Proceedings of Symposia in Pure Mathematics
Vol. 33 (1979), part 1, pp. 185–188

CLASSICAL AND ADELIC AUTOMORPHIC FORMS. AN INTRODUCTION

I. PIATETSKI-SHAPIRO*

1. Classical Hecke theory. Let H be the upper half-plane $\{z \in C | \operatorname{Im} z > 0\}$, Γ a congruence subgroup of $\mathbf{SL}(2, Z)$, i.e., $\Gamma \supset \Gamma_N$ for some integer $N \geqq 0$ (where

$$\Gamma_N = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{SL}(2, Z) \text{ such that } \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} (\operatorname{mod} N)\right\}\Bigg).$$

The group $\mathbf{SL}(2, \boldsymbol{R})$ acts on H by $z \to (az + b)/(cz + d)$. We say that a function $f: H \to C$ is a *modular form of weight* k (with respect to Γ) iff

(a) f is holomorphic on H.

(b) $f((az + b)/(cz + d))(cz + d)^{-k} = f(z)$, for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$, and k some strictly positive integer k.

(c) f is holomorphic also at the cusps of H with respect to Γ.

For example, at ∞, this means that f has the Fourier expansion

$$f(z) = \sum_{n \geqq 0} a_n e^{2\pi i n \lambda z}.$$

We say that f is a *cusp form* if in the Fourier expansion at each cusp, $a_0 = 0$.

Let $\Omega(f)$ be the C-linear span of the set

$$\left\{ f\left(\frac{az + b}{cz + d}\right)(cz + d)^{-k} \,\middle|\, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{GL}^+(2, \boldsymbol{Q})\right\}.$$

Here $G = \mathbf{GL}^+(2, \boldsymbol{Q}) = \{g \in \mathbf{GL}(2, \boldsymbol{Q}) \mid \det(g) > 0\}$. Note that $\mathbf{GL}^+(2, \boldsymbol{R})$ acts on H. There is an obvious representation of G on $\Omega(f)$.

Hecke defined the L-function attached to the modular form f by the formula:

$$L(f, s) = \frac{(2\pi\lambda)^s}{\Gamma(s)} \int_0^\infty f(iy) y^{(s-1)/2} \, dy = \sum_{n=1}^{\infty} a_n n^{-s}$$

(the Dirichlet series corresponding to f). Hecke **[1]** proved the following theorem:

THEOREM. (1) *$L(f, s)$ is a "nice" entire function when f is a cusp form ("nice" means that $L(f, s)$ has a functional equation).*

(2) *$L(f, s)$ has an Euler product if $\Omega(f)$ is algebraically irreducible, i.e., has no invariant linear subspaces under the action of G.*

The second statement is the more interesting; it is equivalent to Hecke's actual

AMS (MOS) subject classifications (1970). Primary 10D20; Secondary 10D10, 22E55.

*The author was supported by an NSF grant while preparing this paper.

statement. He proved that $L(f, s)$ has an Euler product if f is an eigenfunction of the "Hecke operators". This is equivalent to $\Omega(f)$ being irreducible. Already (2) suggests that it might be better to study modular forms from the point of view of representation theory.

This leads us to the main purpose of this paper, which is to motivate the transition to the adelic setting and the systematic use of representation theory in the study of modular forms.

First let us explain the notion of Euler product. $L(s) = L(f, s)$ has an Euler product means that

$$L(s) = \prod_{p \text{ a prime}} L_p(s),$$

where

$$L_p(s) = (1 - \alpha_p p^{-s})^{-1} (1 - \beta_p p^{-s})^{-1}, \qquad \alpha_p, \beta_p \in \boldsymbol{C}.$$

In Hecke's theorem we have a functional equation of the sort: $L(s) = \varepsilon(s)\tilde{L}(1 - s)$, where $L(s)$ (resp. $\varepsilon(s)$) can be written as an infinite product $\prod L_p(s)$ (resp. $\prod \varepsilon_p(s)$). Now Tate's theory of L-functions associated to Grossencharakters and Artin L-functions suggests the problem of finding objects such that the local factors $L_p(s)$ and $\varepsilon_p(s)$ are ε- and L-functions for these objects. It was the beautiful idea of Jacquet-Langlands [2] to take as such objects irreducible representations of $\mathbf{GL}(2,\boldsymbol{Q}_p)$. More precisely, let us look at $\Omega(f)$. Any element $h \in \Omega(f)$ is fixed by a congruence subgroup $\Gamma \subseteq G$. Now we define a topology on G by taking a basis of neighborhoods of the identity to be the set of congruence subgroups of G, and let $\bar{G}$ be the completion of G in this topology. Then $\bar{G} = \{g \in \prod_{\text{finite primes}} \mathbf{GL}(2, \boldsymbol{Q}_p) | \det g_p = r > 0,\ r \in \boldsymbol{Q}\}$, where $\prod$ means restricted direct product, acts on $\Omega(f)$, via the action of G, and it can be shown that this representation π is an infinite tensor product (interpreted in a suitable sense) $\pi = \bigotimes_{\text{finite primes}} \pi_p$, where π_p is a representation of $\mathbf{GL}(2, \boldsymbol{Q}_p)$.

In the author's interpretation, it is the idea of Jacquet-Langlands to attach an L-function to the irreducible representation $\pi = \Omega(f)$, rather than to f, and then to interpret $L_p(s)$ and $\varepsilon_p(s)$ as the L- and ε-factors for the representations π_p of $\mathbf{GL}(2, \boldsymbol{Q}_p)$.

