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"Chevalley Supergroups"

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INTRODUCTION

In his work of 1955, Chevalley provided a combinatorial construction of all simple algebraic groups over any field. In particular, his method led to a proof of the existence theorem for simple algebraic groups and to new examples of finite simple groups which had escaped the attention of specialists in group theory. The groups that Chevalley constructed are now known as *Chevalley groups*. Furthermore, Chevalley's construction provided a description of all simple algebraic groups as group schemes over $\mathbb{Z}$.

In this paper we adapt this philosophy to the setup of supergeometry, so as to give an explicit construction of algebraic supergroups whose Lie superalgebra is of classical type over an arbitrary field (or even ring). Our construction provides at one strike the supergroups corresponding to the families $A(m,n)$, $B(m,n)$, $C(n)$, $D(m,n)$ of basic Lie superalgebras and to the families of strange Lie superalgebras $P(n)$, $Q(n)$, as well as to the exceptional basic Lie superalgebras $F(4)$, $G(3)$, $D(2,1;a)$ — for integral values of a. To our knowledge, supergroups corresponding to the exceptional Lie superalgebras have not previously appeared in the literature.

To explain our work, we first revisit the whole classical construction.

Let $\mathfrak{g}$ be a finite dimensional simple (or semisimple) Lie algebra over an algebraically closed field $\mathbb{K}$ (e.g. $\mathbb{K} = \mathbb{C}$). Fix in $\mathfrak{g}$ a Cartan subalgebra; then a root system is defined, and $\mathfrak{g}$ splits into weight spaces indexed by the roots. Also, $\mathfrak{g}$ has a special basis, called *Chevalley* basis, for which the structure constants are integers, satisfying special conditions in terms of the root system. This defines an integral form of $\mathfrak{g}$, called *Chevalley Lie algebra*.

In the universal enveloping algebra of $\mathfrak{g}$, there is a $\mathbb{Z}$–integral form, called *Kostant algebra*, with a special "PBW-like" basis of ordered monomials, whose factors are divided powers of weight vectors and binomial coefficients of Cartan generators, corresponding to elements of the Chevalley basis of $\mathfrak{g}$.

If V is a faithful $\mathfrak{g}$–module, there is a $\mathbb{Z}$–lattice $M \subseteq V$, which is stable under the action of the Kostant algebra. Hence the Kostant algebra acts on the vector space $V_{\Bbbk} := \Bbbk \otimes_{\mathbb{Z}} M$ for any field $\Bbbk$. Moreover there exists an integral form $\mathfrak{g}_V$ of $\mathfrak{g}$ leaving the lattice invariant and depending only on the representation V and not on the choice of the lattice.

For any root vector X of $\mathfrak{g}$, we take the exponential $\exp(tX) \in \mathrm{GL}(V_{\Bbbk})$, $t \in \Bbbk$ (as X acts as nilpotent, the expression makes sense). The subgroup of $\mathrm{GL}(V_{\Bbbk})$ generated by all the $\exp(tX)$, for all roots and all t, is the *Chevalley group* $G_V(\Bbbk)$, as introduced by Chevalley. This defines $G_V(\Bbbk)$ set-theoretically, as an abstract group; some extra work is required to show it is an algebraic group and to construct its functor of points. We refer the reader to [st,borel,hu] for a comprehensive treatment of all of these aspects.

We want to extend Chevalley's construction to the supergeometric setting.

In supergeometry the best way to introduce supergroups is via their functor of points. Unlikely the classical setting, the points over a field of a supergroup tell us very little of the supergroup itself. In fact such points miss the odd coordinates and describe only the classical part of the supergroup. In other words, over a field we cannot see anything beyond classical geometry. Thus we cannot generalize Chevalley's recipe as it is, but we need to suitably and subtly modify it introducing the functor of points language right at the beginning, reversing the order in which the classical treatment was developed.

The functor of points approach realizes an affine supergroup as a representable functor from the category of commutative superalgebras (salg) to the category of groups (groups). In this work, we shall first construct a functor from (salg) to (groups), and then we shall prove it is representable.

