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Twisted Yang-Baxter equations for linear quantum (super)groups

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Abstract

We consider the modified (or twisted) Yang-Baxter equations for the $SL_q(N)$ groups and $SL_q(N|M)$ supergroups. The general solutions for these equations are presented in the case of the linear quantum (super)groups. The introduction of spectral parameters in the twisted Yang-Baxter equation and its solutions are also discussed.

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1. Introduction

Recently various types of modified Yang-Baxter equation (m-YBE) have been considered. First of all such m-YBE appeared in investigations on special exchange algebras [1]. Another one was explored [2] in the context of the construction of new integrable lattice models which generalize the $SL(3)$ (and in general $SL(N)$) chiral Potts model. After a series of papers (see [3]) devoted to the solutions of the tetrahedron equation, similar modification, but now for the 3-d analogs of YBE, have been used for a construction of new integrable 3-d lattice theories [4]. Further, a variant of YBE has been found as certain cubic relations (for R-matrices being a special set of quantum 6-j symbols) which express the consistency of a quadratic algebra for elements of matrix generating a set of Clebsh-Gordon coefficients [5], [6], [7]. It is interesting that this modification coincides with the one considered in [1] and the corresponding m-YBE and R-matrices essentially depend on the phase space coordinates. Note that the same dependence occures for the classical r-matrices (which are called dynamical r-matrices) in Calogero-Moser type models (see [8], [9] and references therein). On the other hand, we recall that the q-analogs of the 6-j symbols (or Racah coefficients) give the braiding/fusing matrices expressing the property of crossing symmetry for four-point conformal blocks in 2-D conformal field theories (see e.g. [10]). Finally we stress that the analogous 'twisted' YBE has been proposed also in the context of quasitriangular Hopf algebras [11].

In this paper we investigate the m-YBE appeared in [1] as consistency relations for exchange matrices and has been considered in [5], [6], [7] as some relations for $SL_q(2)$ 6-j symbols. Here, in the cases of $SL_q(N)$ and $SL_q(N|M)$ (super)groups, we present the explicit solutions $R(p)$ for such m-YBE. These solutions could be related with 6-j symbols for corresponding quantum groups. Then we show how one can generalize these m-YBE and their solutions by introducing spectral parameters and also present the Yangian type limits for these solutions. Our conjecture is that after introducing the spectral parameter we obtain some objects related to the 6-j symbols for quantum affine Kac-Moody algebras.

2. Quantum deformations of dynamical systems on co-adjoint orbits or Alekseev-Faddeev toy models

At the beginning, to introduce the objects which will be under consideration, we remind some facts from the paper [6]. It is known [12] that apparently all finite dimensional integrable models, like Toda chain or Calogero particles, can be considered as systems on co-adjoint orbits of some Lie groups

G (with Lie algebras $\mathcal{G}$) and described by the Lagrangians:

$$\mathcal{L}(t) = < L|\frac{d}{dt}g\,g^{-1} > -\frac{1}{2} < L|L > + < L - \mu^L|\phi > + < g^{-1}Lg - \mu^R|\psi > ,$$
(1)

where t is a time, $g(t) \in G$, $L(t)$ and constant elements $\mu^{L,R}$ belong to the space $\mathcal{G}^*$ dual to the Lie algebra $\mathcal{G}$, terms with ϕ, $\psi \in \mathcal{G}$ define the momentum mappings and ϕ, ψ are nothing but Lagrange multipliers, $< .|. >$ is a paring of $\mathcal{G}$ and $\mathcal{G}^*$. We also identify $\mathcal{G}$ and $\mathcal{G}^*$ through the invariant Killing metric. The explicit choice of the group G, multipliers ϕ, ψ and elements $\mu^{L,R}$ specifies the dynamical system. If we consider the case for which we can take the matrix representation for $\mathcal{G}$ and $\mathcal{G}^*$ such that the pairing will be defined via operator Tr ($< A|B > \to Tr(AB)$), then, one can find the equations of motion from the Lagrangian (1) and prove that the quantities $I_n = Tr(L^n)$ are integrals of motion. Thus, for appropriate momentum mappings we can expect that the system with Lagrangian (1) yields an example of integrable model. From the Lagrangian (1) we find the following Poisson brackets (see e.g. [6]):

