HOMOTOPY THEORY FOR BEGINNERS

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Abstract. This note contains comments to Chapter 0 in Allan Hatcher's book [5].

Contents

1. Notation and some standard spaces and constructions

We will often refer to these standard spaces:

- $\mathbf{R}$ is the real line and $\mathbf{R}^n = \mathbf{R} \times \cdots \times \mathbf{R}$ is euclidian n-space
- $\mathbf{C}$ is the field of complex numbers and $\mathbf{C}^n = \mathbf{C} \times \cdots \times \mathbf{C}$ is the n-dimensional complex vector space
- $\mathbf{H}$ is the (skew-)field of quaternions and $\mathbf{H}^n = \mathbf{H} \times \cdots \times \mathbf{H}$ is the n-dimensional quaternion vector space
- $S^n = \{x \in \mathbf{R}^{n+1} \mid |x| = 1\}$ is the unit n-sphere in $\mathbf{R}^{n+1}$
- $D^n = \{x \in \mathbf{R}^n \mid |x| \leq 1\}$ is the unit n-disc in $\mathbf{R}^n$
- $I = [0,1] \subset \mathbf{R}$ is the unit interval
- $\mathbf{R}P^n$, $\mathbf{C}P^n$, $\mathbf{H}P^n$ is the topological space of 1-dimensional linear subspaces of $\mathbf{R}^{n+1}$, $\mathbf{C}^{n+1}$, $\mathbf{H}^{n+1}$.
- M_g is the orientable and N_g the nonorientable compact surface of genus g

If X is a topological space, Y is a set, and $p\colon X \to Y$ a surjective map, define the **quotient topology** on Y to be $\{V \subset Y \mid p^{-1}V \text{ is open in } X\}$ (General Topology, 2.74). For instance, if $\sim$ is a relation on X, let $X/\!\sim$ be the set of equivalence classes for the smallest equivalence relation containing the relation $\sim$. We give $X/\!\sim$ the quotient topology for the surjective map $p\colon X \to X/\!\sim$ taking points in X to their equivalence classes. If A is a closed subspace of X, the quotient space X/A is the set $(X - A) \cup \{A\}$ with the quotient topology for the map $p\colon X \to X/A$ taking points of $X - A$ to points of $X - A$ and points of A to $\{A\}$.

Example 1.1. The projective spaces have the quotient topology for the surjective maps from the unit spheres

$$(1.2)\quad p_n\colon S^n = S(\mathbf{R}^{n+1}) \to \mathbf{R}P^n, \qquad p_n\colon S^{2n+1} = S(\mathbf{C}^{n+1}) \to \mathbf{C}P^n, \qquad p_n\colon S^{4n+3} = S(\mathbf{H}^{n+1}) \to \mathbf{H}P^n$$

given by $p_n(x) = Fx \subset F^{n+1}$, $x \in S(F^{n+1})$, $F = \mathbf{R}, \mathbf{C}, \mathbf{H}$.

If both X and Y are topological spaces a **quotient map** is a surjective map $p\colon X \to Y$ if the topology on Y is the quotient topology, ie if for any $V \subset Y$ we have: V is open in $Y \iff p^{-1}V$ is open in X (General Topology, 2.76).

The **topological category**, **Top**, is the category where the objects are topological spaces and the morphisms are continuous maps between topological spaces. Two spaces are isomorphic in the topological

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category if they are homeomorphic. Topology is the study of continuity or the study of the topological category.

In the following, **space** will mean topological space and **map** will mean continuous map.

2. Homotopy

Let X and Y be two (topological) spaces and $f_0, f_1 \colon X \to Y$ two (continuous) maps of X into Y.

Definition 2.1. *The maps f_0 and f_1 are* **homotopic**, *$f_0 \simeq f_1$, if there exists a map, a* **homotopy**, *$F \colon X \times I \to Y$ such that $f_0(x) = F(x,0)$ and $f_1(x) = F(x,1)$ for all $x \in X$.*

Homotopy is an equivalence relation on the set of maps $X \to Y$. We write $[X, Y]$ for the set of homotopy classes of maps $X \to Y$. Since homotopy is well-behaved under composition of maps, in the sense that

$$f_0 \simeq f_1 \colon X \to Y \text{ and } g_0 \simeq g_1 \colon Y \to Z \Longrightarrow g_0 \circ f_0 \simeq g_1 \circ f_1 \colon X \to Z,$$

composition of maps induces composition $[X, Y] \times [Y, Z] \xrightarrow{\circ} [X, Z]$ of homotopy classes of maps.

Example 2.2. The identity map $S^1 \to S^1$ and the map $S^1 \to S^1$ that takes $z \in S^1 \subset \mathbf{C}$ to z^2 are not homotopic (as we shall see). Indeed, none of the maps $z \to z^n$, $n \in \mathbf{Z}$, are homotopic to each other, so that the set $[S^1, S^1]$ is infinite.

Definition 2.3. *A map is* **nullhomotopic** *if it is homotopic to a constant map.*

Definition 2.4. *The spaces X and Y are* **homotopy equivalent**, *$X \simeq Y$, if there are maps,* **homotopy equivalences** *$X \to Y$ and $Y \to X$, such that the two compositions, $X \xrightarrow{f} Y \xrightarrow{g} X$ and $Y \xrightarrow{g} X \xrightarrow{f} Y$, are homotopic to the respective identity maps.*

More explicitly: X and Y are homotopy equivalent if there exist maps $f \colon X \to Y$, $g \colon Y \to X$, and homotopies $H \colon X \times I \to X$, $G \colon Y \times I \to Y$, such that $H(x,0) = x$, $H(x,1) = gf(x)$ for all $x \in X$ and $G(y,0) = y$, $G(y,1) = fg(y)$. If two spaces are homeomorphic they are also homotopy equivalent.

If $X \rightleftarrows Y$ are homotopy equivalences, then the induced maps $[T, X] \rightleftarrows [T, Y]$ and $[X, T] \rightleftarrows [Y, T]$ are bijections for any space T.

Homotopy equivalence is an equivalence relation on spaces. A **homotopy type** is an equivalence class of homotopy equivalent spaces. Here is the most simple homotopy type.

Definition 2.5. *A space is* **contractible** *if it is homotopy equivalent to a one-point space, $X \simeq \{*\}$.*

Proposition 2.6. *The space X is contractible if and only if one of the following equivalent conditions holds:*

- *There is a point $x_0 \in X$ and a homotopy $C \colon X \times I \to X$ such that $C(x,0) = x$ and $C(x,1) = x_0$ for all $x \in X$.*
- *The identity map 1_X of X is nullhomotopic*

If X is contractible then $[T, X] = \{*\}$ and $[X, T]$ is the set of path-components of T for any space T.

Example 2.7. $\mathbf{R}^n$ is contractible, S^n is not contractible. The House with Two Rooms [5, p 4] and the House with One Room are contractible. The infinite dimensional sphere S^∞ is contractible (Example 4.11).

Any map $X \to \mathbf{R}^n$ is nullhomotopic. The standard inclusion $S^n \to S^{n+1}$ is nullhomotopic since it factors through the contractible space $S^{n+1} - \{*\} = \mathbf{R}^{n+1}$.

