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AUTOMORPHIC FORMS AND AUTOMORPHIC REPRESENTATIONS

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Originally, the theory of automorphic forms was concerned only with holomorphic automorphic forms on the upper half-plane or certain bounded symmetric domains. In the fifties, it was noticed (first by Gelfand and Fomin) that these automorphic forms could be viewed as smooth vectors in certain representations of the ambient group G, on spaces of functions on G invariant under the given discrete group Γ. This led to the more general notion of automorphic forms on real semisimple groups, with respect to arithmetic subgroups, on adelic groups, and finally to the direct consideration of the underlying representations. The main purpose of this paper is to discuss the notions of automorphic forms on real or adelic reductive groups, of automorphic representations of adelic groups, and the relations between the two. We leave out completely the passage from automorphic forms on bounded symmetric domains to automorphic forms on groups, which has been discussed in several places (see, e.g., **[2]**, or also **[5]**, **[6]**, **[15]** for modular forms).

1. Automorphic forms on a real reductive group.

1.1. Let G be a connected reductive group over $\mathbf{Q}$, Z the greatest $\mathbf{Q}$-split torus of the center of G and K a maximal compact subgroup of $G(\mathbf{R})$. Let $\mathfrak{g}$ be the Lie algebra of $G(\mathbf{R})$, $U(\mathfrak{g})$ its universal enveloping algebra over $\mathbf{C}$ and $Z(\mathfrak{g})$ the center of $U(\mathfrak{g})$. We let $\mathscr{H}$ or $\mathscr{H}(G(\mathbf{R}), K)$ be the convolution algebra of distributions on $G(\mathbf{R})$ with support in K **[4]** and A_K the algebra of finite measures on K. We recall that $\mathscr{H}$ is isomorphic to $U(\mathfrak{g}) \otimes_{U(\mathfrak{k})} A_K$. An idempotent in $\mathscr{H}$ is, by definition, one of A_K, i.e., a finite sum of measures of the form $d(\sigma)^{-1} \cdot \chi_\sigma \cdot dk$, where σ is an irreducible finite dimensional representation of K, $d(\sigma)$ its degree, χ_σ its character, and dk the normalized Haar measure on K. The algebra $\mathscr{H}$ is called the Hecke algebra of $G(\mathbf{R})$ and K.

1.2. A norm $\| \ \|$ on $G(\mathbf{R})$ is a function of the form $\|g\| = (\mathrm{tr}\, \sigma(g)^* \cdot \sigma(g))^{1/2}$, where $\sigma \colon G(\mathbf{R}) \to \mathrm{GL}(E)$ is a finite dimensional complex representation with finite kernel and image closed in the space $\mathrm{End}(E)$ of endomorphisms of E and $*$ denotes the adjoint with respect to a Hilbert space structure on E invariant under K. It is easily seen that if τ is another such representation, then there exist a constant $C > 0$ and a positive integer n such that

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(1) $$\|x\|_\sigma \leqq C \cdot \|x\|_\tau^n, \quad \text{for all } x \in G(\boldsymbol{R}).$$

A function f on $G(\boldsymbol{R})$ is said to be *slowly increasing* if there exist a norm $\| \ \|$ on $G(\boldsymbol{R})$, a constant C and a positive integer n such that

(2) $$|f(x)| \leqq C \cdot \|x\|^n, \quad \text{for all } x \in G(\boldsymbol{R}).$$

In view of (1) this condition does not depend on the norm (but n does).

REMARK. If $\sigma: G \to \mathrm{GL}(E)$ has finite kernel, but does not have a closed image in End E, then we can either add one coordinate and det $\sigma(g)^{-1}$ as a new entry, or consider the sum of σ and $\sigma^*: g \mapsto \sigma(g^{-1})^*$. The associated square norms are then $|\det \sigma(g)^{-1}|^2 + \mathrm{tr}(\sigma(g)^* \sigma(g))$ and $\mathrm{tr}(\sigma(g)^* \sigma(g)) + \mathrm{tr}(\sigma(g^{-1})\sigma(g^{-1})^*)$.

1.3. Let Γ be an arithmetic subgroup of $G(\boldsymbol{Q})$. A smooth complex valued function f on $G(\boldsymbol{R})$ is an *automorphic form* for (Γ, K), or simply for Γ, if it satisfies the following conditions:

(a) $f(\gamma \cdot x) = f(x)$ $(x \in G(\boldsymbol{R}); \gamma \in \Gamma)$.

(b) There exists an idempotent $\xi \in \mathscr{H}$(cf. 1.1) such that $f * \xi = f$.

(c) There exists an ideal J of finite codimension of $Z(\mathfrak{g})$ which annihilates f: $f * X = 0 \quad (X \in J)$.

(d) f is slowly increasing (1.2).

An automorphic form satisfying those conditions is said to be of type (ξ, J). We let $\mathscr{A}(\Gamma, \xi, J, K)$ be the space of all automorphic forms for (Γ, K) of type (ξ, J).

1.4. EXAMPLE. Let $G = \mathbf{GL}_2$, $\Gamma = \mathbf{GL}_2(\boldsymbol{Z})$, $K = \boldsymbol{O}(2, \boldsymbol{R})$, $\xi = 1$, J the ideal of $Z(\mathfrak{g})$ generated by $(C - \lambda)$, where C is the Casimir operator and $\lambda \in \boldsymbol{C}$. Then f is an eigenfunction of the Casimir operator which is left invariant under Γ, right invariant under K, and invariant under the center Z of $G(\boldsymbol{R})$. The quotient $G(\boldsymbol{R})/Z \cdot K$ may be identified with the Poincaré upper half-plane H. The function f may then be viewed as a Γ-invariant eigenfunction of the Laplace-Beltrami operator on X. It is a "wave-form," in the sense of Maass. See **[2]**, **[5]**, **[6]** for a similar interpretation of modular forms.

1.5. REMARKS. (1) Condition 1.3(b) is equivalent to: f is K-finite on the right, i.e., the right translates of f by elements of K span a finite dimensional space of functions. These functions clearly satisfy 1.3(a), (c), (d) if f does.

(2) Let $r : K \to \mathrm{GL}(V)$ be a finite dimensional unitary representation of K. One can similarly define the notion of a V-valued automorphic form: a smooth function φ: $G \to V$ satisfying (a), (c), (d), where $| \ |$ now refers to the norm in V, and

(b′) $$f(x \cdot k) = r(k^{-1}) \cdot f(x) \qquad (x \in G(\boldsymbol{R}); k \in K).$$

For semisimple groups, this is Harish-Chandra's definition (cf. **[2]**, **[11]**). For $v \in V$, the functions $x \mapsto (\varphi(x), v)$ are then scalar automorphic forms. Conversely, a finite dimensional space E of scalar automorphic forms stable under K yields an E-valued automorphic form.

(3) A similar definition can be given if $G(\boldsymbol{R})$ is replaced by a finite covering H of an open subgroup of $G(\boldsymbol{R})$ and Γ by a discrete subgroup of H whose image in $G(\boldsymbol{R})$ is arithmetic. For instance, modular forms of rational weight can be viewed as automorphic forms on finite coverings of $\mathbf{SL}_2(\boldsymbol{R})$.