$$\begin{aligned}
\{g^1, g^2\} &= 0 \ , \\
\{L^1, g^2\} &= C\,g^2 \ , \\
\{L^1, L^2\} &= -\tfrac{1}{2}[C,\, L^1 - L^2] \ .
\end{aligned}$$
(2)

where as usual $g^1 = g \otimes 1$, $L^2 = 1 \otimes L, \ldots$ and $C = t_a \otimes t_b\,\eta^{ab}$ is an ad-invariant tensor (η^{ab} define Killing metric, and t_a form the basis for Lie algebra $\mathcal{G}$). Then, for the case $G = SL(N)$, the diagonalization of the left L and right $g^{-1}Lg$ momenta can be considered:

$$L = uPu^{-1} \ , \quad g^{-1}Lg = v^{-1}Pv \ ,$$
(3)

and this leads to the diagonalization of the group element g:

$$g = u\,Q^{-1}\,v \ .$$
(4)

Here we have used

$$P = -\frac{i}{2}\mathrm{diag}\{p_1,\, p_2, \ldots, p_N\} \ , \quad \sum_{i=1}^{N} p_i = 0 \ ,$$
(5)

$$Q = \mathrm{diag}\{\exp(ix_1),\, \exp(ix_2), \ldots, \exp(ix_N)\} \ , \quad \sum_{i=1}^{N} x_i = 0$$
(6)

and matrices u, v belong to the homogeneous space G/H where H is a Cartan subgroup associated with P.

In the papers [6], [7], [13] it has been shown that Poisson structure (2), in terms of the new variables $\{u, v, P, Q\}$, acquires the form

$$\{u^1,\, u^2\} = -u^1\,u^2\,r_0(p) \ , \quad \{v^1,\, v^2\} = r_0(p)\,v^1\,v^2 \ ,$$
(7)

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and

$$\{x_i,\,p_j\} = \delta_{ij}\ (1 \le i,j \le N-1)\,,\quad \{u_0,\,v_0\} = 0\,,\quad \{u_0,\,p_i\} = 0 = \{v_0,\,p_i\}\,,$$
(8)

where we have introduced $u = u_0\,Q,\ v = Q\,v_0,$

$$r_0(p) = \sum_\alpha \frac{i}{p_\alpha}(e_\alpha \otimes e_{-\alpha} - e_{-\alpha} \otimes e_\alpha)$$
(9)

and α runs over positive roots of $\mathcal{G}$. Namely we have

$$e_\alpha = e_{jk},\ j < k\,,\quad (e_{ij}e_{kl} = \delta_{kj}e_{il})\,,\quad p_\alpha = (p_j - p_k)\,.$$

The variable p (in $r_0(p)$) means that r_0 depends on all moments p_i. The quantum version of the formulas (7) - (9) has been discussed in [6], [7] for the case of $SL_q(2)$ group and, as it has been pointed out in [7], can be postulated for the general case of $SL_q(N)$ in the same form:

$$R_{12}\,u_1\,u_2 = u_2\,u_1\,R(p)_{12}\,,$$
$$R(p)_{12}\,v_2\,v_1 = v_1\,v_2\,R_{12}\,,$$
(10)

$$[u_0^1,\,v_0^2] = 0\,,\quad [u_0,\,p_i] = 0 = [v_0,\,p_i]\,,\quad [x_i,\,p_j] = i\,h\,\delta_{ij}\ (i,j \le N-1)\,.\ (11)$$