The **homotopy category of spaces**, ho**Top**, is the category where the objects are topological spaces. The morphisms between two spaces X and Y is the set, $[X, Y]$, of homotopy classes of maps of X into Y. Composition in this category is composition of homotopy classes of maps. Two spaces are isomorphic in the homotopy category if they are homotopy equivalent. Topology is the study of the category of topological spaces. Algebraic topology is the study of the homotopy category of spaces.

2.1. Relative homotopy.

Let X be a space and $A \subset X$ a subspace. Suppose that $f_0, f_1 \colon X \to Y$ are maps that agree on A, ie that $f_0(a) = f_1(a)$ for all $a \in A$.

Definition 2.8. *The maps f_0 and f_1 are homotopic relative to A, $f_0 \simeq f_1$ rel A, if there exists a homotopy $F \colon X \times I \to Y$ from f_0 to f_1 such that $f_0(a) = F(a,t) = f_1(a)$ for all $a \in A$ and all $t \in I$.*

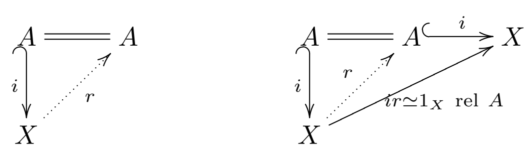
If two maps are homotopic rel A, then they are homotopic.

A **pointed space** is a pair (X, x_0) consisting of space X and one of its points $x_0 \in X$. The pointed topological category, $\mathbf{Top}_*$, is the category where the objects are pointed topological spaces and the morphisms are base-point preserving maps, based maps. The pointed homotopy category, $\mathbf{hoTop}_*$, is the category where the objects are pointed topological spaces and the morphisms are based homotopy classes of based maps.

2.2. **Retracts and deformation retracts.** Let X be a space, $A \subset X$ a subspace, and $i\colon A \to X$ the inclusion map.

Definition 2.9. *A is* **retract** *of X if the the identity map of A can be extended to X. A is* **deformation retract** *of X if the identity map of A can be extended to map that is homotopic relative to A to the identity map of X.*

Definition 2.10. *A* **retraction** *of X onto A is a map $r\colon X \to A$ such that $r(a) = a$ for all $a \in A$. If also $ir \simeq 1_X$ rel A then r is a* **deformation retraction** *of X onto A.*



Proposition 2.11. *A is a deformation retract of X if and only if there exists a homotopy $R\colon X \times I \to X$ such that $R(x, 0) = x$, $R(x, 1) \in A$ for all $x \in X$, and $R(a, t) = a$ for all $a \in A$, $t \in I$*

Example 2.12. We'll later prove that S^1 is not a retract of D^2. S^1 is a retract, but not a deformation retract, of the torus $S^1 \times S^1$. The wedge of two circles, $S^1 \vee S^1$, is a deformation retract of a punctured torus. Any Seifert surface deformation retracts to a graph. If X deformation retracts onto one of its points then X is contractible but there are contractible spaces that do not deformation retract to any of its points.

There are several other good examples of (deformation) retracts in [5]. Any retract A of a Hausdorff space X is closed as $A = \{x \in X \mid r(x) = x\}$ is the equalizer of two continuous maps [7, Ex 31.5].

If $r\colon X \to A$ is a deformation retraction of X onto A, then r is a homotopy equivalence with the inclusion map $i\colon A \hookrightarrow X$ as homotopy inverse because $ri = 1_A \simeq 1_A$ and $ir \simeq 1_X$ rel A. Conversely, if the inclusion map is a homotopy equivalence, there exists a map $r\colon X \to A$ such that $ri \simeq 1_A$ and $ir \simeq 1_X$. This is not quite the same as saying that A is a deformation retract of X since r may not fix the points of A and, even if it does, the points in A may not be fixed under the homotopy from ri to the identity of A. However, surprisingly enough, the converse does hold if the pair (X, A) is sufficiently nice (4.6.(2)).

2.3. **Constructions on topological spaces.** We mention some standard constructions on topological spaces and maps.

Example 2.13 (Mapping cylinders, mapping cones, and suspensions)**.** Let X be a space. The **cylinder** on X is the product

$$X \times I$$

of X and the unit interval. $X = X \times \{1\}$ is a deformation retract of the cylinder. What is the cylinder on the n-sphere S^n?

The **cone** on X

$$CX = \frac{X \times I}{X \times 1}$$

is obtained by collapsing one end of the cylinder on X. The cone is always contractible. What is the cone on the n-sphere S^n?

The (unreduced) **suspension** of X

$$SX = \frac{X \times I}{(X \times 0, X \times 1)}$$

is obtained by collapsing both ends of the cylinder on X. What is the suspension of the n-sphere S^n? (General Topology, 2.147) The cylinder, cone, and suspension are endofunctors of the topological category.

Let $f\colon X \to Y$ be a map. The **cylinder** on f or **mapping cylinder** of f

$$M_f = \frac{(X \times I) \amalg Y}{(x,0) \sim f(x)}$$

is obtained by gluing one end of the cylinder on X onto Y by means of the map f. The mapping cylinder deformation retracts onto its subspace Y. What is the mapping cylinder of $z \to z^2\colon S^1 \to S^1$?

The **mapping cone** on f

$$C_f = \frac{CX \amalg Y}{(x,0) \sim f(x)}$$

is obtained by gluing the cone on X onto Y by means of the map f. What is the mapping cone of $z \to z^2\colon S^1 \to S^1$?

There is a sequence of maps

$$X \xrightarrow{f} Y \longrightarrow C_f \longrightarrow SX \xrightarrow{Sf} SY \longrightarrow C_{Sf} \longrightarrow SSX \longrightarrow \cdots$$

where the map $C_f \to SX$ is collapse of $Y \subset C_f$.

Example 2.14 (Wedge sum and smash product of pointed spaces)**.** Let (X, x_0) and (Y, y_0) be pointed spaces. The **wedge sum** and the **smash product** of X and Y are

$$X \vee Y = X \times \{y_0\} \cup \{x_0\} \times Y \subset X \times Y, \qquad X \wedge Y = \frac{X \times Y}{X \vee Y}$$

The (reduced) **suspension** of the pointed space (X, x_0) is the smash product

$$\Sigma X = X \wedge S^1 = X \wedge I/\partial I = \frac{X \times I}{X \times \partial I \cup \{x_0\} \times I}$$

of X and a pointed circle $(S^1, 1) = (I/\partial I, \partial I/\partial I)$. (The last equality holds when X is a locally compact Hausdorff space (General Topology, 2.171).) Suspension is an endo-functor of $\mathbf{Top}_*$. How does the reduced suspension differ from the unreduced suspension?

What is the smash product $X \wedge I$? What is the smash product $S^m \wedge S^n$? (General Topology, 2.171)

Proposition 2.15. *Any map factors as an inclusion map followed by a homotopy equivalence.*

Proof. For any map $f\colon X \to Y$ there is a commutative diagram using the mapping cylinder

$$\begin{array}{ccc} & & M_f \\ & {\scriptstyle x \to (x,1)} \nearrow & \simeq \big\downarrow {\scriptstyle (x,t) \to (x,0)} \\ X & \xrightarrow{f} & Y \end{array}$$

where the slanted map is an inclusion map and the vertical map is a homotopy equivalence (even a deformation retraction). □

Example 2.16 (Adjunction spaces)**.** See (General Topology, 2.85). From the input $X \supset A \xrightarrow{\varphi} Y$ consisting of a map φ defined on a closed subspace A of X, we define the **adjunction space** $X \cup_\varphi Y$ as $(X \amalg Y)/\sim$ where $A \ni a \sim f(a) \in Y$.

