula for v-codeterminants, to

$$\sum_{\mu,a,b} C^{\mathrm{wt}(j),i,j}_{\mu,a,b} \widehat{Y}^{\mu}_{a,b}.$$

The claim now follows from the definition of the elements Υ. ∎

Applications

An important application of Theorem 3.5 is that it gives an easy proof of the semistandard basis theorem, which is one of the main results in [3]. This application is explained in detail in [5, §5].

Another application of the theorem is that it helps a great deal in proving that the structure constants for Du's canonical basis for $S_q(n,r)$ lie in $\mathbf{N}[v, v^{-1}]$. This is carried out in full in [7].

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