Here q - is a deformation parameter, h is a Planck constant, R_{12} is the well known R-matrix for the $GL_q(N)$ group (see [14], [15]) and we introduce new R-matrix $R(p)_{12}$ which nontrivially depends on the moments $p_i,\ \forall i$. For simplicity we remove from eqs. (10) nonessential factor $(q^{-1/N})$ which transforms $GL_q(N)$ R- matrix to the $SL_q(N)$ one. Here and below we use R- matrix formalism which was developed in [15]. We note that $u-$ and $v-$ algebras (10) can be identified via relation $u = v^{-1}$. Let us recall that the $GL_q(N)$ R -matrix satisfies the YBE:

$$\hat{R}\,\hat{R}'\,\hat{R} = \hat{R}'\,\hat{R}\,\hat{R}'$$
(12)

and the Hecke relation:

$$\hat{R}_{12}^2 = \lambda\hat{R}_{12} + 1\,,\quad \lambda = q - q^{-1}\,.$$
(13)

where $\hat{R} = \hat{R}_{12} = P_{12}R_{12},\ \hat{R}' = P_{23}R_{23}$ and P_{12} is a permutation matrix. Using relation (13) we immediately derive from eqs. (10) that $\hat{R}(p) = \hat{R}(p)_{12} = P_{12}\,R(p)_{12}$ also obeys the Hecke relation:

$$\hat{R}(p)^2 = \lambda\hat{R}(p) + 1\,,$$
(14)

Considering third order monomials in u (or in v) and using the commutation relations (10), (11) give the analogue of the YBE [1], [5], [6], [7] for the new objects $R(p_i)_{12}$:

$$(Q_1)^{-1}\,R(p)_{23}\,Q_1\,R(p)_{13}\,(Q_3)^{-1}\,R(p)_{12}\,Q_3 =$$
$$R(p)_{12}\,(Q_2)^{-1}\,R(p)_{13}\,Q_2\,R(p)_{23}$$
(15)

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This equation can be rewritten in the form (cf. with (12)):

$$\hat{R}(p)\,\tilde{R}(p)'\,\hat{R}(p) = \tilde{R}(p)'\,\hat{R}(p)\,\tilde{R}(p)' \tag{16}$$

where the matrix $\tilde{R}(p)' = Q_3\,\hat{R}(p)_{23}\,(Q_3)^{-1}$ obviously also satisfies the Hecke condition. For searching solutions of equations (16) it is convenient to rewrite them as

$$(Q_3^{-1}\,\hat{R}(p)\,Q_3)\,\hat{R}(p)'\,(Q_3^{-1}\,\hat{R}(p)\,Q_3) = \hat{R}(p)'\,(Q_3^{-1}\,\hat{R}(p)\,Q_3)\,\hat{R}(p)' \tag{17}$$

where $\hat{R}(p)' = \hat{R}(p)_{23}$. We call eqs. (15) - (17) modified or twisted Yang-Baxter equations. Note that ¿from eq. (16) one can obtain the relations which are similar to the reflection equations:

$$L(p)\,\tilde{R}(p)'\,L(p)\,\tilde{R}(p)' = \tilde{R}(p)'\,L(p)\,\tilde{R}(p)'\,L(p)\ ,$$

$$\tilde{L}(p)\,\hat{R}(p)\,\tilde{L}(p)\,\hat{R}(p) = \hat{R}(p)\,\tilde{L}(p)\,\hat{R}(p)\,\tilde{L}(p)\ ,$$

where $L(p) = \hat{R}(p)^2$, $\tilde{L}(p) = (\tilde{R}'(p))^2$.

In the papers [5], [6] and [7] the explicit form for the matrix $R(p)$ in $SL_q(2)$ case has been presented. There was stressed also that the elements of the matrix $R(p)$ give the special set of the 6-j symbols for $SL_q(2)$ group. Below we resolve equations (16) (and, thus, derive the explicit formulas for $R(p)$) in the case of groups $SL_q(N)$ and supergroups $SL_q(N|M)$. Note, that special solutions of (17) for arbitrary simple quantum groups (including $SL_q(N)$) are presented in [1]. Our solutions are multiparametric and more general. The specific choices of these parameters lead our solutions to the $R(p)$ which could be interpreted as corresponding 6-j symbols [5] or as exchange R- matrix for the vertex algebra [1].