Example 2.17 (The n-cellular extension of a space)**.** Let X be a space and $\phi\colon \coprod S^{n-1}_\alpha \to X$ a map from a disjoint union of spheres into X. The adjunction space $X \cup_\varphi \coprod D^n$ is the push-out of the diagram

$$\begin{array}{ccc} \coprod S^{n-1}_\alpha & \xrightarrow{\phi} & X \\ \big\downarrow & & \big\downarrow \\ \coprod D^n_\alpha & \xrightarrow{\overline{\phi}} & X \cup_\phi \coprod D^n_\alpha \end{array}$$

and it is called the **n-cellular extension** of X with **attaching map** ϕ and **characteristic map** $\overline{\phi}$. (Alternatively, $X \cup_\phi \coprod D^n_\alpha$ is the mapping cone on φ.) This space is the disjoint union of the closed subspace X and the **n-cells** $e^n_\alpha = \overline{\varphi}(D^n_\alpha - S^{n-1}_\alpha)$.

3. CW-complexes

The CW-complexes are a class of spaces that are particularly well suited for the methods of algebraic topology. CW-complexes are built inductively out of cells.

Definition 3.1. *A CW-complex is a space X with an ascending filtration of subspaces (called* **skeleta***)*

$$\emptyset = X^{-1} \subset X^0 \subset X^1 \subset \cdots \subset X^{n-1} \subset X^n \subset \cdots \subset X = \bigcup X^n$$

such that

- *X^0 is a discrete set of points*
- *X^n is (homeomorphic to) an n-cellular extension of X^{n-1} for $n \geq 1$*
- *The topology on X is coherent with the filtration in the sense that*

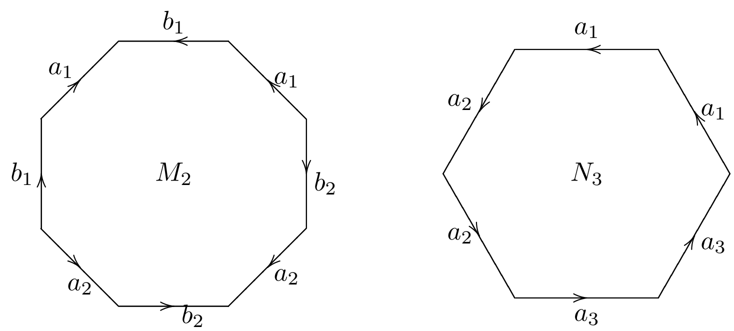
$$A \text{ is closed (open) in } X \iff A \cap X^n \text{ is closed (open) in } X^n \text{ for all } n$$

for any subset A of X.

X^0 is a discrete topological space. X^1 is a topological space since it is a 1-cellular extension of X^0. In fact, all the skeleta X^n are topological spaces and X^i is a subspace of X^j for $i < j$. The purpose of the third item of the defintion is to equip the union of all the skeleta with a topology. (Check that it is a topology!)

A CW-complex X is finite-dimensional if $X = X^n$ for some n. Any CW-complex is the disjoint union of its cells. Any CW-complex is a normal topological space [7, Exercise 35.8] (solution). CW-decompositions are not unique; there are generally many CW-decompositions of a given space X – consider for instance $X = S^2$.

Example 3.2 (Compact surfaces as CW-complexes)**.**



The closed orientable surface $M_g = (S^1 \times S^1)\# \cdots \#(S^1 \times S^1)$ of genus $g \geq 1$ is a CW-complex

$$M_g = \bigvee_{1 \leq i \leq g} S^1_{a_i} \vee S^1_{b_i} \cup_{\prod [a_i, b_i]} D^2$$

with 1 0-cell, $2g$ 1-cells, and 1 2-cell. (Picture of M_2.)

The closed nonorientable surface $N_h = \mathbf{R}P^2 \# \cdots \# \mathbf{R}P^2$ of genus $h \geq 1$ is a CW-complex

$$N_h = \bigvee_{1 \leq i \leq h} S^1_{a_i} \cup_{\prod a_i^2} D^2$$

with 1 0-cell, h 1-cells, and 1 2-cell.

Example 3.3 (Spheres as CW-complexes)**.** Points on the n-sphere $S^n \subset \mathbf{R}^{n+1} = \mathbf{R}^n \times \mathbf{R}$ have coordinates of the form (x, u). Let $D^n_\pm$ be the images of the embeddings $D^n \to S^n \colon x \to (x, \pm\sqrt{1 - |x|^2})$. Then

$$S^n = S^{n-1} \cup D^n_+ \cup D^n_- = S^{n-1} \cup_{\mathrm{id} \amalg \mathrm{id}} (D^n \amalg D^n)$$

is obtained from S^{n-1} by attaching two n-cells. The infinite sphere S^∞ is an infinite dimensional CW-complex

$$S^0 \subset S^1 \subset \cdots \subset S^{n-1} \subset S^n \subset \cdots \subset S^\infty = \bigcup_{n=0}^{\infty} S^n$$

with two cells in each dimension. A subspace A of S^∞ is closed iff $A \cap S^n$ is closed in S^n for all n.

Example 3.4 (Projective spaces as CW-complexes)**.** Real projective n-space $\mathbf{R}P^n$ can be obtained from $\mathbf{R}P^{n-1}$ by attaching one n-cell along the canonical quotient map $p_{n-1}\colon S^{n-1}\to\mathbf{R}P^{n-1}$. To see this, regard S^n as a subspace of $\mathbf{R}^{n+1}=\mathbf{R}^n\times\mathbf{R}$ and let D^n_+ be the image of the embedding $D^n\to S^n\colon x\to(x,\sqrt{1-|x|^2})$. Since every point of $\mathbf{R}P^n=S^n/\sim$ has a representative in the subspace $S^{n-1}\cup D^n_+$ of S^n we see that

$$\mathbf{R}P^n=S^n/\sim=(S^{n-1}\cup D^n_+)/\sim=\mathbf{R}P^{n-1}\cup_{p_{n-1}}D^n$$

where we also use that $S^{n-1}\hookrightarrow D^n\to D^n_+\to\mathbf{R}P^n$ is the projection $S^{n-1}\to\mathbf{R}P^{n-1}$ to identify the attaching map. Since $\mathbf{R}P^m=\mathbf{R}P^{m-1}\cup_{p_{m-1}}D^m$ for all $m\le n$, $\mathbf{R}P^n$ is a finite CW-complex with one cell in each dimension 0 through n. In particular, $\mathbf{R}P^0$ is a point and $\mathbf{R}P^1=\mathbf{R}P^0\cup D^1=S^1$ is a 1-sphere. The infinite real projective space $\mathbf{R}P^\infty=\bigcup_{n=0}^\infty\mathbf{R}P^n$ is an infinite dimensional CW-complex

$$*=\mathbf{R}P^0\subset S^1=\mathbf{R}P^1\subset\cdots\subset\mathbf{R}P^{n-1}\subset\mathbf{R}P^n\subset\cdots\subset\mathbf{R}P^\infty$$

when equipped with the coherent topology.