Our initial datum is a simple Lie superalgebra of classical type (or a direct sum of finitely many of them, if one prefers), say $\mathfrak{g}$: in our construction it plays the role of the simple (or semisimple) Lie algebra in Chevalley's setting. We start by proving some basic results on $\mathfrak{g}$ (previously known only partially, cf. [ik,sw]) like the existence of *Chevalley bases*, and a PBW-like theorem for the Kostant $\mathbb{Z}$–form of the universal enveloping superalgebra.

Next we take a faithful $\mathfrak{g}$–module V, and we show that there exists a lattice M in V fixed by the Kostant superalgebra and also by a certain integral form $\mathfrak{g}_V$ of $\mathfrak{g}$, which again depends on V only. We then define a group-valued functor G_V, from the category of commutative superalgebras to the category of sets, as follows. For any commutative superalgebra A, $G_V(A)$ is the subgroup of $\mathrm{GL}(V(A))$ — the general linear supergroup on V — generated by the homogeneous one-parameter unipotent subgroups (acting on M) associated to the root vectors, together with the multiplicative one-parameter subgroups (formally corresponding to exponentials of elements in the Cartan subalgebra). In this supergeometric setting, one must carefully define the homogeneous one-parameter subgroups, which may have three possible superdimensions: $1|0$, $0|1$ and $1|1$. This also will be discussed.

As a group-theoretical counterpart of the $\mathbb{Z}_2$–splitting $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$, we find a factorization $G_V(A) = G_0(A)\, G_1^{<}(A) \cong G_0(A) \times G_1^{<}(A)$. Here $G_0(A)$ is (roughly) a classical Chevalley-like group attached to $\mathfrak{g}_0$ and V, while $G_1^{<}(A)$ may be euristically thought of as exponential of $A_1 \otimes \mathfrak{g}_1$. In fact we show that the functor $G_1 : A \mapsto G_1(A)$ is representable and isomorphic to $\mathbb{A}_{\Bbbk}^{0|dim(\mathfrak{g}_1)}$.

Actually, our result is more precise: indeed, $\mathfrak{g}_1$ in turn splits into $\mathfrak{g}_1 = \mathfrak{g}_1^+ \oplus \mathfrak{g}_1^-$ according to the splitting of odd roots into positive and negative ones, and so at the group level we have $G_1^{<}(A) \cong G_1^{-,<}(A) \times G_1^{+,<}(A)$ and $G(A) \cong G_1^{-,<}(A) \times G_0(A) \times G_1^{+,<}(A)$, resembling the classical "big cell" decomposition, which however in this context holds *globally*.

Despite the analogy with Chevalley construction, G_V is not a representable functor, hence it is not an algebraic supergroup. This is a phenomenon already observed at the classical level: one-parameter subgroups, defined via their functor of points, do not generate Chevalley groups over an arbitrary commutative ring. Hence we need to consider the sheafification $\mathbf{G}_V$ of the functor G_V, which coincides with G_V on local superalgebras (we provide at the end an appendix with a brief treatment of sheafification of functors). In particular, $\mathbf{G}_V$ inherits the factorization $\mathbf{G}_V = \mathbf{G}_0\,\mathbf{G}_1 \cong \mathbf{G}_0 \times \mathbf{G}_1$, with $\mathbf{G}_1 = G_1$ and $\mathbf{G}_0$ being a classical (reductive) Chevalley-like group-scheme associated to $\mathfrak{g}_0$ and V. More in detail, we find the finer factorization $\mathbf{G}_V(A) = \mathbf{G}_0(A) \times \mathbf{G}_1^{-,<}(A) \times \mathbf{G}_1^{+,<}(A)$ with $\mathbf{G}_1(A) = \mathbf{G}_1^{-,<}(A) \times \mathbf{G}_1^{+,<}(A)$ and $\mathbf{G}_1^{\pm,<}(A) = G_1^{\pm,<}(A)$. As $\mathbf{G}_1 = G_1$ and $\mathbf{G}_0$ are representable, the above factorization implies that $\mathbf{G}_V$ is representable too, and so it is an algebraic supergroup. We then take it to be, by definition, our "Chevalley supergroup".

In the end, we prove the functoriality in V of our construction, and that, over any field $\Bbbk$, the Lie superalgebra $\mathrm{Lie}(\mathbf{G}_V)$ is just $\Bbbk \otimes \mathfrak{g}_V$ as one expects.

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