3. Solutions of the modified YBE for the case of the linear quantum groups.

The m-YBE and their solutions for the case of $SL_q(N)$ have been firstly considered in [1]. Here we obtain more general multiparametric solution for the case of linear quantum groups and supergroups. We will consider the case of $GL_q(N)$ and $GL_q(K|N-K)$. In the case of special q-groups, solutions $R(p)$ can be obtained by multiplying $R(p)$ on some factor which is a simple function of q (see below).

Let us search the solution of m-YBE (17) in the form

$$\hat{R}_{12} = \hat{R}(p)^{i_1 i_2}_{j_1 j_2} = \delta^{i_1}_{j_2}\delta^{i_2}_{j_1}\,a_{i_1 i_2}(p) + \delta^{i_1}_{j_1}\delta^{i_2}_{j_2}\,b_{i_1 i_2}(p)\ . \tag{18}$$

Without limitation of generality one can put $b_{ii}(p) = 0$. Now the condition that $R(p)$ (18) satisfy the Hecke relation (14) gives the following constraints:

$$b_{ij} + b_{ji} = \lambda\ ,\ \ i \neq j\ , \tag{19}$$

$$a_{ij}\, a_{ji} - b_{ij}\, b_{ji} = 1,\ i \neq j\ , \qquad (20)$$

$$a_i^2 - \lambda a_i - 1 = 0 \Rightarrow a_i - \lambda = \frac{1}{a_i}\ , \quad a_i \equiv a_{ii}\ . \qquad (21)$$

Note that the eq. (21) has two solutions: $a_i = \pm q^{\pm 1}$ and therefore coefficients a_i are independent of the parameters p_k. If we take $a_i = q$, $\forall i$ (or $a_i = -q^{-1}$, $\forall i$) then we will have the case of the group $GL_q(N)$ (or $GL_{-q^{-1}}(N)$). But if we consider the mixing case: $a_i = q$ for $1 \leq i \leq K$ and $a_i = -q^{-1}$ for $K + 1 \leq i \leq N$ then we come to the case of supergroups $GL_q(K|N-K)$.

Now let us use the relations (cf. with (11))

$$\exp(-i\, x_j)\, p_k\, \exp(i\, x_j) = p_k + h\delta_{kj}\ , \quad (1 \leq k, j \leq N) \qquad (22)$$

and substitute (18) to the m-YBE (17). As a result, in addition to the relations (19) - (21), we obtain new constraints on the functions $a_{ij}(p)$ and $b_{ij}(p)$. First of all we deduce:

$$a_{ij}(p_1, \ldots, p_N) = a_{ij}(p_i, p_j)\ , \quad b_{ij}(p_1, \ldots, p_N) = b_{ij}(p_i, p_j)\ , \qquad (23)$$

and, as a consequence, relations (19), (20) have to be rewritten in the form:

$$b_{ij}(p_i, p_j) + b_{ji}(p_j, p_i) = \lambda\ ,$$

$$a_{ij}(p_i, p_j)\, a_{ji}(p_j, p_i) - b_{ij}(p_i, p_j)\, b_{ji}(p_j, p_i) = 1\ .$$

Then we have the constraint:

$$b_{ij}\, b_{jk}\, b_{ki} + b_{ik}\, b_{kj}\, b_{ji} = 0\ , \quad i \neq j \neq k \neq i \qquad (24)$$

(where there is no summation over i, j, k) and equations

$$b_{ij}(p_i + h, p_j) = \frac{b_{ij}(p_i, p_j)\, a_i}{1/a_i + b_{ij}(p_i, p_j)}\ , \qquad (25)$$

$$b_{ij}(p_i, p_j + h) = \frac{b_{ij}(p_i, p_j)/a_j}{a_j - b_{ij}(p_i, p_j)}\ . \qquad (26)$$

