# Category theory

**The theory of categories, functors and natural transformations (CFNT), often labelled with the short name of 'Category Theory', is concerned with the abstract concept of *morphism*, or *arrow*, which is a fundamental notion in all of mathematics. Thus, category theory can be described as "a general mathematical theory of structures and sytems of structures" specified through the actions of morphisms.**

Morphisms can be composed, or linked together, into *diagrams via* a *composition law* or rule, such as " $*$ ", and they are also subject to an *associativity axiom*, that is, if **f, g** and **h** are thre morphisms of a category **C** that can be composed, then $(f * g) * h = f * (g * h)$. *Identity morphisms* **1** are assumed to exist such that $f * 1 = f = 1 * f$ for any morphism **f** of the category **C**; obviously, in the general case, all identity morphisms are distinct (unlike the particular case of a group where the identity is unique for the whole group).

In the case of Set Theory, a morphism is a set-theoretical mapping or a mathematical function. It can also be represented as a relation between, or among, sets. In the beginning, the theory of categories was developed in 1945 by Eilenberg and MacLane based on the theory of sets. However, in later years, in order to avoid certain antimonies related to 'sets of sets', an independent axiomatic foundation of CFNT was proposed called the Elementary Theory of Abstract Categories (ETAC). ETAC has eight axioms in addition to the (nonelementary) *axiom of completeness*, without making use of the set-membership relation required by the older set theory; unfortunately, the intial version of ETAC was still subject to the *axiom of choice* ('Axiom 5') which has been the subject of controvercy among modern set theorists that are concerned with the foundations of a new set theory. As in the general theory of categories, the terms that remain undefined in ETAC are those of *mapping, domain, codomain*, and *composition of mappings*.

Subsequently, a foundation of mathematics in the category of categories was proposed in 1966 by William F. Lawvere that reduces the number of undefined terms, and has also more general axioms than those of ETAC.

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# Abstract Categories

In *abstract categories* one is concerned only with morphisms and their higher dimensional, or higher level, types, such as functors and natural transformations. Thus, and abstract category is defined as a class of linked morphisms with certain associativity and composition rules (such as those already introduced above), that are very similar to those encountered for mathematical functions.

## Duality

Reversing the direction of all arrows in a category **A** results in a category $A^{op}$ called the *dual* category of **A**. The universal properties of $A^{op}$ are mirror images of those of **A**; thus, for every theorem which is valid for category **A** there is a corresponding theorem that holds for the dual category.

## Functors

A *functor* **F** is defined as an arrow between two categories **A** and **B**, and can be thus intuitively perceived as a comparison, or 'mapping', of the two categories; the composition rules for functors are similar to those for morphisms, and there exist identity functors $1_A : A \to A$ that 'leave the category **A** unchanged'. Moreover, categorical diagrams can also be defined by means of a special functor over a scheme.

There are two different kinds of functors:

covariant functors that convert arrows $A \to B$ into arrows $F(A) \to F(B)$ thus preserving the direction of the arrows, whereas

contravariant functors that reverse arrows: $A \to B$ becomes $F(B) \to F(A)$; for example, the *fundamental theorem of Galois theory* describes a contravariant functor from the extensions of a given field to subgroups of its *Galois group*.

Another example is the *forgetful functor* that simply 'forgets' the structure of an object: one can convert a group into a set by simply forgetting the group law.

An even more interesting example is that of the *fundamental group*, or *Poincaré group functor*: it takes any topological space to the first and simplest of its homotopy groups--the *fundamental group* which encodes information about the loops in a topological space. The fundamental groups of homeomorphic spaces are isomorphic, and in fact, the fundamental group only depends on the homotopy type of the space. We recall here that two mappings are said to be *homotopically equivalent* if one can be continuously deformed into the other. Therefore, *homotopy groups* are employed in Algebraic Topology to classify topological spaces; such groups encode the information about the holes of a topological space as the continuous deformation of paths is prevented by the holes present in the space.

Categories of *dynamical graphs* or dynamic networks may represent dynamic system transformations in terms of functors between networks or dynamical graphs. An important example is that of functors between *spin networks* that represent certain dynamic transformations diagrams in quantum gravity which are called *spin foams*. Because a spin network is equivalent to an one-dimensional CW-complex the spin foams lead directly to an equivalent topological category of CW-complexes.