Complex projective n-space $\mathbf{C}P^n$ can be obtained from $\mathbf{C}P^{n-1}$ by attaching one $2n$-cell along the canonical quotient map $p_n\colon S^{2n-1}\to\mathbf{C}P^{n-1}$. To see this, regard S^{2n+1} as a subspace of $\mathbf{C}^{n+1}=\mathbf{C}^n\times\mathbf{C}$ and let D^{2n}_+ be the image of the embedding $D^{2n}\to S^{2n+1}\colon x\to(x,\sqrt{1-|x|^2})$. Since every point of $\mathbf{C}P^n=S^{2n+1}/\sim$ has a representative in the subspace $S^{2n-1}\cup D^{2n}_+$ of S^{2n+1} we see that

$$\mathbf{C}P^{2n}=S^{2n+1}/\sim=(S^{2n-1}\cup D^{2n}_+)/\sim=\mathbf{C}P^{n-1}\cup_{p_{n-1}}D^{2n}$$

where we also use that $S^{2n-1}\hookrightarrow D^{2n}\to D^n_+\to\mathbf{C}P^n$ is the projection $S^{2n-1}\to\mathbf{C}P^{n-1}$ to identify the attaching map. view the points of

Since $\mathbf{C}P^m=\mathbf{C}P^{m-1}\cup_{p_{m-1}}D^{2m}$ for all $m\le n$, $\mathbf{C}P^n$ is a finite CW-complex with one cell in each dimension 0 through n. In particular, $\mathbf{C}P^0$ is a point and $\mathbf{C}P^1=\mathbf{C}P^0\cup D^2=S^2$ is a 2-sphere. The infinite complex projective space $\mathbf{C}P^\infty=\bigcup_{n=0}^\infty\mathbf{C}P^n$ is an infinite dimensional CW-complex

$$\{*\}=\mathbf{C}P^0\subset S^2=\mathbf{C}P^1\subset\cdots\subset\mathbf{C}P^{n-1}\subset\mathbf{C}P^n\subset\cdots\subset\mathbf{C}P^\infty$$

when equipped with the coherent topology.

Similarly, quaternion projective n-space $\mathbf{H}P^n$ can be obtained from $\mathbf{H}P^{n-1}$ by attaching one $4n$-cell along the canonical quotient map $p_{n-1}\colon S^{4n-1}\to\mathbf{H}P^{n-1}$. Thus $\mathbf{H}P^n$ is a CW-complex with one cell in each of the dimensions $0,4,\dots,4n$. In particular, $\mathbf{H}P^0$ is a point and $\mathbf{H}P^1=\mathbf{H}P^0\cup D^4=S^4$ is a 4-sphere

The *Hopf maps* are the maps

$$S^0\to S^1\to S^1,\qquad S^1\to S^3\to S^2,\qquad S^3\to S^7\to S^4, \tag{3.5}$$

that we obtain from the canonical maps (1.2) when $n=1$.

Definition 3.6. *A subcomplex of a CW-complex X is a closed subspace that is a union of cells of X.*

Any subcomplex A of a CW-complex X is a CW-complex. The 0-skeleton A^0 is a subset of X^0, and A^n is obtained by attaching to A^{n-1} a subset of the n-cells of X. The quotient space of a CW-complex by a subcomplex is a CW-complex. For instance, $D^n/S^{n-1}=S^n$, $M_g/M_g^1=S^2$, and, more generally, $X^n/X^{n-1}=\bigvee S^n$ for any CW-complex X.

The product of two CW-complexes X and Y is a CW-complex whose cells are $e^n_\alpha\times f^m_\beta$ where e^n_α is a cell of X and e^m_β a cell of Y. (There is a little warning here as the CW-topology on $X\times Y$ may not equal the product topology when X and Y are infinite complexes.)

Definition 3.7. *Let A be any topological space. A relative CW-complex on A is a space X with an ascending filtration of subspaces (called* **skeleta***)*

$$A=X^{-1}\subset X^0\subset X^1\subset\cdots\subset X^{n-1}\subset X^n\subset\cdots\subset X=\bigcup X^n$$

such that

- *X^0 is the union of A and a discrete set of points*
- *X^n is (homeomorphic to) an n-cellular extension of X^{n-1} for $n\ge1$*
- *The topology on X is coherent with the filtration in the sense that*

$$B \text{ is closed (open) in } X \iff B\cap X^n \text{ is closed (open) in } X^n \text{ for all } n$$

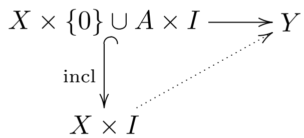
for any subset B of X.

4. The Homotopy Extension Property

The Homotopy Extension Property will be very important to us. Let X be a space with a subspace $A \subset X$.

Definition 4.1. [1, VII.1] *The pair (X, A) has the Homotopy Extension Property (HEP) if any partial homotopy $A \times I \to Y$ of a map $X \to Y$ into any space Y can be extended to a (full) homotopy of the map.*

Diagrammatically, (X, A) has the HEP if it is always possible to complete the diagram



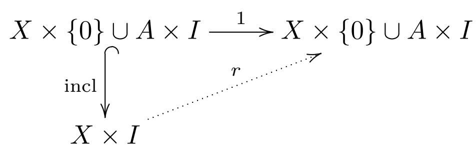
for any space Y and any partial homotopy of a map $X \to Y$.

The pair $(X, \emptyset)$ always has the HEP. A **nondegenerate base point** is a point $x_0 \in X$ such that $(X, \{x_0\})$ has the HEP.

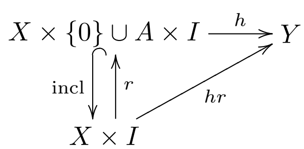
Proposition 4.2. [5, p 14, Ex 0.26] *Let X a space and $A \subset X$ a subspace. The following three conditions are equivalent*

1. *(X, A) has the HEP*
2. *$X \times \{0\} \cup A \times I$ is a retract of $X \times I$ on X.*
3. *$X \times \{0\} \cup A \times I$ is a deformation retract of $X \times I$ on X.*

Proof. (1) $\Longrightarrow$ (2): We apply the HEP to the universal example. If (X, A) has the HEP then the identity map of the partial cylinder $X \times \{0\} \cup A \times I$ extends to a retraction of the cylinder $X \times I$ onto the partial cylinder:



(2) $\Longrightarrow$ (1): If the inclusion of the partial cylinder into the cylinder has a left inverse r then it is very easy



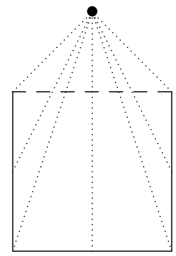
to find an extension of any partial homotopy h. This shows that (1) $\iff$ (2).
(3) $\Longrightarrow$ (2): Clear.
(2) $\Longrightarrow$ (3): Given a retraction $r \colon X \times I \to X \times \{0\} \cup A \times I$, written as $r(x, t) = (r_1(x, t), r_2(x, t))$, we can manufacture a deformation retraction $H \colon X \times I \times I \to X \times I$ by [2, p 329]

$$H(x, t, s) = (r_1(x, st), (1 - s)t + r_2(x, st))$$

We check that $H(x, t, 0) = (r_1(x, t), t + r_2(x, 0)) = (x, t + 0) = (x, t)$, $H(x, t, 1) = (r_1(x, t), r_2(x, t)) = r(x, t) \in X \times \{0\} \cup A \times I$, $H(x, 0, s) = (x, 0)$, and $H(a, t, s) = (r_1(a, ts), (1 - s)t + r_2(a, st)) = (a, (1 - s)t + st) = (a, t)$ for all $a \in A$ so that H is indeed a deformation retraction of the cylinder $X \times I$ onto the cylinder on the inclusion, $X \times \{0\} \cup A \times I$. □