Using (25) and (26) leads to the following general relations:

$$b_{ij}(p_i + n\, h, p_j + m\, h) = \frac{a_i^n\, a_j^{-m}\, b_{ij}(p_i, p_j)}{a_i^{-n}\, a_j^m + b_{ij}(p_i, p_j)\, (a_i^n a_j^{-m} - a_i^{-n} a_j^m)/\lambda} =$$

$$= \frac{\lambda\, a_i^n\, a_j^{-m}\, b_{ij}(p_i, p_j)}{a_i^n\, a_j^{-m}\, b_{ij}(p_i, p_j) + a_i^{-n}\, a_j^m\, b_{ji}(p_j, p_i)}\ . \qquad (27)$$

From these relations one can immediately find the general solution for the coefficients $b_{ij}(p)$:

$$b_{ij}(p_i, p_j) = \frac{a_i^{p_i/h}\, a_j^{-p_j/h}\, b_{ij}^0}{a_i^{-p_i/h}\, a_j^{p_j/h} + b_{ij}^0\, (a_i^{p_i/h} a_j^{-p_j/h} - a_i^{-p_i/h} a_j^{p_j/h})/\lambda} = \qquad (28)$$

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$$= \frac{\lambda \, a_i^{p_i/h} \, a_j^{-p_j/h} \, b_{ij}^0}{a_i^{p_i/h} \, a_j^{-p_j/h} \, b_{ij}^0 + a_i^{-p_i/h} \, a_j^{p_j/h} \, b_{ji}^0} \; ,$$

where constants $b_{ij}^0 = b_{ij}(0,0)$ have to obey the algebraical relations:

$$b_{ii}^0 = 0 \; , \quad b_{ij}^0 + b_{ji}^0 = \lambda \; ,$$

$$b_{ij}^0 \, b_{jk}^0 \, b_{ki}^0 + b_{ik}^0 \, b_{kj}^0 \, b_{ji}^0 = 0 \; , \tag{29}$$

which can be deduced by the substitution (28) into eqs. (19) and (24).

It is clear now that if $a_i = a_j$ (indices i and j 'have the same grading') then $b_{ij}(p_i, p_j) = b_{ij}(p_i - p_j)$, but if $a_i = -1/a_j$ (the case of supergroups when indices i and j 'have the opposite grading') then we deduce that $b_{ij}(p_i, p_j) = b_{ij}(p_i + p_j)$. Note that the only conditions on the parameters $a_{ij}(p)$ needed for fulfillment of m-YBE are listed in (20) and (23).

Let us consider the case of the $GL_q(N)$ group. In this case we have $a_i = q \; \forall i$ and relation (28) takes the form:

$$b_{ij}(p_i - p_j) = \frac{q^{(p_i - p_j)/h} \, b_{ij}^0}{q^{(-p_i + p_j)/h} + [\frac{p_i - p_j}{h}]_q \, b_{ij}^0} \; , \tag{30}$$

while for the functions $a_{ij}(p_i, p_j)$ we obtain ¿from (20) the following relations:

$$a_{ij}(p_i, p_j) \, a_{ji}(p_j, p_i) = 1 + \frac{\lambda^2 \, b_{ij}^0 \, b_{ji}^0}{(q^{(p_i - p_j)/h} \, b_{ij}^0 + q^{(p_j - p_i)/h} \, b_{ji}^0)^2} \; . \tag{31}$$

