Introduction to Lie groups and algebras

Definitions, examples and problems.

Fall 2006

1 Lie groups

1.1 Lie groups, direct products

Notation: by a manifold and submanifold we will always mean C^∞–smooth manifolds and submanifolds. We use the following definition of a submanifold. A subset $S \subset M$ of a manifold M is a submanifold if for every point $x \in S$ (not $x \in M$!) there exists a chart U of x in M diffeomorphic to an affine space V such that $U \cap S$ is diffeomorphic to a subspace of V.

Definition 1. A *real Lie group* is a real manifold G together with a group structure such that the multiplication map

$$G \times G \to G; \quad (x, y) \mapsto xy$$

and the inversion map

$$G \to G; \quad x \to x^{-1}$$

are smooth. A *complex Lie group* is defined analogously: G must be a complex manifold and the multiplication map and the inversion map must be complex differentiable. In the same way, one can define a Lie group over any field K as long as the notion of a manifold over K makes sense.

Note that any complex Lie group of dimension n can be regarded as a real Lie group of dimension $2n$.

Examples:

1. Any real or complex vector space. The group operation is addition of vectors.
2. The multiplicative groups $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$ and $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$.
3. The circle $\mathbb{T} = \{z \in \mathbb{C}^* : |z| = 1\}$ is a real Lie group.
4. Any finite or countable group with discrete topology regarded as a 0-dimensional manifold is a Lie group.

5. Less trivial examples include *linear groups* familiar to us from linear algebra, e.g. *general linear groups* $GL_n(\mathbb{R})$ and $GL_n(\mathbb{C})$, that are the groups of invertible linear operators in $\mathbb{R}^n$ and $\mathbb{C}^n$, respectively. A Lie group is called *linear* if it is a subgroup of some general linear group. Classical examples include:

   - special linear group SL_n = all operators preserving a nondegenerate alternating n-form,
   - orthogonal group O_n = all operators preserving a nondegenerate symmetric 2-form, which in the real case must also be positive definite,
   - symplectic group Sp_{2n} = all operators preserving a nondegenerate skew-symmetric 2-form (which exists only for even-dimensional vector spaces),
   - unitary group $U_n(\mathbb{C})$ = all operators in $GL_n(\mathbb{C})$ preserving a non-degenerate positive definite hermitian form.
   - pseudo orthogonal group $O_{p,q}(\mathbb{R})$ = all operators preserving a nondegenerate symmetric 2-form of signature (p, q) on a $(p+q)$-dimensional vector space

   It is not evident why all these groups are manifolds. We will prove this later. Other classical groups are special orthogonal group $SO_n = O_n \cap SL_n$, special pseudo orthogonal group $SO_{p,q}(\mathbb{R}) = O_{p,q} \cap SL_{p+q}(\mathbb{R})$ and special unitary group $SU_n(\mathbb{C}) = U_n(\mathbb{C}) \cap SL_n(\mathbb{C})$.

One can get new examples of Lie groups by taking *direct products*.

Definition 2. Let G_1 and G_2 be Lie groups. Their *direct product* $G_1 \times G_2$ is Cartesian product of the manifolds G_1 and G_2 together with group structure of direct product.

Examples:

1. A real torus $\mathbb{T}^n = \underbrace{\mathbb{T}^1 \times \ldots \times \mathbb{T}^1}_{n}$ is a direct product of circles.

2. A complex torus $(\mathbb{C}^*)^n$ is a direct product of multiplicative groups $\mathbb{C}^*$.

Exercises:

1. Show that the group G of non-degenerate upper-triangular $n \times n$ matrices over a field K (the field K here is either $\mathbb{R}$ or $\mathbb{C}$) is a Lie group and find its dimension. Show then that G is diffeomorphic as a manifold but not isomorphic as a group to the direct product of several copies of K^* and K.

2. Show that if the multiplication map in Definition 1 is smooth then so is the inversion map (i.e. the second condition in the definition of a Lie group can be dropped).

3. Show that the direct product of Lie groups is a Lie group.

4. Show that SL_2, SO_2 and $SU_2(\mathbb{C})$ are Lie groups and find their dimensions.

5. Show that all operators from the symplectic group Sp_{2n} have determinant one.

6. Which of the classical groups are compact? Connected?

7. Show that the tangent bundle of any Lie group is trivial (i.e. it is diffeomorphic to the direct product of the Lie group and a vector space of the same dimension).

8. Which of the following manifolds can be endowed with the structure of a Lie group?
   (a) S^1 (b) S^2 (c*) S^3 (d*) $\mathbb{RP}^3$ (e) $\mathbb{T}^2$ (f) Klein bottle

9. Show that
   (a) $SO_2(\mathbb{R})$ is isomorphic to $\mathbb{T}^1$
   (b) $SO_{1,1}(\mathbb{R})$ is isomorphic to $\mathbb{R}^*$
   (c) $SU_2(\mathbb{C})$ is diffeomorphic to S^3
   (d) $SO_3(\mathbb{R})$ is diffeomorphic to $\mathbb{RP}^3$

1.2 Subgroups, homomorphisms

Definition 3. A subgroup H of a Lie group G is called a *Lie subgroup* of G if $H \subset G$ is a submanifold.