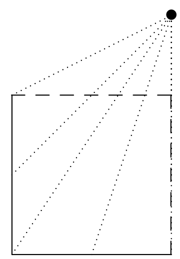
Corollary 4.3. *The pair (D^n, S^{n-1}) has the HEP for all $n \geq 1$. More generally, the pair (CX, X) has the HEP for all spaces X.*

Proof. For instance, $D^1 \times I \subset \mathbf{R} \times I \subset \mathbf{R}^2$ (deformation) retracts onto $D^1 \times \{0\} \cup S^0 \times I$ by radial projection from $(0, 2)$. as indicated in this picture:



In fact, $D^n \times I \subset \mathbf{R}^n \times I \subset \mathbf{R}^{n+1}$ (deformation) retracts onto $D^n \times \{0\} \cup S^{n-1} \times I$ by a radial projection from $(0, \dots, 0, 2)$.

More generally, for any space X, the pair (CX, X) has the HEP because $CX \times \{0\} \cup X \times I$ is a retract of $CX \times I$. The below picture indicates a retraction $R \colon I \times I \to \{0\} \times I \cup I \times \{0\}$, sending all of $\{1\} \times I$ to $(1, 0)$.



The map $\mathrm{id} \times R \colon X \times I \times I \to X \times \{0\} \times I \cup X \times I \times \{0\}$ factors through

$$\begin{array}{ccc} X \times I \times I & \xrightarrow{\mathrm{id} \times R} & X \times \{0\} \times I \cup X \times I \times \{0\} \\ \downarrow & & \downarrow \\ \frac{X \times I}{X \times \{1\}} \times I & \dashrightarrow & X \times \{0\} \times I \cup \frac{X \times I}{X \times \{1\}} \times \{0\} \end{array}$$

to give the required retraction $CX \times I \to X \times I \cup CX \times \{0\}$. (Remember that the left vertical map is a quotient map by the Whitehead Theorem from General Topology.) □

Proposition 4.4. *If (X, A) has the HEP and X is Hausdorff, then A is a closed subspace of X.*

Proof. $X \times \{0\} \cup A \times I$ is a closed subspace of $X \times I$ since it is a retract. Now look at X at level $\frac{1}{2}$ inside the cylinder $X \times I$. □

See [1, VII.1.5] for a necessary condition for an inclusion to have the HEP.

Example 4.5. (A closed subspace that does not have the HEP) (General topology exam, Problem 3). (I, A) where $A = \{0\} \cup \{\frac{1}{n} | n = 1, 2, \dots\}$ does *not* have the HEP since $I \times \{0\} \cup A \times I$ is not a retract of $I \times I$. Indeed, assume that $r \colon I \times I \to I \times \{0\} \cup A \times I$ is a retraction. For each $n \in \mathbf{Z}_+$, the map $t \to (t \times 1)$, $t \in [\frac{1}{n+1}, \frac{1}{n}]$, is a path in $I \times I$ from $\frac{1}{n+1} \times 1$ to $\frac{1}{n} \times 1$ and its image under the retraction, $t \mapsto r(t \times 1)$, $\frac{1}{n+1} \le t \le \frac{1}{n}$, is a path in A connecting the same two points. Such a path must pass through all points of $(\frac{1}{n+1}, \frac{1}{n}) \times \{0\} \subset I \times \{0\}$ because $\pi_1 r([\frac{1}{n+1}, \frac{1}{n}] \times \{1\}) \supset [\frac{1}{n+1}, \frac{1}{n}]$ by connectedness. Thus there is a point $t_n \in (\frac{1}{n+1}, \frac{1}{n})$ such that $r(t_n \times 1) \in (\frac{1}{n+1}, \frac{1}{n}) \times \{0\}$. This contradicts continuity of r for $t_n \times 1$ converges to 0×1 and $r(t_1 \times 1)$ converges to $0 \times 0 \neq r(0 \times 1)$. (A similar, but simpler, argument shows that there is no retraction $r \colon A \times I \to A \times \{0\} \cup \{0\} \times I$ so that 0 is a degenerate base-point of A.)

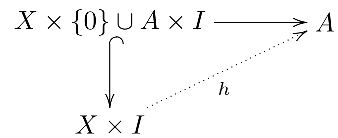
The following proposition show the usefulness of the HEP.

Proposition 4.6. [5, Proposition 0.17] [5, Proposition 0.18, Ex 0.26] [1, VII.4.5] *Suppose that (X, A) has the HEP.*

1. *If the inclusion map has a homotopy left inverse then A is a retract of X.*
2. *If the inclusion map is a homotopy equivalence then A is a deformation retract of X.*

(3) *If A is contractible then the quotient map $X \to X/A$ is a homotopy equivalence.*
(4) *Let Y be any space and $\varphi_0 \colon A \to Y$ any map. Then the homotopy type of the adjunction space $Y \cup_{\varphi_0} X$ only depends on the homotopy class of the attaching map $\varphi_0 \colon A \to Y$.*

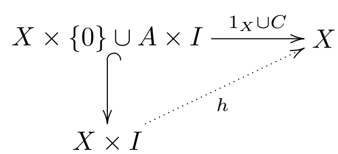
Proof. (1) Assume that $r \colon X \to A$ is a map such that $ri \simeq 1_A$. We must change r on A so that it actually fixes points of A. There is a map $X \times \{0\} \cup A \times I \to A$ which on $X \times \{0\}$ is r and on $A \times I$ is a homotopy from ri to the identity of A. Using the HEP we may complete the commutative diagram



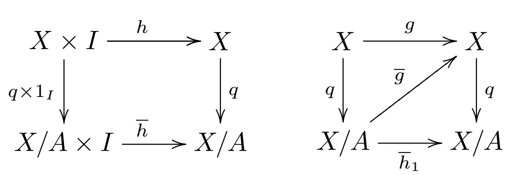
and get a homotopy $h \colon X \times I \to A$. The (corestriction of the) end-value of this homotopy is a map $h_1 \colon X \to A$ such that $h_1 i = 1_A$ (a retract).
(2) Let $i \colon A \to X$ be the inclusion map. The assumption is that there exists a map $r \colon X \to A$ such that $ri \simeq 1_A$ and $ir \simeq 1_X$. By point (1) we can assume that $ri = 1_A$, ie that A is a retract of X. Let $G \colon X \times I \to X$ be a homotopy with start value $G_0 = 1_X$ and end value $G_1 = ir$. For $a \in A$, $G(a,0) = a$ and $G(a,1) = a$ but we have no control of $G(a,t)$ when $0 < t < 1$. We want to modify G into a deformation retraction, that is a homotopy from 1_X to ir *relative to* A. Since (X,A) has the HEP so does $(X,A) \times (I, \partial I) = (X \times I, A \times I \cup X \times \partial I)$ ((4.9).(3)). Let $H \colon X \times I \times I \to X \times I$ be an extension (a homotopy of homotopies) of the map $X \times I \times \{0\} \cup A \times I \times I \cup X \times \partial I \times I$ given by

$$\begin{aligned} H(x,t,0) &= G(x,t) \\ H(a,t,s) &= G(a, t(1-s)) \text{ for } a \in A \\ H(x,0,s) &= x \\ H(x,1,s) &= G(ir(x), 1-s) \end{aligned}$$