In (30) we have used the standard notation $[x]_q = (q^x - q^{-x})/(q - q^{-1})$. One can obtain from these expressions the solutions discussed in [1], [5], [6] and [7] if we take the proper normalization of a_{ij} (e.g. Faddeev's or unitary normalization $a_{ij}(p_i - p_j) = a_{ji}(p_j - p_i)$) and consider, in (30), (31), the limit $b_{ij}^0 \to \infty$ $(i < j)$. This limit can be performed selfconsistently such that it does not cancel the conditions (29). Then we recall that our consideration was done for the case of the general groups $GL_q(K|N - K)$. In fact one can obtain SL-matrices $R(p)$ multiplying them by the functions $q^{1/(N-K)-1/K}$ which are needed for obtaining the identity $Sdet_q(R) = 1$ (for the $SL_q(N)$ case we have to multiply $R(p)$ by $q^{-1/N}$). As a result for the $SL_q(N)$ case we have the matrix $[q^{-1/N} \cdot R(p)]$ (18) with substitution:

$$b_{ij}(p_i - p_j) = \frac{q^{(p_i - p_j)/h}}{[(p_i - p_j)/h]_q} = \lambda - b_{ji}(p_j - p_i) \; , \tag{32}$$

$$a_{ij}(p_i - p_j) = \frac{([(p_i - p_j)/h + 1]_q \, [(p_i - p_j)/h - 1]_q)^{1/2}}{[(p_i - p_j)/h]_q \, \epsilon_{ij}} = a_{ji}(p_j - p_i) \tag{33}$$

where $\epsilon_{ij} = \pm 1$ $(i < j)$, $= \mp 1$ $(j > i)$. This choice of a_{ij} leads to the unitary condition (for real q and $p_i^\dagger = p_i$):

$$R(p)_{12}^\dagger = R(p)_{12}^{t_1 t_2} = R(p)_{21}.$$

Analogously, for the $SL_q(K|N-K)$ case, we obtain the matrix $q^{1/(N-K)-1/K} \cdot R(p)$ (18):

$$\hat{R}(p)_{SL_q(K|N-K)} = q^{1/(N-K)-1/K} \left(\delta_{j_2}^{i_1} \delta_{j_1}^{i_2} \left[(-1)^{(i_1)} q^{1-2(i_1)} \delta^{i_1 i_2} + \right.\right.$$

$$+ a(p_i - p_j)(\theta_{K+1,i}\, \theta_{K+1,j} + \theta_{i,K}\theta_{j,K}) + a(p_i + p_j)(\theta_{K+1,i}\, \theta_{j,K} + \theta_{i,K}\theta_{K+1,j})] +$$

$$+ \delta_{j_1}^{i_1} \delta_{j_2}^{i_2} \left[b(p_i - p_j)\theta_{K+1,i}\, \theta_{K+1,j} + b(p_j - p_i)\theta_{i,K}\, \theta_{j,K} + \right.$$

$$b(p_i + p_j)\theta_{K+1,i}\, \theta_{j,K} + b(-p_i - p_j)\theta_{i,K}\, \theta_{K+1,j}] \left.\right)$$

$$(34)$$

where $(i) = 0$ for $1 \le i \le K$ and $= 1$ for $K+1 \le i \le N$, $\theta_{ij} = 1$ for $i > j$ and $= 0$ for $i \le j$. The functions $a(p_i - p_j) = a_{ij}(p_i - p_j)$, $b(p_i - p_j) = b_{ij}(p_i - p_j)$ are defined in (32), (33).

To conclude this section we stress that we have found the more general solution of the m-YBE (15) - (17) then obtained in [1], [5]. Namely, our solutions depend on the set of arbitrary parameters b_{ij}^0 constrained by the conditions (29). The role of these parameters is still to be clarified. Taking some special limits and choosing the normalization of a_{ij} one leads to the known solutions $R(p)$ of the papers [1] and [5] (see e.g. (32), (33)).

4. Modified (twisted) YBE with spectral parameters.

In this section we show that m-YBE (15) - (17) can be generalized by introducing spectral parameters $y, z, \ldots$. We demonstrate that every solution $R(p)$ which has been found in the previous section will lead to the solution $R(p, y)$ for the m-YBE with the spectral parameters. It is interesting to note that such introducing of the spectral parameters can be done in complete analogy with the usual way of obtaining the trigonometric solutions

$$\hat{R}(y) = y^{-1}\, \hat{R} - y\, \hat{R}^{-1} \tag{35}$$

of the YBE from the R- matrices $\hat{R}$ related to the $GL_q(N)$ groups. On the other hand, we know that the trigonometric solutions (35) are related to the quantum Kac-Moody algebras [16]. In this connection (following the statements of the papers [5], [6], [7]) it is natural to conjecture that $R(p, y)$ could be interpreted as a special set of 6-j symbols for the q-deformations of linear affine algebras.