Note that not every subgroup of a Lie group is a Lie subgroup. E.g. the cyclic subgroup $H = \langle e^{2\pi i\sqrt{2}} \rangle \subset \mathbb{T}^1$ is everywhere dense in $\mathbb{T}^1$ and. hence, is not a submanifold.

Examples:

1. A subspace of a vector space is a Lie subgroup.

2. All n-th roots of unity in $\mathbb{C}$ for a given n form a Lie subgroup of the circle $\mathbb{T}^1$.

3. The real torus $\mathbb{T}^n$ is a Lie subgroup of the complex torus $(\mathbb{C}^*)^n$ (regarded as a real Lie group) since $\mathbb{T}^1$ is a subgroup of $\mathbb{C}^*$.

4. The group of non-degenerate upper-triangular square matrices, the group of non-degenerate diagonal square matrices and all classical groups are Lie subgroups of general linear groups.

Definition 4. Let G_1 and G_2 be Lie groups. A map $G_1 \to G_2$ is called a *Lie group homomorphism* if it is a group homomorphism and is also smooth.

Note that the image of a Lie group under a Lie group homomorphism is not always a Lie subgroup. E.g. the map $h : \mathbb{R} \to \mathbb{T}^2$; $h : x \to (e^{2\pi i x}, e^{2\pi i\sqrt{2}x})$ is a Lie group homomorphism but $h(\mathbb{R}) \subset \mathbb{T}^2$ (an irrational winding of the torus) is everywhere dense in $\mathbb{T}^2$ and, hence, is not a Lie subgroup.

Examples:

1. An exponential homomorphism: $\exp : \mathbb{R} \to \mathbb{R}^*$; $\exp : x \mapsto e^x$.
2. Another exponential homomorphism: $\exp : \mathbb{R} \to \mathbb{T}$; $\exp : x \mapsto e^{ix}$.
3. Determinant: $\det : GL_n(\mathbb{R}) \to \mathbb{R}^*$; $\det : A \mapsto \det(A)$.
4. For any Lie group G and any element $g \in G$ there is an inner automorphism: $a(g) : G \to G$; $a(g) : x \mapsto gxg^{-1}$.
5. A Lie group homomorphism $G \to GL_n$ is called an n-dimensional *linear representation* of a Lie group G.

Exercises:

1. For each pair of real numbers α and β define the subgroup $H \subset \mathbb{T}^2$ as follows:
$$H = \{(e^{i\alpha x}, e^{i\beta x}), x \in \mathbb{R}\}.$$
Under what conditions on α and β is H a Lie subgroup?
2. Show that every Lie subgroup is a Lie group.
3. Let H be a subgroup of a Lie group G. Show that if there exists a neighborhood $U_e \subset G$ of the identity element $e \in G$ such that $H \cap U_e$ is a submanifold of G, then H is a Lie subgroup of G.
4. Prove that a Lie subgroup is a closed submanifold (note that in Definition 3 we do not require that a Lie subgroup be a closed submanifold). See also Problem 11 in the end of this section.
5. For any Lie group G denote by G^0 the connected component of the identity element. Show that G^0 is a normal Lie subgroup of G.
6. Find the differential $d_e\det$ at the identity of the determinant homomorphism $\det : GL_n(\mathbb{R}) \to \mathbb{R}^*$.
7. Show that the kernel of a Lie group homomorphism is a Lie subgroup.

1.3 Actions

Definition 5. Let G be a Lie group, and M a manifold. A homomorphism α from G to the group of diffeomorphisms of M is called an *action* of G on M if the map
$$G \times M \to M; \quad (g, x) \to \alpha(g)x$$
is smooth. The *orbit* Gx of a point $x \in M$ is the image of G under the map
$$\alpha_x : g \to \alpha(g)x.$$

The *stabilizer* G_x of x is the preimage of x under the map α_x, i.e.

$$G_x = \{g \in G : \alpha(g)x = x\}$$

Examples:

1. There are three important types of actions of a Lie group G on itself:
   - Left action: $l(g) : x \to gx$;
   - Right action: $r(g) : x \to xg^{-1}$;
   - Adjoint action: $a(g) : x \to gxg^{-1}$.
2. Any linear representation of a Lie group G on a vector space V provides an action of G on V.
3. The group SO_n acts on the unit sphere in $\mathbb{R}^n$.

Exercises:

1. Find all orbits and stabilizers of the adjoint action of $GL_2(\mathbb{C})$.
2. Show that for any point $x \in M$ the map α_x from Definition 5 is smooth of constant rank. Show also that if the rank of α_x is k, then
   (a) The stabilizer G_x is a Lie subgroup of codimension k in G. The tangent space T_eG_x of G_x at the identity is the kernel of the differential $d_e\alpha_x : T_eG \to T_x(Gx)$.
   (b) For some sufficiently small neighborhood $U_e \subset G$ of the identity element $e \in G$ the set $\alpha(U_e)x$ is a submanifold of dimension k in M.
   (c) If the orbit Gx is a submanifold of M, then its dimension is k.