1.6. *Growth condition.* (1) Let A be the identity component of a maximal $\boldsymbol{Q}$-split torus S of G, and ${}_{\boldsymbol{Q}}\Phi$ the system of $\boldsymbol{Q}$-roots of G with respect to S. Fix an ordering on ${}_{\boldsymbol{Q}}\Phi$ and let Δ be the set of simple roots. Given $t > 0$ let

(1) $$A_t = \{a \in A : |\alpha(a)| \geq t, (a \in \Delta)\}.$$

Let f be a function satisfying 1.3(a), (b), (c). Then the growth condition (d) is equivalent to:

(d′) Given a compact set $R \subset G(\boldsymbol{R})$, and $t > 0$, there exist a constant $C > 0$ and a positive integer m such that

(2) $$|f(x \cdot a)| \leq C \cdot |\alpha(a)|^m, \quad \text{for all } a \in A_t, \alpha \in \Delta, x \in R.$$

This follows from reduction theory [**11**, §2]. More precisely, let G' be the derived group of G. Then A is the direct product of $Z(\boldsymbol{R})^\circ$ and $A' = A \cap G'(\boldsymbol{R})$. For a function satisfying 1.3(a), (b), the growth condition (d′) is equivalent to (d) for $a \in A'_t$; but says nothing for $a \in Z(\boldsymbol{R})^\circ$. However condition (c) implies that f depends polynomially on $z \in Z(\boldsymbol{R})$, and this takes care of the growth condition on $Z(\boldsymbol{R})$.

(2) Assume f satisfies 1.3(a), (b), (c) and

(3) $$f(z \cdot x) = \chi(z) f(x) \qquad (z \in Z(\boldsymbol{R}), x \in G(\boldsymbol{R}))$$

where χ is a character of $Z(\boldsymbol{R})/(Z(\boldsymbol{R}) \cap \Gamma)$. Then $|f|$ is a function on $Z(\boldsymbol{R}) \cdot \Gamma\backslash G(\boldsymbol{R})$. If $|f| \in L^p(Z(\boldsymbol{R})\Gamma\backslash G(\boldsymbol{R}))$ for some $p \geqq 1$, then f is slowly increasing, hence is an automorphic form. In view of the fact that $Z(\boldsymbol{R})\Gamma\backslash G(\boldsymbol{R})$ has finite invariant volume, it suffices to prove this for $p = 1$. In that case, it follows from the corollary to Lemma 9 in [**11**], and from the existence of a K-invariant function $\alpha \in C_c^\infty(G(\boldsymbol{R}))$ such that $f = f * \alpha$ (a well-known property of K-finite and $Z(\mathfrak{g})$-finite elements in a differentiable representation of $G(\boldsymbol{R})$, which follows from 2.1 below).

1.7. THEOREM [**11**, THEOREM 1]. *The space $\mathscr{A}(\Gamma, \xi, J, K)$ is finite dimensional.*

This theorem is due to Harish-Chandra. Actually the proof given in [**11**] is for semisimple groups, but the extension to reductive groups si easy. In fact, it is implicitly done in the induction argument of [**11**] to prove the theorem. For another proof, see [**13**, Lemma 3.5]. At any rate, it is customary to fix a quasi-character χ of $Z(\boldsymbol{R})/(\Gamma \cap Z(\boldsymbol{R}))$ and consider the space $\mathscr{A}(\Gamma, \xi, J, K)_\chi$ of elements in $\mathscr{A}(\Gamma, \xi, J, K)$ which satisfy 1.6(3). For those, the reduction to the semisimple case is immediate. Note that since the identity component $Z(\boldsymbol{R})^\circ$ of $Z(\boldsymbol{R})$ (sometimes called the split component of $G(\boldsymbol{R})$) has finite index in $Z(\boldsymbol{R})$ and $Z(\boldsymbol{R})^\circ \cap \Gamma = \{1\}$, it is substantially equivalent to require 1.6(3) for an arbitrary quasi-character of $Z(\boldsymbol{R})^\circ$.

The space $\mathscr{A}(\Gamma, \xi, J, K)$ is acted upon by the center $C(G(\boldsymbol{R}))$ of $G(\boldsymbol{R})$, by left or right translations. Since it is finite dimensional, we see that *any automorphic form is* $C(G(\boldsymbol{R}))$-finite.

1.8. *Cusp forms.* A continuous (resp. measurable) function on $G(\boldsymbol{R})$ is cuspidal if

(1) $$\int_{(\Gamma \cap N(\boldsymbol{R}))\backslash N(\boldsymbol{R})} f(n \cdot x)\, dn = 0,$$

for all (resp. almost all) x in $G(\boldsymbol{R})$, where N is the unipotent radical of any proper parabolic $\boldsymbol{Q}$-subgroup of G. It suffices in fact to require this for any proper maximal parabolic $\boldsymbol{Q}$-subgroup [**11**, Lemma 3].

A *cusp form* is a cuspidal automorphic form. We let ${}^\circ\mathscr{A}(\Gamma, \xi, J, K)$ be the space of cusp forms in $\mathscr{A}(\Gamma, \xi, J, K)$.

Let f be a smooth function on $G(\boldsymbol{R})$ satisfying the conditions (a), (b), (c) of 1.3. Assume that f is cuspidal and that there exists a character χ of $Z(\boldsymbol{R})$ such that 1.6(3) is satisfied. Then the following conditions are equivalent:

(i) f is slowly increasing, i.e., f is a cusp form;

(ii) f is bounded;

(iii) $|f|$ is square-integrable modulo $Z(\boldsymbol{R}) \cdot \Gamma$

(cf. [**11**, §4]). In fact, one has much more: $|f|$ decreases very fast to zero at infinity on $Z(\boldsymbol{R})\Gamma\backslash G(\boldsymbol{R})$, so that if g is any automorphic form satisfying 1.6(3), then $|f \cdot g|$ is integrable on $Z(\boldsymbol{R}) \cdot \Gamma\backslash G(\boldsymbol{R})$ (loc. cit.).

The space ${}^\circ\mathscr{A}(\Gamma, \xi, J, K)_\chi$ of the functions in ${}^\circ\mathscr{A}(\Gamma, \xi, J, K)$ satisfying 1.6(3) may then be viewed as a closed subspace of bounded functions in the space $L^2(\Gamma\backslash G(\boldsymbol{R}))_\chi$ of functions on $\Gamma\backslash G(\boldsymbol{R})$ satisfying 1.6(3), whose absolute value is square-integrable on $Z(\boldsymbol{R})\Gamma\backslash G(\boldsymbol{R})$. Since $Z(\boldsymbol{R})\Gamma\backslash G(\boldsymbol{R})$ has finite measure, this space is finite dimensional by a well-known lemma of Godement [**11**, Lemma 17]. This proves 1.7 for $\mathscr{A}(\Gamma, \xi, J, K)_\chi$ when $Z(\boldsymbol{R})\Gamma\backslash G(\boldsymbol{R})$ is compact, and is the first step of the proof of 1.7 in general.