Note that H is well-defined since the first line, $H(x,1,0) = G(x,1) = ir(x)$, and the fourth line, $H(x,1,0) = G(ir(x),1) = irir(x) = ir(x)$, yield the same result. The end value of H, $(x,t) \mapsto H(x,t,1)$, is a homotopy rel A of $H(x,0,1) = x$ to $H(x,1,1) = G(ir(x),0) = ir(x)$. This is a homotopy rel A since $H(a,t,1) = G(a,0) = a$ for all $a \in A$.
(3) What we need is a homotopy inverse to the projection map $q \colon X \to X/A$ and this is more or less the same thing as a homotopy $X \times I \to X$ from the identity to a map that collapses A inside A. How can we get such a homotopy? Well, precisely from the HEP! (This could be used as the motivation for HEP.) Let $C \colon A \times I \to A \subset X$ be a contraction of A, a homotopy of the identity map to a constant map. Use the HEP to extend the contraction of A and the identity on X



to a homotopy $h \colon X \times I \to X$ such that h_0 is the identity map of X, h_t sends A to A for all $t \in I$, and $g := h_1$ sends A to a point of A. By the universal property of quotient maps (General Topology, 2.81), the homotopy h induces a homotopy $\overline{h}$ and the map g induces a map $\overline{g}$ such that the diagrams



commute. (The product map $q \times 1 \colon X \times I \to X/A \times I$ is quotient since I is locally compact Hausdorff (General Topology, 2.87).) We claim that $X \underset{\overline{g}}{\overset{q}{\rightleftarrows}} X/A$ are homotopy inverse to each other. This is because $\overline{g}q = g = h_1 \simeq h_0 = 1_X$ and $q\overline{g} = \overline{h}_1 \simeq \overline{h}_0 = 1_{X/A}$.

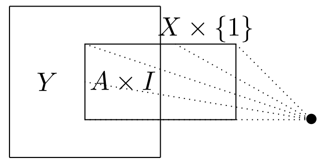


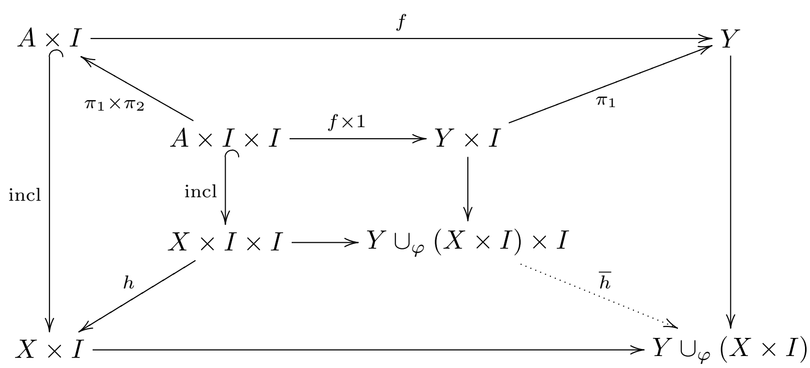
FIGURE 1. $Y \cup_\varphi (X \times I)$ deformation retracts onto $Y \cup_{\varphi_1} X$

(4) Let $\varphi\colon A \times I \to Y$ be a homotopy from φ_0 to φ_1. We want to show that $Y \cup_{\varphi_0} X$ and $Y \cup_{\varphi_1} X$ are homotopy equivalent. The point is that both $Y \cup_{\varphi_0} X$ and $Y \cup_{\varphi_1} X$ are deformation retracts of $Y \cup_\varphi (X \times I)$. We get the deformation retraction from the deformation retraction of $X \times I$ onto $A \times I \cup X \times \{0\}$ and $A \times I \cup X \times \{1\}$ as indicated in Figure 1.

We intend to show that the inclusions

$$Y \cup_{\varphi_0} X = Y \cup_\varphi (X \times \{0\} \cup A \times I) \subset Y \cup_\varphi (X \times I) \supset Y \cup_\varphi (X \times \{1\} \cup A \times I) = Y \cup_{\varphi_1} X$$

are homotopy equivalences rel Y. (The equality signs are there because all points of $A \times I$ have been identified to points in Y.) The left inclusion is a homotopy equivalence because the subspace is a deformation retract of the big space. The deformation retraction $\overline{h}$ of $Y \cup_\varphi (X \times I)$ onto $Y \cup_{\varphi_0} X$ is induced by the universal property of adjunction spaces (General Topology, 14.17) as in the diagram



from a deformation retraction $h\colon X \times I \times I \to X \times I$ of $X \times I$ onto $X \times \{0\} \cup A \times I$ (4.2.(3)). Here, the outer square is the push-out diagram for $Y \cup_\varphi (X \times I)$ and the inner square is just this diagram crossed with the unit interval. The homotopy $h\colon X \times I \times I \to X \times I$ starts as the identity map, is constant on the subspace $X \times \{0\} \cup A \times I \subset X \times I$, and ends as a retraction of $X \times I$ onto this subspace. The induced homotopy $\overline{h}\colon Y \cup_\varphi (X \times I) \times I \to Y \cup_\varphi (X \times I)$ starts as the identity map, is constant on the subspace $Y \cup_\varphi (X \times \{0\} \cup A \times I) = Y \cup_{\varphi_0} X$, and ends as a retraction onto this subspace. We conclude that $Y \cup_\varphi (X \times I)$ deformation retracts onto its subspace $Y \cup_{\varphi_0} X$. Similarly, $Y \cup_\varphi (X \times I)$ deformation retracts onto its subspace $Y \cup_{\varphi_1} X$. Thus $Y \cup_{\varphi_0} X$ and $Y \cup_{\varphi_1} X$ are homotopy equivalent spaces. □

Example 4.7 (Are the Hopf maps (3.5) nullhomotopic?)**.** In Example 2.7 we claimed that the squaring map $2\colon S^1 \to S^1$ is not homotopic to the constant map $0\colon S^1 \to S^1$. To prove this it suffices to show that the mapping cones $C_2 = S^1 \cup_2 D^2 = \mathbf{R}P^2$ and $C_0 = S^1 \cup_0 D^2 = S^1 \vee S^2$ are not homotopy equivalent.

The complex projective plane $\mathbf{C}P^2 = S^2 \cup_\varphi D^4$ is obtained by attaching a 4-cell to the 2-sphere along the Hopf map $S^3 \to S^2$ (3.5). If the attaching map is nullhomotopic then $\mathbf{C}P^2$ is homotopy equivalent to $S^2 \cup_* D^4 = S^2 \vee S^4$. We shall later develop methods to show that $\mathbf{C}P^2$ and $S^2 \vee S^4$ are not homotopy equivalent.

Example 4.8 (A closed subspace that does not have the HEP)**.** Let C be the quasi-circle [5, Ex 1.3.7]. Collapsing the interval $A = [-1, 1] \subset C$ lying on the vertical axis gives a quotient map $C \to S^1$ [6, §28] which is not a homotopy equivalence (since $\pi_1(S^1) = \mathbf{Z}$ and $\pi_1(C)$ is trivial) even though A is contractible. Thus (C, A) does not have the HEP.