   In part (c), could it be that the orbit is not a submanifold?
3. Let a compact Lie group act on a manifold M. Show that all orbits are closed submanifolds of M.
4. Prove that SL_n is a Lie subgroup of GL_n and find its dimension. Describe explicitly the tangent space T_eSL_n (this is some vector subspace in the space of all linear operators).
5. Do the same for O_n. (Hint: consider the action of GL_n on the space of positive definite symmetric bilinear forms.)
6. Do the same for the other classical groups.

1.4 Exponential map and one-parameter subgroups

The case of linear Lie groups. Denote by $\mathfrak{gl}_n = T_e GL_n$ the space of all linear operators on an n-dimensional vector space.

Definition 6. *Define the exponential map* $\exp : \mathfrak{gl}_n \to GL_n$ *by the formula* $\exp(A) = e^A$*, where* e^A *is the matrix exponent:*

$$e^A = I + A + \frac{A^2}{2!} + \ldots + \frac{A^n}{n!} + \ldots .$$

Exercises:

1. (a) Show that the power series defining the matrix exponent e^A converges for every operator $A \in \mathfrak{gl}_n$.
   (b) Show that if $AB = BA$, then $e^A e^B = e^{A+B}$.
   (c) Show that $\det(e^A) = e^{\text{trace}(A)}$
2. Show that the exponential map is a diffeomorphism of some neighborhood of 0 in $\mathfrak{gl}_n$ to some neighborhood of the identity element e in GL_n.
3. Show that the exponential map for $GL_n(\mathbb{C})$ is surjective. Is it injective?
4. Show that the exponential map for $SL_2(\mathbb{R})$ is not surjective.
5. Verify that for each classical group G the image of the tangent space $T_e G$ under the exponential map lies in G. Show also that the exponential map provides a diffeomorphism between some neighborhood of 0 in $T_e G$ and some neighborhood of e in G.
6. Verify that for any $A \in \mathfrak{gl}_n$ the map

$$\varphi : \mathbb{R} \to GL_n; \quad \varphi : t \to e^{At}$$

   satisfies the differential equation

$$\frac{d\varphi(t)}{dt} = A\varphi(t); \quad \varphi(0) = e, \left.\frac{d\varphi(t)}{dt}\right|_{t=0} = A.$$

   The product $A\varphi(t)$ here is the composition of linear operators A and $\varphi(t)$.

General case

Definition 7. Let G be a Lie group. A Lie group homomorphism $\varphi : \mathbb{R} \to G$ is called a *one-parameter subgroup.*

Examples:

1. $G = \mathbb{R}^*$; $\varphi : t \to e^t$;

2. $G = \mathbb{C}^*$; $\varphi : t \to e^{at}$, where $a \in \mathbb{C}$ is any complex number;

3. $G = GL_n$; $\varphi : t \to e^{At}$,

4. G is any Lie group; $\varphi_v(t)$ is the solution of the differential equation

$$\frac{d\varphi(t)}{dt} = \varphi(t)v; \quad \varphi(0) = e, \left.\frac{d\varphi(t)}{dt}\right|_{t=0} = v,$$

   where v is any vector in the tangent space T_eG and vg is a shorthand notation for the image of v in T_gG under the differential $d_er(g) : T_eG \to T_gG$.

Exercise:

1. Show that the one-parameter subgroup $\varphi_v(t)$ in the last example is well-defined (i.e. the solution of the differential equation exists for all $t \in \mathbb{R}$ and provides a homomorphism $\mathbb{R} \to G$).

Definition 8. Let G be a Lie group. Denote by $\mathfrak{g}$ the tangent space T_eG. Define the exponential map $\exp : \mathfrak{g} \to G$ as follows:

$$\exp(v) = \varphi_v(1).$$

Exercises:

1. Show that for the classical groups this definition is equivalent to Definition 6.

2. Show that the exponential map is a diffeomorphism of some neighborhood of 0 in $\mathfrak{g}$ to some neighborhood of the identity element e in G.

Problems:

1. Classify all connected real Lie groups of dimension 1.

2. Classify all connected complex Lie groups of dimension 1.

3. Show that a compact Lie group of positive dimension has Euler characteristic zero.

4. Find all compact real Lie groups of dimension 2.

5. Denote by $\widetilde{G}$ the universal cover of a Lie group G. Show that $\widetilde{G}$ can be endowed with the structure of a Lie group.

6. A discrete normal subgroup of a connected Lie group G lies in the center of G.

7. The fundamental group of a connected Lie group is Abelian.

8. Show that if G is a connected commutative Lie group then the exponential map $\exp : \mathfrak{g} \to G$ is a Lie group homomorphism (the group operation on the vector space $\mathfrak{g}$ is vector addition). Use this to classify all commutative connected Lie groups.

9. Show that any connected compact complex Lie group is commutative.

10. Is it true that the intersection of two submanifolds is also a submanifold? Prove that the intersection of Lie subgroups is always a Lie subgroup.

11. (very difficult!) Prove that if a subgroup of a Lie group is closed then it is a Lie subgroup.