1.9. Let $a \in G(\boldsymbol{Q})$. Then ${}^a\Gamma = a \cdot \Gamma \cdot a^{-1}$ is an arithmetic subgroup of $G(\boldsymbol{Q})$, and the left translation l_a by a induces an isomorphism of $\mathscr{A}(\Gamma, \xi, J, K)$ onto $\mathscr{A}({}^a\Gamma, \xi, J, K)$. Let Σ be a family of arithmetic subgroups of $G(\boldsymbol{Q})$, closed under finite intersection, whose intersection is reduced to $\{1\}$. The union $\mathscr{A}(\Sigma, \xi, J, K)$ of the spaces $\mathscr{A}(\Gamma, \xi, J, K)$ $(\Gamma \in \Sigma)$ may be identified to the inductive limit of those spaces:

(1) $$\mathscr{A}(\Sigma, \xi, J, K) = \operatorname{ind} \lim_{\Gamma \in \Sigma} \mathscr{A}(\Gamma, \xi, J, K),$$

where the inductive limit is taken with respect to the inclusions

(2) $$j_{\Gamma''\Gamma'}: \mathscr{A}(\Gamma', \xi, J, K) \to \mathscr{A}(\Gamma'', \xi, J, K) \qquad (\Gamma'' \subset \Gamma')$$

associated to the projections $\Gamma''\backslash G(\boldsymbol{R}) \to \Gamma'\backslash G(\boldsymbol{R})$.

Assume Σ to be stable under conjugation by $G(\boldsymbol{Q})$. Then $G(\boldsymbol{Q})$ operates on $\mathscr{A}(\Sigma, \xi, J, K)$ by left translations. Let us topologize $G(\boldsymbol{Q})$ by taking the elements of Σ as a basis of open neighborhoods of 1. Then this representation is admissible (every element is fixed under an open subgroup, and the fixed point set of every open subgroup is finite dimensional). By continuity, it extends to a continuous admissible representation of the completion $G(\boldsymbol{Q})_\Sigma$ of $G(\boldsymbol{Q})$ for the topology just defined. For suitable Σ, the passage to $\mathscr{A}(\Sigma, \xi, J, K)$ amounts essentially to considering all adelic automorphic forms whose type at infinity is prescribed by ξ, J, K; the group $G(\boldsymbol{Q})_\Sigma$ may be identified to the closure of $G(\boldsymbol{Q})$ in $G(\boldsymbol{A}_f)$ and its action comes from one of $G(\boldsymbol{A}_f)$. See 4.7.

1.10. Finally, we may let ξ and J vary and consider the space $\mathscr{A}(\Sigma, J, K)$ spanned by the $\mathscr{A}(\Sigma, \xi, J, K)$ and the space $\mathscr{A}(\Sigma, K)$ spanned by the $\mathscr{A}(\Sigma, J, K)$. They are $G(\boldsymbol{Q})_\Sigma$-modules and $(\mathfrak{g}, K)$-modules, and these actions commute. Again, this has a natural adelic interpretation (4.8).

1.11. *Hecke operators.* Let $\mathscr{H}(G(\boldsymbol{Q}), \Gamma)$ be the Hecke algebra, over $\boldsymbol{C}$, of $G(\boldsymbol{Q})$

mod Γ. It is the space of complex valued functions on $G(\boldsymbol{Q})$ which are bi-invariant under Γ and have support in a finite union of double cosets mod Γ. The product may be defined directly in terms of double cosets (see, e.g., **[17]**) or of convolution (see below). This algebra operates on $\mathscr{A}(\Gamma, \xi, J, K)$. The effect of $\Gamma a \Gamma$ $(a \in G(\boldsymbol{Q}))$ is given by $f \mapsto \sum_{b \in (\Gamma \cap {}^a\Gamma) \backslash \Gamma} l_b f$. More generally, let $\mathscr{H}(G(\boldsymbol{Q}), \Sigma)$ be the Hecke algebra spanned by the characteristic functions of the double cosets $\Gamma' a \Gamma''$ $(\Gamma', \Gamma'' \in \Sigma$, $a \in G(\boldsymbol{Q}))$ **[17**, Chapter 3]. It may be identified with the Hecke algebra $\mathscr{H}(G(\boldsymbol{Q})_\Sigma)$ of locally constant compactly supported functions on $G(\boldsymbol{Q})_\Sigma$. This identification carries $\mathscr{H}(G(\boldsymbol{Q}), \Gamma)$ onto $\mathscr{H}(G(\boldsymbol{Q})_\Sigma, \bar{\Gamma})$, where $\bar{\Gamma}$ is the closure of Γ in $G(\boldsymbol{Q})_\Sigma$ **[12]**. The product here is ordinary convolution (which amounts to finite sums in this case). Since $\mathscr{A}(\Sigma, \xi, J, K)$ is an admissible module for $G(\boldsymbol{Q})_\Sigma$, the action of $G(\boldsymbol{Q})_\Sigma$ extends in the standard way to one of $\mathscr{H}(G(\boldsymbol{Q})_\Sigma)$. The space $\mathscr{A}(\Gamma, \xi, J, K)$ is the fixed point set of $\bar{\Gamma}$, and the previous operation of $\mathscr{H}(G(\boldsymbol{Q}), \Gamma)$ on this space may be viewed as that of $\mathscr{H}(G(\boldsymbol{Q})_\Sigma, \bar{\Gamma})$. For an adelic interpretation, see 4.8.

2. Automorphic forms and representations of $G(\boldsymbol{R})$. The notion of automorphic form has a simple interpretation in terms of representations (which in fact suggested its present form). To give it, we need the following known lemma (cf. **[18]** for the terminology).

2.1. LEMMA. *Let (π, V) be a differentiable representation of $G(\boldsymbol{R})$. Let $v \in V$ be K-finite and $Z(\mathfrak{g})$-finite. Then the smallest $(\mathfrak{g}, K)$-submodule of V containing v is admissible.*

Indeed, $\mathscr{H} \cdot v$ is a finite sum of spaces $\mathscr{H}^\circ \cdot w$, where $\mathscr{H}^\circ$ is the Hecke algebra of the identity component $G(\boldsymbol{R})^\circ$ of $G(\boldsymbol{R})$ and $K^\circ = K \cap G(\boldsymbol{R})^\circ$, and w is K°-finite and $Z(\mathfrak{g})$-finite. It suffices therefore to show that $\mathscr{H}^\circ \cdot v$ is an admissible $(\mathfrak{g}, K^\circ)$-module. By assumption, there exist an ideal R of finite codimension of the enveloping algebra $U(\mathfrak{k})$ of the Lie algebra $\mathfrak{k}$ of K and an ideal J of finite codimension of $Z(\mathfrak{g})$ which annihilate v and moreover $U(\mathfrak{k})/R$ is a semisimple $\mathfrak{k}$-module. Then $\mathscr{H}^\circ \cdot v$ may be identified with $U(\mathfrak{g})/U(\mathfrak{g}) \cdot R \cdot J$. By a theorem of Harish-Chandra (see **[19**, 2.2.1.1]), $U(\mathfrak{g})/U(\mathfrak{g}) \cdot R$ is $\mathfrak{k}$-semisimple and its $\mathfrak{k}$-isotypic submodules are finitely generated $Z(\mathfrak{g})$-modules. Hence their quotients by J are finite dimensional.