## Natural Transformations

Furthermore, *natural transformations* $\eta$, or arrows between functors, can also be defined in the *category of functors* $C_F$, but the resulting constructions need to be subject to a *naturality condition* in the form of a square commutative diagram. Thus, natural transformations convert a functor into another one while respecting the *internal structure* (that is, the composition of morphisms) of the categories involved; this naturality condition is precisely defined as follows.

*Definition 0.2.* Let $\mathcal{C}$ and $\mathcal{D}$ be two categories, and let $\Phi, \Psi : \mathcal{C} \to \mathcal{D}$ be two covariant functors between them. Then assume that for every object $A$ in $\mathcal{C}$ one has a morphism $\eta_A : \Phi(A) \to \Psi(A)$ in $\mathcal{D}$ such that for every morphism $\alpha : A \to B$ in $\mathcal{C}$ the following diagram

$$\begin{tikzcd}
\Phi(A) \arrow[r, "\eta_A"] \arrow[d, "\Phi(f)"'] & \Psi(A) \arrow[d, "\Psi(f)"] \\
\Phi(B) \arrow[r, "\eta_B"'] & \Psi(B)
\end{tikzcd}$$

Figure 1: Naturality condition

is commutative.

Then, one writes $\eta : S \dot{\to} T \quad \text{or} \quad \eta : S \Rightarrow T \quad \text{or} \quad \eta : S \to T$, and calls $\eta$ a *natural trasformation* of $S$ into $T$.

An identity natural transformation is called a **natural equivalence** because it defines in an abstract, general sense, an *equivalence of two categories* that can have very different objects and morphisms but are in a certain, mathematical sense, equivalent in their categorical, or *universal* properties. An important example is that found in the fundamental theorem of Galois theory: the functor from a subgroup of the Galois group of a *field* to its fixed field is an equivalence of categories; because it is a contravariant functor, it reverses the arrows.

# Concrete Categories

On the other hand, one can also define *concrete categories* that have *objects* in addition to arrows, or morphisms. However, an object **A** of such a concrete category can be assimilated with its *identity morphism* $1_A : A \rightarrow A$ that leaves the object unchanged. Thus, more generally, morphisms that are not identities can also be regarded as *transformations* of objects.

## Examples of categories

A short list of examples of categories is as follows:

- **Group**
- **Groupoid**
- **Monoid**
- **Abelian Category**
- **Category of groups and group homomorphisms**
- **Category of groupoids and groupoid homomorphisms**
- **Category of topological spaces and homeomorphisms**
- **Category of automata and automaton homomorphisms**
- **Category of Metabolic--Replication, or (M,R), Systems**
- **Category of Neural Nets**
- **Category of Genetic Networks**
- **Category of Cell Groupoids**.

In the case of the *category of sets*, **Set**, the objects are sets and the morphisms are set-theoretical mappings between sets. In the case of the *category of groups*, **Group**, the objects are mathematical groups and the morphisms are group homomorphisms. The *category of abelian groups*, *Ab*, consists of a class of abelian groups together with a class of abelian group homomorphisms that are subject to six axioms denoted by *Ab1* to *Ab3* and *Ab1** to *Ab3**.

The following is the definition of an *Abelian category* according to Barry Mitchell (1965).

*Definition 0.1.* An Abelian category, **Ab**, is an exact additive category with finite products. The following theorem from Mirtchell (1968) is also relevant as it relates key properties of Abelian categories:

*Theorem 0.1. "The following statements are equivalent:*

> **Ab** is an Abelian category;
>
> **Ab** has kernels, cokernels, finite products, finite coproducts, and is both normal and comormal;
>
> **Ab** has pushouts and pullbacks and is both normal and conormal".

In the case of the *category of topological spaces*, **Top**, the objects are topological spaces and the morphisms are topological transformations or *homeomorphisms*. The *category of groupoids* (concrete categories with all invertible morphisms), $G_{pd}$, consists of the class $Ob(G_{pd})$ of groupoids-- considered as its objects-- together with the class of groupoid homomorphisms, $Mor(G_{pd})$, considered as arrows in $G_{pd}$ that are also subject to the commutativity and associativity axioms; the existence of a unique identity is also assumed for each groupoid in $Ob(G_{pd})$.

The *category of automata* consists of the class of automata, or sequential machines, together with the class of automata homomorphisms that are also subject to two associativity and composition conditions taken as axioms; it also has a unique identity for each automaton that leaves the automaton unchaged thus playing the role of an identity transformation for the automaton. This category has an alternative definition in terms of semigroups (of automaton states) and semigroup homomorphisms.

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