Proposition 4.9. *The HEP property is preserved under some constructions.*

(1) *(Transitivity) If $X_0 \subset X_1 \subset X_2$ and both pairs (X_2, X_1) and (X_1, X_0) have the HEP, then (X_2, X_0) has the HEP. More generally, if $X = \bigcup X_k$ has the coherent topology with respect to its subspaces $X_0 \subset X_1 \subset \cdots \subset X_{k-1} \subset X_k \subset \cdots$ where each pair of consecutive subspaces has the HEP, then (X, X_0) has the HEP.*
(2) *If (X, A) has the HEP then $Y \times (X, A) = (Y \times X, Y \times A)$ has the HEP for all spaces Y.*
(3) *If (X, A) has the HEP then $(X, A) \times (I, \partial I) = (X \times I, X \times \partial I \cup A \times I)$ has the HEP.*
(4) *If (X, A) has the HEP then $(Y \cup_\varphi X, Y \cup_\varphi A)$ has the HEP for all spaces Y and all maps $\varphi \colon B \to Y$ defined on a subspace B of A. In particular, $(Y \cup_\varphi X, Y)$ has the HEP for any attaching map $\varphi \colon A \to Y$. (See Figure 2)*
(5) *The n-cellular extension $(Y \cup_\varphi \coprod D^n, Y)$ of any space Y has the HEP for any attaching map $\varphi \colon \coprod S^{n-1} \to Y$. More generally, the pair (C_f, Y) has the HEP for any map $f \colon X \to Y$.*

Proof. (1) In the first case, there are retractions $r_2 \colon X_2 \times I \to X_1 \times I \cup X_2 \times \{0\}$ and $r_1 \colon X_1 \times I \cup X_2 \times \{0\} \to X_0 \times I \cup X_2 \times \{0\}$. Then $r_1 r_2$ is a retraction of $X_2 \times I$ onto $X_0 \times I \cup X_2 \times \{0\}$.

In the general case, there are retractions $r_k \colon X_k \times I \cup X \times \{0\} \to X_{k-1} \times I \cup X \times \{0\}$. There is a well-defined retraction $X \times I \to X_0 \times I \cup X \times \{0\}$ that on $X_k \times I \cup X \times \{0\}$ is

$$X_k \times I \cup X \times \{0\} \xrightarrow{r_k} X_{k-1} \times I \cup X \times \{0\} \xrightarrow{r_{k-1}} \cdots \xrightarrow{r_2} X_1 \times I \cup X \times \{0\} \xrightarrow{r_1} X_0 \times I \cup X \times \{0\}$$

This retraction $X \times I \to X_0 \times I \cup X \times \{0\}$ is continuous because the product topology on $X \times I$ is coherent with the filtration $X_k \times I$, $k = 0, 1, \ldots$. (Verify this claim!)
(2) We use 4.2. Let $r \colon X \times I \to X \times I$ be a retraction onto $X \times \{0\} \cup A \times I$. Then the product map $1_Y \times r$ is a retraction of $(Y \times X) \times I$ onto $(Y \times X) \times \{0\} \cup (Y \times A) \times I$.
(3) [2, 7.5 p. 330]
(4) We use 4.2 again. Let $r \colon X \times I \to X \times I$ be a retraction onto $X \times \{0\} \cup A \times I$. The universal property of quotient maps provides a factorization, $\overline{1_{Y \times I} \amalg r}$, of $1_{Y \times I} \amalg r$

$$\begin{array}{ccc} (Y \amalg X) \times I & \xrightarrow{1_{Y \times I} \amalg r} & (Y \amalg X) \times I \\ {\scriptstyle q \times 1_I} \downarrow & & \downarrow {\scriptstyle q \times 1_I} \\ (Y \cup_\varphi X) \times I & \xrightarrow[\overline{1_{Y \times I} \amalg r}]{} & (Y \cup_\varphi X) \times I \end{array}$$

that is a retraction of $Y \cup_\varphi X \times I$ onto $Y \cup_\varphi X \times \{0\} \cup Y \cup_\varphi A \times I$. To prove continuity, note that the left vertical map is a quotient map since I is locally compact Hausdorff (General Topology, 2.87). This shows that $(Y \cup_\varphi X, Y \cup_\varphi A)$ has the HEP. If the attaching map φ is defined on all of A, we have that $(Y \cup_\varphi X, Y \cup_\varphi A) = (Y \cup_\varphi X, Y)$ so this pair has the HEP.
(5) This is a special case of (4) since $(\coprod D^n, \coprod S^{n-1})$ has the HEP (4.3). $\square$

Corollary 4.10. *Any relative CW-complex (X, A) (Definition 3.7) has the HEP. In particular, any CW-pair (X, A) has the HEP.*

Proof. There is a filtration of X

$$A \subset A \cup X^0 \subset A \cup X^1 \subset \cdots \subset A \cup X^{n-1} \subset A \cup X^n \subset \cdots \subset X$$

where $A \cup X^n$ is obtained from $A \cup X^{n-1}$ by attaching the n-cells of X that are not in A. Since a cellular extension has the HEP, transitivity implies that also (X, A) has the HEP. $\square$

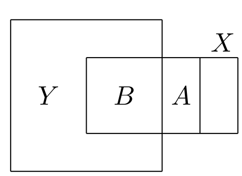


FIGURE 2. The pair $(Y \cup_\varphi X, Y \cup_\varphi A)$

Example 4.11 (S^∞ is contractible)**.** Choose $* = 1$ as the base-point of $\mathbf{R} \supset S^0 \subset S^\infty$. The inclusion map $S^0 \hookrightarrow S^1$ is homotopic rel $*$ to the constant map $*\colon S^0 \to S^1$ because it factors through $S^1_+ = D^1$ which contains $*$ as a deformation retract. Let $S^0 \times I \to S^1$ be a homotopy rel $*$ from the inclusion to the constant map. Use the HEP to extend to a homotopy $S^1 \times I \to S^1$ from the identity map of S^1 to some self-map of S^1 that is constant on S^0. This map $S^1 \to S^1$ composed with the inclusion $S^1 \hookrightarrow S^2$ is homotopic rel S^0 to the constant map $S^1 \to S^2$ because it factors through $S^2_+ = D^2$ which contains $*$ as a deformation retract. Let $S^1 \times I \to S^2$ be a homotopy rel S^0 to the constant map. Use the HEP to extend to a homotopy $S^2 \times I \to S^2$ from the identity map of S^2 to some self-map of S^2 that is constant on S^1. Continue in this way. Figure 3

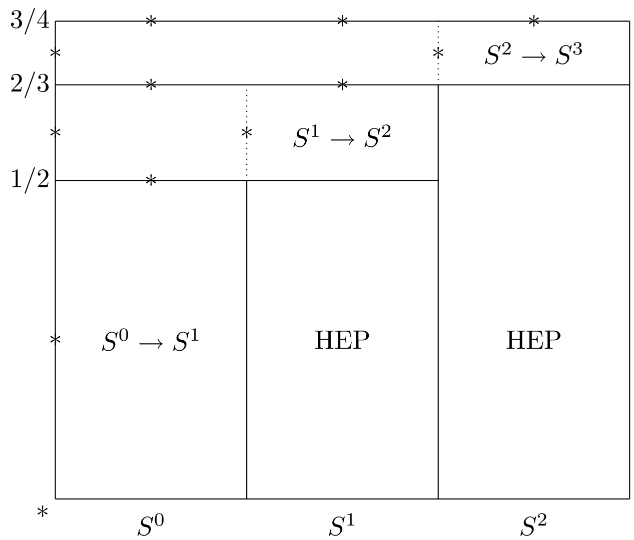


FIGURE 3. S^∞ is contractible

shows the beginning of a homotopy $S^\infty \times I \to S^\infty$ rel $*$ between the identity map and the constant map. It is continuous because the area where it is constant (indicated by the dotted lines) gets larger and larger as we approach $S^\infty \times \{1\}$.