2.2. We apply this to $C^\infty(\Gamma \backslash G(\boldsymbol{R}))$, acted upon by $G(\boldsymbol{R})$ via right translations. Therefore, if f is automorphic form, then $f * \mathscr{H}$ is an admissible $\mathscr{H}$- or $(\mathfrak{g}, K)$-module. This module consists of automorphic forms. In fact, 1.3(a) is clear, and 1.3(b) follows from 2.1; its elements are annihilated by the same ideal of $Z(\mathfrak{g})$ as v, whence (d). Finally, there exists $\alpha \in C_c^\infty(G)$ such that $f * \alpha = f$ so that $f * X$ satisfies 1.2(c) (with the same exponent as f) for all $X \in U(\mathfrak{g})$ **[11**, Lemma 14]. Thus the spaces

$$\mathscr{A}(\Gamma, J, K) = \Sigma_\xi \mathscr{A}(\Gamma, \xi, J, K), \qquad \mathscr{A}(\Gamma, K) = \Sigma_J \mathscr{A}(\Gamma, J, K),$$

are $(\mathfrak{g}, K)$-modules and unions of admissible $(\mathfrak{g}, K)$-modules.

If f is a cusp form, then $f * \mathscr{H}$ consists of cusp forms. Thus the subspace ${}^\circ\mathscr{A}(\Gamma, K)$ of cusp forms is also an $\mathscr{H}$-module. If χ is a quasi-character of Z, then the space ${}^\circ\mathscr{A}(\Gamma, K)_\chi$ of eigenfunctions for Z with character χ is a direct sum of irreducible admissible $(\mathfrak{g}, K)$-modules, with finite multiplicities. In fact, after a twist by $|\chi|^{-1}$, we may assume χ to be unitary, and we are reduced to the Gelfand-Piatetski-Shapiro theorem (**[7]**, see also **[11**, Theorem 2], **[13**, pp. 41–42]) once we identify ${}^{\circ}\mathscr{A}(\Gamma, K)_\chi$ to the space of K-finite and $Z(\mathfrak{g})$-finite elements in the space ${}^{\circ}L^2(\Gamma\backslash G(\boldsymbol{R}))_\chi$ of cuspidal functions in $L^2(\Gamma\backslash G(\boldsymbol{R}))_\chi$ (see 1.8 for the latter).

3. Some notation. We fix some notation and conventions for the rest of this paper.

3.1. F is a global field, O_F the ring of integers of F, V or V_F (resp. V_∞, resp. V_f) the set of places (resp. archimedean places, resp. nonarchimedean places) of F, F_v the completion of F at $v \in V$, O_v the ring of integers of F_v if $v \in V_f$. As usual, A or A_F (resp. A_f) is the ring of adèles (resp. finite adèles) of F.

3.2. G is a connected reductive group over F, Z the greatest F-split torus of the center of G, $\mathscr{H}_v$ the Hecke algebra of $G_v = G(F_v)$ $(v \in V)$ [4]. Thus $\mathscr{H}_v$ is of the type considered in §1 if $v \in V_\infty$ and is the convolution algebra of locally constant compactly supported functions on $G(F_v)$ if $v \in V_f$. We set

$$(1) \qquad \mathscr{H}_\infty = \bigotimes_{v\in V_\infty} \mathscr{H}_v, \qquad \mathscr{H}_f = \bigotimes_{v\in V_f} \mathscr{H}_v, \qquad \mathscr{H} = \mathscr{H}_\infty \otimes \mathscr{H}_f,$$

where the second tensor product is the restricted tensor product with respect to a suitable family of idempotents [4]. Thus $\mathscr{H}$ is the global Hecke algebra of $G(A)$ [4]. If F is a function field, then V_∞ is empty and $\mathscr{H} = \mathscr{H}_f$.

If L is a compact open subgroup of $G(A_f)$, we denote by ξ_L the associated idempotent, i.e., the characteristic function of L divided by the volume of L (relative to the Haar measure underlying the definition of $\mathscr{H}_f$). Thus $f * \xi_L = f$ if and only if f is right invariant under L.

The right translation by $x \in G(A)$ on $G(A)$, or on functions on $G(A)$, is denoted r_x or $r(x)$.

3.3. A continuous (resp. measurable) function on $G(A)$ is cuspidal if

$$\int_{N(F)\backslash N(A)} f(nx)\, dn = 0,$$

for all (resp. almost all) $x \in G(A)$, where N is the unipotent radical of any proper parabolic F-subgroup P of G. It suffices to chcek this condition when P runs through a set of representatives of the conjugacy classes of proper maximal parabolic F-subgroups.

4. Groups over number fields.

4.1. In this section, F is a number field. An element $\xi \in \mathscr{H}$ is said to be *simple* if it is of the form

$$(1) \qquad \xi = \xi_\infty \otimes \xi_f, \qquad \xi_f \in \mathscr{H}_f,\ \xi_\infty \text{ idempotent in } \mathscr{H}_\infty.$$

We let $G_\infty = \prod_{v\in V_\infty} G_v$ and $\mathfrak{g}_\infty$ be the Lie algebra of G_∞, viewed as a real Lie group. We recall that G_∞ may be viewed canonically as the group of real points $H(\boldsymbol{R})$ of a connected reductive group H, namely the group $H = R_{F/\boldsymbol{Q}}\, G$ obtained from G by restriction of scalars from F to $\boldsymbol{Q}$. This identification is understood when we apply the results and definitions of §§1, 2 to G_∞.

The group $G(A)$ is the direct product of G_∞ by $G(A_f)$. A complex valued function on $G(A)$ is *smooth* if it is continuous and, if viewed as a function of two arguments $x \in G_\infty$, $y \in G(A_f)$, it is C^∞ in x (resp. locally constant in y) for fixed y (resp. x).

4.2. *Automorphic forms.* Fix a maximal compact subgroup K_∞ of G_∞. A smooth function f on $G(A)$ is a K_∞-automorphic form on $G(A)$ if it satisfies the following conditions:

(a) $f(\gamma \cdot x) = f(x)$ $(\gamma \in G(F),\ x \in G(A))$.

(b) There is a simple element $\xi \in \mathscr{H}$ such that $f * \xi = f$.

(c) There is an ideal J of finite codimension of $Z(\mathfrak{g}_\infty)$ which annihilates f.

(d) For each $y \in G(A_f)$, the function $x \mapsto f(x \cdot y)$ on G_∞ is slowly increasing.

We shall sometimes say that f is then of type (ξ, J, K_∞). We let $\mathscr{A}(\xi, J, K_\infty)$ be the space of automorphic forms of type (ξ, J, K_∞).

There exists a compact open subgroup L of $G(A_f)$ such that $\xi_f * \xi_L = \xi_f$. Then $\mathscr{A}(\xi, J, K_\infty) \subset \mathscr{A}(\xi_\infty * \xi_L, J, K_\infty)$. We could therefore assume $\xi_f = \xi_L$ for some L without any real loss of generality.