The natural assumption about the form of the m-YBE dependent on the spectral parameters is the following:

$$\hat{R}(p,\,y)\,\tilde{R}'(p,\,y\cdot z)\,\hat{R}(p,\,z) = \tilde{R}'(p,\,z)\,\hat{R}(p,\,y\cdot z)\,\tilde{R}'(p,\,y) \qquad (36)$$

Now it is not difficult to check by using m-YBE (16) and the Hecke relations (14) that the following matrices

$$\hat{R}(p,\,y) = y^{-1}\,\hat{R}(p) - y\,\hat{R}(p)^{-1}\,,$$

$$\tilde{R}'(p,\,y) = y^{-1}\,\tilde{R}'(p) - y\,(\tilde{R}'(p))^{-1} = Q_3\,\hat{R}'(p,\,y)\,Q_3^{-1}\,, \qquad (37)$$

are the solutions of the new m-YBE (36). We note that solutions (37) satisfy the identity

$$\hat{R}(p,\,y)\,\hat{R}(p,\,y^{-1}) = \left(\lambda^2 - (y - y^{-1})^2\right)\,,$$

which is a kind of unitary condition for $R(p)$ (if $y^* = y^{-1}$). It is clear that the analog of the relations (10) dependent on the spectral parameters has the form

$$\hat{R}(y)\,u_1(yz)\,u_2(z) = u_1(z)\,u_2(yz)\,\hat{R}(p,\,y)\,.$$

Now let us put $q = exp(\gamma\,h)$ [6], [7] and $y = exp((-1/2)\lambda\,\theta)$. Following [6] we consider two different cases: deformed classical case ($h = 0,\ \gamma \neq 0$) and quantum non-deformed case ($\gamma = 0,\ h \neq 0$). In the first case we obtain that $\hat{R}(p,\,y)/\lambda$ tends to the Yangian R- matrix $\hat{R}(\theta) = \theta\,P_{12} - 1$ which satisfies usual YBE

$$\hat{R}(\theta)\,\hat{R}'(\theta + \theta')\,\hat{R}(\theta') = \hat{R}'(\theta')\,\hat{R}(\theta + \theta')\,\hat{R}'(\theta)\,.$$

In the second case we derive:

$$\lim_{\gamma\to 0}\frac{\hat{R}(p,\,y)}{\lambda} = \hat{R}(p,\,\theta) = \theta\,\hat{R}^0(p) - 1\,, \qquad (38)$$

where $\hat{R}^0(p)$ is represented in the form (18) with the following parameters (cf. with [6, 7]):

$$b_{ij} = \frac{h}{p_i - p_j} = -b_{ji}\,,$$

$$a_{ij} = \frac{((p_i - p_j + h)(p_i - p_j - h))^{1/2}}{\epsilon_{ij}(p_i - p_j)} = a_{ji}\,,$$

and matrix $\hat{R}(p,\,\theta)$ (38) satisfies twisted YBE:

$$\hat{R}(p,\,\theta)\,\tilde{R}'(p,\,\theta + \theta')\,\hat{R}(p,\,\theta') = \tilde{R}'(p,\,\theta')\,\hat{R}(p,\,\theta + \theta')\,\tilde{R}'(p,\,\theta)\,. \qquad (39)$$

To conclude this paper we note that it would be extremely interesting to use the twisted YBE (36), (39) and their solutions (37), (38) for formulating

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the integrable models e.g. via box construction of [2] or to relate these solutions with the braiding matrices describing generalized statistics (see e.g. [17]).

Acknowledgments

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