Exercise 4.12. The Dunce cap [3] is the quotient of the of the 2-simplex by the identifications indicated in Fig 4. Show that the Dunce cap is contractible, in fact, homotopy equivalent to D^2.

Example 4.13. The unreduced suspension SX and the reduced suspension $\Sigma X = SX/\{x_0\}\times I$ are homotopy equivalent for all CW-complexes X based at a 0-cell $\{x_0\}$.

Example 4.14. [4, 21.21] (Homotopic maps have homotopy equivalent mapping cones). The pair (CX, X) has the HEP (4.3) and therefore the homotopy type of the mapping cone C_f only depends on the homotopy class of $f\colon X \to Y$.

Example 4.15 (The pair $(X \cup CA, CA)$)**.** If (X, A) has the HEP so does the pair $(X \cap CA, CA)$ obtained by attaching X to CA (Proposition 4.9.(4). Since the cone on A is contractible,

$$X \cup CA \to X \cup CA/CA = X/A$$

is a homotopy equivalence by Proposition 4.6.(3): The cone on the inclusion and the quotient space are homotopy equivalent and we may replace the quotient space X/A be $X \cup CA$.

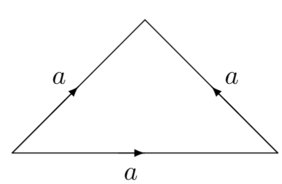


FIGURE 4. The dunce cap

Example 4.16 (The homotopy type of X/A when A contractible in X)**.** Suppose that (X, A) has the HEP and that the inclusion map $A \hookrightarrow X$ is homotopic to the constant map $0\colon A \to X$, ie. A is contractible in X. Then there are homotopy equivalences

$$X/A = X \cup CA/CA \overset{4.6}{\leftarrow} X \cup CA = C_i \overset{4.14}{\simeq} C_0 = X \vee SA$$

For instance, $S^n/S^i \simeq S^n \vee S^{i+1}$ for all $i \leq 0 < n$. (The inclusion $S^i \to S^n$, $0 < i < n$, is nullhomotopic since it factors through the contractible space $S^n - * = \mathbf{R}^n$.) See [5, Exmp 0.8] for an illustration of $S^2/S^0 \simeq S^2 \cup CS^0 \simeq S^2 \vee S^1$.

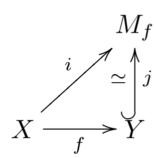
Example 4.17. (HEP for mapping cylinders.) Let $f\colon X \to Y$ be a map. We apply 4.9 in connection with the mapping cylinder $M_f = Y \cup_f (X \times I)$.

$$(I, \partial I) \text{ has the HEP } \overset{4.9.(2)}{\Longrightarrow} (X \times I, X \times \partial I) \text{ has the HEP } \overset{4.9.(4)}{\Longrightarrow} (M_f, X \cup Y) \text{ has the HEP}$$

$$(I, \{0\}) \text{ has the HEP } \overset{4.9.(2)}{\Longrightarrow} (X \times I, X \times \{0\}) \text{ has the HEP } \overset{4.9.(4)}{\Longrightarrow} (M_f, Y) \text{ has the HEP}$$

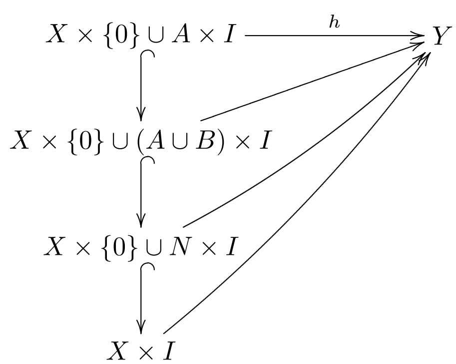
The fact that $(M_f, X \cup Y)$ has the HEP implies that also (M_f, X) has the HEP (simply take a constant homotopy on Y). See 4.18 below for another application.

The proof of [5, Proposition 0.19] uses the homotopy commutative mapping cylinder diagram



which shows that any map is an inclusion (satisfying the HEP), up to a homotopy equivalence.

Example 4.18. (HEP for subspaces with mapping cylinder neighborhoods [5, Example 0.15]) For another application of 4.9, suppose that the subspace $A \subseteq X$ has a mapping cylinder neighborhood. This means that A has a closed neighborhood N containing a subspace B (thought of as the boundary of N) such that $N - B$ is an open neighborhood of A and $(N, A \cup B)$ is homeomorphic to $(M_f, A \cup B)$ for some map $f\colon B \to A$. Then (X, A) has the HEP. To see this, let $h\colon X \times \{0\} \cup A \times I \to Y$ by a partial homotopy of a map $X \to Y$. Extend it to a partial homotopy on $X \times \{0\} \cup (A \cup B) \times I$ by using the constant homotopy on $B \times I$. Since $(N, A \cup B)$ has the HEP, we can extend further to a partial homotopy defined on $X \times \{0\} \cup N \times I$. Finally, extend to $X \times I$ by using a constant homotopy on $X - (N - B) \times I$. In this way we get extensions



The final map is continuous since it restricts to continuous maps on the closed subspaces $X - (N - B) \times I$ and $N \times I$ with union $X \times I$.

For instance, the subspace ABC $\subseteq \mathbf{R}^2$ consisting of the three thin letters in the figure on [5, p. 1] is a subspace with a mapping cylinder neighborhood, namely the three thick letters. Thus $(\mathbf{R}^2, \text{ABC})$ has the HEP.

References

[1] Glen E. Bredon, *Topology and geometry*, Graduate Texts in Mathematics, vol. 139, Springer-Verlag, New York, 1993. MR 94d:55001

[2] James Dugundji, *Topology*, Allyn and Bacon Inc., Boston, Mass., 1966. MR 33 #1824

[3] George K. Francis, *A topological picturebook*, Springer-Verlag, New York, 1987. MR 88a:57002

[4] Marvin J. Greenberg and John R. Harper, *Algebraic topology*, Benjamin/Cummings Publishing Co. Inc. Advanced Book Program, Reading, Mass., 1981, A first course. MR 83b:55001

[5] Allen Hatcher, *Algebraic topology*, Cambridge University Press, Cambridge, 2002. MR 2002k:55001

[6] Jesper M. Møller, *General topology*, http://www.math.ku.dk/~moller/e03/3gt/notes/gtnotes.pdf.

[7] James R. Munkres, *Topology. Second edition*, Prentice-Hall Inc., Englewood Cliffs, N.J., 2000. MR 57 #4063

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