4.3. We want now to relate these automorphic forms to automorphic forms on G_∞. For this we may (and do) assume $\xi = \xi_\infty \otimes \xi_L$ for some compact open subgroup L of $G(A_f)$. There exists a finite set $C \subset G(A)$ such that $G(A) = G(F) \cdot C \cdot G_\infty \cdot L$ [1]. We assume that C is a set of representatives of such cosets and is contained in $G(A_f)$. Then the sets $G(F) \cdot c \cdot G_\infty \cdot L$ form a partition of $G(A)$ into open sets. For $c \in C$, let

$$\Gamma_c = G(F) \cap (G_\infty \times c \cdot L \cdot c^{-1}). \tag{1}$$

It is an arithmetic subgroup of G_∞.

Given a function f on $G(A)$ and $c \in C$, let f_c be the function $x \mapsto f(c \cdot x)$ on G_∞. Suppose f is right invariant under L. Then it is immediately checked that f is left invariant under $G(F)$ if and only if f_c is left invariant under Γ_c for every $c \in C$. More precisely, the map $f \mapsto (f_c)_{c \in C}$ yields a bijection between the spaces of functions on $G(F) \backslash G(A)/L$ and on $\coprod_c (\Gamma_c \backslash G_\infty)$. It then follows from the definitions that it also induces an isomorphism

$$\mathscr{A}(\xi_\infty \otimes \xi_L, J, K_\infty) \xrightarrow{\sim} \bigoplus_{c \in C} \mathscr{A}(\Gamma_c, \xi_\infty, J, K_\infty), \tag{2}$$

so that the results mentioned in §1 transcribe immediately to properties of adelic automorphic forms. In particular, 1.6, 1.7 imply:

(i) The space $\mathscr{A}(\xi, J, K_\infty)$ is finite dimensional.

(ii) A smooth function f on $G(A)$ satisfying 4.2(a), (b), (c), and

$$f(z \cdot x) = \chi(z) \cdot f(x) \qquad (z \in Z(A),\ x \in G(A)), \tag{3}$$

for some character χ of $Z(A)/Z(F)$, such that $|f| \in L^p(Z(A)G(F) \backslash G(A))$ for some $p \geqq 1$, is slowly increasing.

(iii) For a smooth function satisfying 4.2(a), (b), (c), the growth condition 4.2(d) is equivalent to 1.6(d′), where R is now a compact subset of $G(A)$.

(iv) Any automorphic form is $C(G(A))$-finite, where $C(G(A))$ is the center of $G(A)$.

We note also that one can also define directly slowly increasing functions on $G(A)$, as in 1.2, using adelic norms: given an F-morphism $G \to \mathbf{GL}_n$ with finite kernel define, for $x \in G(A)$,

$$\|x\| = \sup_{v \in V} \max_{ij} \left(|\sigma(g)_{ij}|_v, |\sigma(g^{-1})_{ij}|_v \right) \tag{4}$$

(or simply $\max |\sigma(g_{ij})|_v$ if $\sigma(G)$ is closed as a subset of the space of $n \times n$ matrices). For continuous functions satisfying 4.2(a), (b), this is equivalent to 4.2(d).

4.4. A *cusp form* is a cuspidal automorphic form. We let ${}^\circ\mathscr{A}(\xi, J, K_\infty)$ be the space of cusp forms of type (ξ, J, K_∞).

The group N is unipotent, hence satisfies strong approximation, i.e., we have for *any* compact open subgroup Q of $N(A_f)$

$$(2) \qquad N(A) = N(F)\cdot N_\infty \cdot Q; \quad \text{hence } N(F)\backslash N(A) \cong (N(F) \cap (N_\infty \times Q))\backslash N_\infty$$

[1]. Let now f be a continuous function on $G(A)$ which is left invariant under $G(F)$ and right invariant under L. From (2) it follows by elementary computations that f is cuspidal if and only if the f_c's (notation of 4.3) are cuspidal on G_∞. Hence, the isomorphism of 4.3(2) induces an isomorphism

$$(3) \qquad {}^\circ\mathscr{A}(\xi_\infty \otimes \xi_L, J, K_\infty) \xrightarrow{\sim} \bigoplus_{c \in C} {}^\circ\mathscr{A}(\Gamma_c, \xi_\infty, J, K_\infty),$$

so that the results of 1.8 extend to adelic cusp forms. In particular, assume that f satisfies 4.2(a), (b), (c), and also 4.3(3) for a character χ of $Z(A)/Z(F)$. Then the following conditions are equivalent:

(i) f is slowly increasing, i.e., f is a cusp form;

(ii) f is bounded;

(iii) $|f|$ is square-integrable modulo $Z(A)\cdot G(F)$.

REMARK. In 4.3, 4.4, we have reduced statements on adelic automorphic forms to the corresponding ones for automorphic forms on $G(\boldsymbol{R})$, chiefly for the convenience of references. However, it is also possible to prove them directly in the adelic framework, and then deduce the results at infinity as corollaries via 4.3(2), 4.4(3). In particular, as in 1.8, one proves using (ii) above and Godement's lemma that ${}^\circ\mathscr{A}(\xi, J, K_\infty)$ is finite dimensional.

4.5. PROPOSITION. *Let f be a smooth function on $G(A)$ satisfying* 4.2(a), (b), (d). *Then the following conditions are equivalent*:

(1) *f is an automorphic form.*

(2) *For each infinite place v, the space $f * \mathscr{H}_v$ is an admissible $\mathscr{H}_v$-module.*

(3) *For each place $v \in V$, the space $f * \mathscr{H}_v$ is an admissible $\mathscr{H}_v$-module.*

(4) *The space $f * \mathscr{H}$ is an admissible $\mathscr{H}$-module.*

PROOF. The implications (4) $\Rightarrow$ (3) $\Rightarrow$ (2) $\Rightarrow$ (1) are obvious; (1) $\Rightarrow$ (4) follows from 4.3(i).

4.6. *Automorphic representations.* An irreducible representation of $\mathscr{H}$ is *automorphic* (resp. *cuspidal*) if it is isomorphic to a subquotient of a representation of $\mathscr{H}$ in the space of automorphic (resp. cusp) forms on $G(A)$. It follows from 4.5 that such a representation is always admissible. It will also be called an automorphic representation of G or $G(A)$, although, strictly speaking, it is not a $G(A)$-module. However, it is always a $G(A_f)$-module. More generally, a topological $G(A)$-module E will be said to be *automorphic* if the subspace of admissible vectors in E is an automorphic representation of $\mathscr{H}$. In particular, if χ is a character of $Z(A)/Z(F)$, any G-invariant irreducible closed subspace of

$$L^2(G(F)\backslash G(A))_\chi = \{f \in L^2(G(F)\cdot Z(A)\backslash G(A))|, f(z\cdot x) = \chi(z)\cdot f(x),\ z \in Z(A),\ x \in G(A))\}$$

is automorphic in this sense, in view of [4, Theorem 4]. By a theorem of Gelfand and

Piatetski-Shapiro **[7]** (see also **[8]**), the subspace ${}^{\circ}L^2(G(F)\backslash G(A))_\chi$ of cuspidal functions of $L^2(G(F)\backslash G(A))_\chi$ is a discrete sum with finite multiplicities of closed irreducible invariant subspaces. Those give then, up to isomorphisms, all cuspidal automorphic representations in which $Z(A)$ has character χ. The admissible vectors in those subspaces are all the cusp forms satisfying 4.3(3).

4.7. Let $L \supset L'$ be compact open subgroups of $G(A_f)$. Then $\xi_L * \xi_{L'} = \xi_L$; hence $\mathscr{A}(\xi_\infty \otimes \xi_L; J, K_\infty) \subset \mathscr{A}(\xi_\infty \otimes \xi_{L'}, J, K_\infty)$. The space $\mathscr{A}(\xi_\infty, J, K_\infty)$ spanned by all automorphic forms of types $(\xi_\infty \otimes \xi_f, J, K_\infty)$, with ξ_f arbitrary in $\mathscr{H}_f$, may then be identified to the inductive limit

$$\mathscr{A}(\xi_\infty, J, K_\infty) = \operatorname{ind}\lim_{L} \mathscr{A}(\xi_\infty \otimes \xi_L, J, K_\infty), \tag{1}$$

where L runs through the compact open subgroups of $G(A_f)$, the inductive limit being taken with respect to the above inclusions. The group $G(A_f)$ operates on $\mathscr{H}_f$ by inner automorphisms. Let us denote by ${}^x\xi$ the transform of $\xi \in \mathscr{H}_f$ by Int x. We have in particular

$$ {}^x(\xi_L) = \xi_{({}^xL)} \qquad (x \in G(A_f),\ \xi_L \text{ as in } 4.1); \tag{2}$$

if f is a continuous (or measurable) function on $G(A)$, then

$$ (r_x f) * \xi = r_x(f * {}^{x^{-1}}\xi) \qquad (x \in G(A_f),\ \xi \in \mathscr{H}_f). \tag{3}$$

Therefore, $G(A_f)$ operates on $\mathscr{A}(\xi_\infty, J, K_\infty)$ by right translations. It follows from 4.3(i) that this representation is admissible.

In view of 4.3(3), $\mathscr{A}(\xi_\infty, J, K_\infty)$ is the adelic analogue of the space $\mathscr{A}(\Sigma, \xi_\infty, J, K_\infty)$ of 1.9, where Σ is the family of arithmetic subgroups of $G(F)$ of the form $\Gamma_L = G(F) \cap (G_\infty \times L)$, where L is a compact open subgroup of $G(A_f)$. These are the *congruence arithmetic subgroups* of $G(F)$, i.e., those subgroups which, for an embedding $G \hookrightarrow \mathbf{GL}_n$ over F, contain a congruence subgroup of $G \cap \mathbf{GL}_n(O_F)$. This analogy can be made more precise when G satisfies strong approximation, which is the case in particular when G is semisimple, simply connected, almost simple over F, and G_∞ is noncompact **[16]**. In that case, as recalled in 4.4(2), we have $G(A) = G(F)\cdot G_\infty \cdot L$ for any compact open subgroup of $G(A_f)$, so that we may take $C = \{1\}$ in 4.3. Then 4.3(2) provides an isomorphism

$$\mathscr{A}(\xi_\infty \otimes \xi_L, J, K_\infty) \xrightarrow{\sim} \mathscr{A}(\Gamma_L, \xi_\infty, J, K_\infty), \tag{4}$$

for any L, whence

$$\mathscr{A}(\xi_\infty, J, K_\infty) \cong \mathscr{A}(\Sigma, \xi_\infty, J, K_\infty), \tag{5}$$

where Σ is the set of congruence arithmetic subgroups of $G(F)$. Moreover, the projection of $G(F)$ in $G(A_f)$ is dense in $G(A_f)$ and $G(A_f)$ may be identified to the completion $G(F)_\Sigma$ of $G(F)$ with respect to the topology defined by the subgroups Γ_L. It is easily seen that the isomorphism (5) commutes with $G(\boldsymbol{Q})$, where, on the left-hand side $x \in G(\boldsymbol{Q})$ acts as in 1.9, via left translations, and, on the right-hand side, x acts as an element of $G(A_f)$ by right translations. It follows that the isomorphism (5) commutes with the actions of $G(F)_\Sigma = G(A_f)$ defined here and in 1.9. Also, the isomorphism $G(F)_\Sigma \xrightarrow{\sim} G(A_f)$ induces one of the Hecke algebra $\mathscr{H}(G(F)_\Sigma)$ (see 1.10) onto $\mathscr{H}_f$ and, again, (5) is compatible with the actions defined here and in 1.10. Note also that Γ_L is dense in L, by strong approximation, so that this isomorphism of Hecke algebras induces one of $\mathscr{H}(G(F)_\Sigma, \bar{\Gamma}_L)$, which is equal to $\mathscr{H}(G(F), \Gamma_L)$, onto $\mathscr{H}(G(A_f), L)$. [Strictly speaking, this applies at first for $F = \boldsymbol{Q}$, but we can reduce the general case to that one, if we replace G by $R_{F/\boldsymbol{Q}}G$ (4.1).]

In the general case, the isomorphisms 4.3(2) for various ξ_L are compatible with the action of $G(F)$ defined here and in 1.9 respectively, and this extends by continuity to the closure in $G(A_f)$ of the projection of $G(F)$.

4.8. Let $\mathscr{A}(J, K_\infty)$ be the span of the spaces $\mathscr{A}(\xi_\infty, J, K_\infty)$ and $\mathscr{A}(K_\infty)$ the span of the $\mathscr{A}(J, K_\infty)$. These spaces are $\mathscr{H}$-modules, and union of admissible $\mathscr{H}$-submodules. When 4.6(5) holds, they are isomorphic to the spaces $\mathscr{A}(\Sigma, J, K_\infty)$ and $\mathscr{A}(\Sigma, K_\infty)$ of automorphic forms on $G(\boldsymbol{R})$ defined in 1.10. Otherwise, the relationship is more complex, and would have to be expressed by means of the isomorphisms 4.3(2).

5. Groups over function fields. In this section, F is a function field of one variable over a finite field. A function on $G(A)$ is said to be smooth if it is locally constant.

5.1. Let χ be a quasi-character of $Z(A)/Z(F)$ and K an open subgroup of $G(A)$. We let ${}^\circ\mathscr{V}(\chi, K)$ be the space of complex valued functions on $G(A)$ which are right invariant under K, left invariant under $G(F)$, satisfy

$$f(z \cdot x) = \chi(z) \cdot f(x) \qquad (z \in Z(A),\ x \in G(A)), \tag{1}$$

and are cuspidal (3.3). [These functions are cusp forms, in a sense to be defined below (5.7), but the latter notion is slightly more general.] We need the following:

5.2. Proposition (G. Harder). *Let K and χ be given. Then there exists a compact subset C of $G(A)$ such that every element of ${}^\circ\mathscr{V}(\chi, K)$ has support in $Z(A)\cdot G(F)\cdot C$. In particular, ${}^\circ\mathscr{V}(\chi, K)$ is finite dimensional.*

This follows from Corollary 1.2.3 in **[10]** when G is split over F and semisimple (the latter restriction because (1) is not the condition imposed in **[10]** with respect to the center). However, since $G(A)$ can be covered by finitely many Siegel sets, the argument is general (see [**9**, p. 142] for $\mathbf{GL}_n$).

5.3. Corollary. *Let X be a finite set of quasi-characters of $Z(A)/Z(F)$ and m a positive integer. Then the space ${}^\circ\mathscr{V}(X, m, K)$ of cuspidal functions which are right invariant under K and satisfy the condition*

$$\prod_{\chi\in X} (r(z) - \chi(z))^m \cdot f = 0 \tag{1}$$

is finite dimensional.

Proof. The space ${}^\circ\mathscr{V}(X, m, K)$ is the direct sum of the spaces ${}^\circ\mathscr{V}(\{\chi\}, m, K)$; hence we may assume that X consists of one quasi-character χ. By 5.3, ${}^\circ\mathscr{V}(X, 1, K)$ is finite dimensional. We then proceed by induction and assume that ${}^\circ\mathscr{V}(X, s, K)$ is finite dimensional for some $s \geqq 1$.

The group $Z(A)/Z(F)\cdot K'$, where $K' = Z(A) \cap K$, is finitely generated. Let $(z_j)_{1\leqq j\leqq q}$ be a generating set. Set $A_{s+1,0} = {}^\circ\mathscr{V}(X, s+1, K)$ and, for $t = 1, \dots, q$:

$$A_{s+1,t} = \{f \in A_{s+1,0} \mid (r(z_j) - \chi(z_j))^s \cdot f = 0 \ (j = 1, \dots, t)\}.$$

Then

$$A_{s+1,0} \supset A_{s+1,1} \supset \ldots \supset A_{s+1,t} \supset \ldots \supset A_{s+1,q} = {}^\circ\mathscr{V}(X, s, K).$$

Fix t ($0 \leqq t < q$). Then $f \mapsto (r(a_{t+1}) - \chi(a_{t+1}))\cdot f$ maps $A_{s+1,t}$ into $A_{s+1,t+1}$ and its kernel is contained in $A_{s+1,t+1}$. It follows, by descending induction on t, that $A_{s+1,0}$ is finite dimensional.

5.4. REMARK. If H is a commutative group, let $\boldsymbol{C}[H]$ be its group algebra over $\boldsymbol{C}$. Every complex representation (π, W) of H extends canonically to one of $\boldsymbol{C}[H]$. An element $w \in W$ is H-finite if and only if it is annihilated by some ideal I of finite codimension of $\boldsymbol{C}[H]$. If I is such an ideal, there exist a finite set X of quasi-characters of H and a positive integer m such that

$$(1) \qquad \prod_{\chi\in X}(\pi(h) - \chi(h))^m\cdot w = 0, \quad \text{for all } h\in H \text{ and all } w \text{ annihilated by } I.$$

If H is finitely generated, then, conversely, all elements $w \in W$ satisfying (1) for all $h \in H$ are H-finite, and annihilated by some ideal of finite codimension of $\boldsymbol{C}[H]$. Therefore, 5.3 implies that the space ${}^\circ\mathscr{V}(I, K)$ of cuspidal functions on $G(F)\backslash G(\boldsymbol{A})/K$ which are annihilated by some ideal I of finite codimension of $\boldsymbol{C}[Z(\boldsymbol{A})/Z(F)]$ is finite dimensional. In fact, since $Z(\boldsymbol{A})/Z(F)\cdot(K \cap Z(\boldsymbol{A}))$ is finitely generated, these two statements are equivalent.

5.5. Let E be a local field, R a connected reductive group over E, $\mathscr{H}_R$ the Hecke algebra of $R(E)$. A left ideal J of $\mathscr{H}_R$ is said to be *admissible* if the natural representation of $\mathscr{H}_R$ on $\mathscr{H}_R/J$ is admissible.

LEMMA. *Let J be an admissible ideal of $\mathscr{H}_R$ and K a compact open subgroup of $R(E)$. Let P be a parabolic E-subgroup of H, N the unipotent radical of P and M a Levi E-subgroup of P. There is an admissible ideal J_M in the Hecke algebra $\mathscr{H}_M$ of $M(E)$ with the following property: if π is a smooth representation of $R(E)$ on a space W, and $w \in W$ is K-fixed, annihilated by J, then the image $\bar{w}$ of w in the space W_N (cf.* [3]) *is annihilated by J_M.*

PROOF. Let $\varphi_0 \in \mathscr{H}_R$ be the characteristic function of K, v_0 its image in $\mathscr{H}_R/J$ and $\bar{v}_0$ the image of v_0 in $(\mathscr{H}_R/J)_N$. The representation of $R(E)$ on $\mathscr{H}_R/J$ is admissible by assumption; therefore the representation of $M(E)$ on $(\mathscr{H}_R/J)_N$ is admissible [3], and the annihilator J_M of $\bar{v}_0$ in $\mathscr{H}_M$ is admissible. We claim that it has the required properties. In fact, if π and w are as in the lemma, then there is a unique $R(E)$-morphism $\mathscr{H}_R \to W$ taking φ_0 to w. It maps $\xi \in \mathscr{H}_R$ onto a scalar multiple of $\pi(\xi)\cdot w$. Therefore, it factors through an $R(E)$-morphism $\mathscr{H}_R/J \to W$, mapping v_0 onto w, whence an $M(E)$-morphism $(\mathscr{H}_R/J)_N \to W_N$ mapping $\bar{v}_0$ onto $\bar{w}$. It follows that $\bar{w}$ is annihilated by J_M.

5.6. THEOREM. *Let K be a compact open subgroup of $G(\boldsymbol{A})$. Let $v \in V$ and J an admissible ideal of $\mathscr{H}_v$. Then the space $\mathscr{V}(G, v, J, K)$ of complex valued functions f on $G(\boldsymbol{A})$ which are left invariant under $G(F)$, right invariant under K and annihilated by J is finite dimensional.*

PROOF. Since the representation of G_v on $\mathscr{H}_v/J$ is admissible (and finitely generated), there exists an ideal I_v of finite codimension of $\boldsymbol{C}[Z(F_v)]$ which annihilates $\mathscr{H}_v/J$.

Let $f \in \mathscr{V}(G, v, J, K)$. Then $h \mapsto f * \check{h}$ ($h \mapsto \check{h}$ being the canonical involution of $\mathscr{H}_v$) gives a G_v-morphism $\mathscr{H}_v/J \to f * \mathscr{H}_v$. Therefore $\mathscr{V}(G, v, J, K)$ is annihilated by I_v. Let $Z' = Z(A) \cap K$ and regard $Z(F_v)$ as a subgroup of $Z(A)$. Then $Z(F) \cdot Z(F_v) \cdot Z'$ has *finite index* in $Z(A)$. As a consequence, there exists an ideal I of finite codimension of $Z(A)/Z(F)$ which annihilates $\mathscr{V}(G, v, J, K)$. The space ${}^\circ\mathscr{V}$ of cuspidal elements in $\mathscr{V}(G, v, J, K)$ is then contained in ${}^\circ\mathscr{V}(I, K)$ (notation of 5.4), hence is finite dimensional.

We now prove the theorem by induction on the F-rank $\mathrm{rk}_F G'$ of the derived group G' of G. If $\mathrm{rk}_F G' = 0$, then $\mathscr{V}(G, v, J, K) = {}^\circ\mathscr{V}$, and our assertion is already proved. So assume $\mathrm{rk}_F G' \geqq 1$ and the theorem proved for groups of strictly smaller semisimple F-rank. Let now $P = M \cdot N$ vary through a set $\mathscr{P}$ of representatives of the conjugacy classes of proper parabolic F-subgroups of G, where M is a Levi F-subgroup and N the unipotent radical of P. For each such P, let C_P be a set of representatives of $P(A)\backslash G(A)/K$. It is finite. The intersection of the kernels of the maps $f \mapsto f_{P,c}$, where $f_{P,c}(m) = \int_{N(F)\backslash N(A)} f(n \cdot m \cdot c)\, dn$ ($c \in C_P$, $P \in \mathscr{P}$) ($f \in \mathscr{V}(G, v, J, K)$) is then ${}^\circ\mathscr{V}$, hence is finite dimensional. It suffices therefore to show that, for given $P \in \mathscr{P}$, $c \in C_P$, the functions $f_{P,c}$ vary in a fixed finite dimensional space. After having replaced J and K by conjugates, we may assume that $c = 1$. We write f_P for $f_{P,1}$. Let now U_G (resp. U_M) be the space of functions on $G(F)\backslash G(A)$ (resp. $M(F)\backslash M(A)$) which are right invariant under some compact open subgroup (depending on the function). The representation r of $G(A)$ by right translations on U_G is smooth. If $x \in N(A)$ then $f_P = (r_x f)_P$; hence $\mu_P: f \mapsto f_P$ factors through $(U_G)_{N(F_v)}$. It follows then from 5.5 that the elements f_P ($f \in \mathscr{V}(G, v, J, K)$) are all annihilated by some admissible ideal J' of the Hecke algebra of $M(F_v)$. Since these elements are right invariant under $K' = K \cap M(A)$, it follows that μ_P maps $\mathscr{V}(G, v, J, K)$ into $\mathscr{V}(M, v, J', K')$. Since this last space is finite dimensional by our induction assumption, the proof is now complete.

5.7. COROLLARY. *Let f be a function on $G(A)$ which is left invariant under $G(F)$ and right invariant under some compact open subgroup of $G(A)$. Then the following conditions are equivalent*:

(1) *There is a place v of F such that the representation of G_v on the $G(F_v)$-invariant subspace generated by $r(G_v) \cdot f$ is admissible.*

(2) *For each place v of F, the representation of G_v on the space generated by $r(G_v) \cdot f$ is admissible.*

(3) *The representation of $G(A)$ on the space spanned by $r(G(A)) \cdot f$ is admissible.*

PROOF. Clearly (3) $\Rightarrow$ (2) $\Rightarrow$ (1). Assume (1). Let v be as in (1). Then f is annihilated by an admissible ideal J of $\mathscr{H}_v$. Let $U = f * \mathscr{H}$. We have to prove that U^K is finite dimensional for any compact open subgroup K of $G(A)$. There is no harm in replacing K by a smaller group, so we may assume that K fixes f. We may also assume that $K = K_v \times K^v$, where K_v is compact open in G_v and K^v is compact open in the subgroup G^v of elements in $G(A)$ with v-component equal to 1. We also have $\mathscr{H} = \mathscr{H}_v \otimes \mathscr{H}^v$ where $\mathscr{H}^v$ is the Hecke algebra of G^v. Let ξ_v (resp. ξ^v) be the idempotents associated to K_v (resp. K^v) (3.3). Then $\xi_v \otimes \xi^v = \xi_K$ is the idempotent associated to K. Any element g in U is a finite linear combination of elements of the form $f * \alpha * \beta$ ($\alpha \in \mathscr{H}^v$, $\beta \in \mathscr{H}_v$). If such an element is fixed under K, then $g * \xi_K =$

g; hence we may assume that each summand is fixed under K, and that $\alpha * \xi^v = \alpha$, $\beta * \xi_v = \beta$. Since f is fixed under K, it follows that $f * \alpha$ is fixed under K. The elements $f * \alpha$ then belong to the space $\mathscr{V}(G, v, J, K)$, which is finite dimensional by the theorem. For each such element $f * \alpha * \beta$ is contained in the space of K_v-fixed vectors in the admissible $\mathscr{H}_v$-module $f * \alpha * \mathscr{H}_v = f * \mathscr{H}_v * \alpha$, whence our assertion.

5.8. DEFINITIONS. An *automorphic form* on $G(A)$ is a function which is left invariant under $G(F)$, right invariant under some compact open subgroup, and satisfies the equivalent conditions of 5.7. A *cusp form* is a cuspidal automorphic form. Any automorphic form is $Z(A)$-finite (as follows from 5.7(3)).

An *irreducible representation of* $G(A)$ *is automorphic* if it is isomorphic to a subquotient of the $G(A)$-module $\mathscr{A}$ of all automorphic forms on $G(F)\backslash G(A)$. It follows from 5.7 that it is always admissible.

More generally, a topologically irreducible continuous representation of $G(A)$ in a topological vector space is automorphic if the submodule of smooth vectors is automorphic.

As in 4.6, it follows from **[4**, Theorem 4] that if χ is a character of $Z(A)/Z(F)$, then any G-invariant closed irreducible subspace of $L^2(G(F)\backslash G(A))_\chi$ is automorphic.

5.9. PROPOSITION. *Let f be a function on $G(F)\backslash G(A)$. Then the following conditions are equivalent*:

(1) *f is a cusp form.*

(2) *f is $Z(A)$-finite, cuspidal* (3.3), *and right invariant under some compact open subgroup of $G(A)$.*

PROOF. That (1) $\Rightarrow$ (2) is clear. Assume (2). Then f is annihilated by an ideal I of finite codimension of $C[Z(A)/Z(F)]$. Let U be the space of functions spanned by $r(G(A)) \cdot f$. Every element of U is cuspidal, annihilated by I and right invariant under some compact open subgroup. If L is any compact open subgroup, then U^L is contained in the space ${}^\circ\mathscr{V}(I, L)$ (notation of 5.4), hence is finite dimensional. Therefore U is an admissible $G(A)$-module and (1) holds.

5.10. It also follows in the same way that the space ${}^\circ\mathscr{A}(I)$ (resp. ${}^\circ\mathscr{A}(X, m)$) of all cusp forms which are annihilated by an ideal I of finite codimension of $C[Z(A)/Z(F)]$ (resp. which satisfy 5.3(1)) is an admissible $G(A)$-module. Moreover, if X consists of one element χ, and if $m = 1$, in which case we put $\mathscr{A}(X, m) = {}^\circ\mathscr{A}_\chi$, then this space is a direct sum of irreducible admissible $G(A)$-modules, with finite multiplicities. To see this we may, after twisting with $|\chi|^{-1}$, assume that χ is unitary. Then, since ${}^\circ\mathscr{A}_\chi$ consists of elements with compact support modulo $Z(A)$,

$$(f, g) = \int_{Z(A)G(F)\backslash G(A)} f(x) \cdot \overline{g(x)}\, dx$$

defines a nondegenerate positive invariant hermitian form on ${}^\circ\mathscr{A}_\chi$. Our assertion follows from this and admissibility. This is the counterpart over function fields of the Gelfand-Piatetski-Shapiro theorem (4.6).

We note that, by **[14]**, every automorphic representation transforming according to χ is a constituent of a representation induced from a cuspidal automorphic representation of a Levi subgroup of some parabolic F-subgroup, for any global field F.

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