Finally, let us give the following additional motivation for introducing the adelic framework into the study of modular forms. We wish to study modular forms with respect to different congruence subgroups simultaneously, and every such form is stabilized by some congruence subgroup. To each inclusion relation $\Gamma \subset \Gamma_1$ corresponds a projection $H/\Gamma \to H/\Gamma_1$. A small computation shows that the projective limit of this system,

$$\operatorname*{proj\,lim}_{\Gamma \text{ a congruence subgroup}} H/\Gamma = K_\infty \backslash \mathbf{SL}(2, A)/\mathbf{SL}(2, \boldsymbol{Q}).$$

Hence the study of modular forms has been transformed into the study of a certain space of functions on this double coset space.

2. Automorphic forms for adelic groups. Assume G is a connected reductive algebraic group over a global field k. For such a G we can define the group G_A (cf. Springer's paper for the construction), and $G_k \subset G_A$ as a discrete subgroup.

We need some basic facts from reduction theory. First let $G_A^0 = \bigcap\{\ker|\chi|, \chi$ a rational character of $G\}$. Then $G_k\backslash G_A^0$ is compact iff there are no additive unipotent elements. (This is due to Borel-Harish-Chandra-Mostow-Tamagawa-Harder.)

Let $G_A = NAK$ be the global Iwasawa decomposition (which follows directly from the local Iwasawa decompositions), where K is a standard maximal compact subgroup of G_A, A the set of adelic points of a maximal split torus, and N is a maximal unipotent subgroup.

EXAMPLE. $G = \mathbf{GL}(2, k)$. Then

$$\begin{aligned} K_p &= O(2) && \text{if } p \text{ is real,} \\ &= U(2) && \text{if } p \text{ is complex,} \\ &= \mathbf{GL}(2, \mathbf{Q}_p) && \text{for a finite place;} \end{aligned}$$

$$N = \left\{\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}\right\}, \quad A = \left\{\begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}\right\}.$$

Note that $A \supset C = Z(G_A)$, and that we can write $A = CA^0$ for some subgroup A^0 s.t. $C \cap A^0$ is finite.

We say that a set $S \subset A^0$ is semibounded iff for any relatively compact subset $N_0 \subset N$, the set $\bigcup_{s \in S} sN_0s^{-1}$ is relatively compact.

EXAMPLE. $G = \mathrm{GL}(2)$. Then $A^0 = \{(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix})\}$, $S = \{(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}) \in A^0 |\ |a_p|_p \geqq c_p$ for all places p, and $c_p = 1$ for almost all $p\}$. Then S is semibounded.

We now give the important definition of a Siegel set $\mathfrak{S}$.[1]

$\mathfrak{S} = N_0SK$, when N_0 is an open compact subset of N, S is semibounded, and K as above. Then the main result of reduction theory says:

THEOREM. *There exists a Siegel set $\mathfrak{S}$ s.t. $G_A = G_k\mathfrak{S}C$. Hence there exists a fundamental domain for G_k contained in $\mathfrak{S}C$.*

We now come to the definition of a cusp form (the definition of an automorphic form is given elsewhere in these PROCEEDINGS): A function f: $G_k\backslash G_A \to \mathbf{C}$ is said to be a cusp form iff

(1) f is an eigenfunction with respect to C: $f(cg) = \omega(c)f(g)$.

(2) f is smooth (i.e., f is C^∞ at archimedean places and locally constant at the finite places).

(3) $\int_{CG_k\backslash G_A} |f(g)|^2\, dg < \infty$ (here we suppose that ω is unitary).

(4) $\int_{Z_k\backslash Z_A} f(zg)\, dz = 0$, where Z is the unipotent radical of any parabolic subgroup P.

EXAMPLE. For $\mathbf{GL}_2$ it is sufficient to consider the standard unipotent group $(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix})$.

For **GL** one has the following conjugacy classes of unipotent radicals:

$$\begin{pmatrix} 1 & * & * \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix}.$$

In general, for $\mathbf{GL}_n$, one associates to each partition $n_1 + \dots + n_r, = n$, a conjugacy class of unipotent radicals

[1]For simplicity we give the definition only for split groups. There is a similar definition in general.

$$\begin{bmatrix} I_{n_1} & & * \\ & I_{n_2} & \\ 0 & & \\ & & I_{n_r} \end{bmatrix}.$$

We shall write $L_0^2(\omega)$ for the space of cusp forms with respect to ω.

The first main result is the following.

THEOREM (GELFAND, PIATETSKI-SHAPIRO, HARDER). $L_0^2(\omega)$ *is a countable sum of irreducible admissible representations of* G_A, *each occurring with finite multiplicity.*

We now describe a naive form of Langlands' philosophy. Let π be a representation of G_A occurring in $L_0^2(\omega)$. Then one can attach to π a function $L(\pi, s)$ which is a product $\prod L(\pi_p, s)$, and furthermore $L(\pi, s)$ is "nice" (i.e., it is meromorphic, with a finite set of poles, and satisfies a functional equation), and the ε-factor $\varepsilon(\pi, s) = \prod \varepsilon(\pi_p, s)$.

A precise exposition of Langlands' philosophy will be given by Borel in his article [**5**].

3. The case of $\mathbf{GL}_2$. Now we shall explain what follows from Langlands' conjecture for $\mathbf{GL}_2$.

Let π be an irreducible cuspidal automorphic representation of $\mathbf{GL}_2(A)$. Then $\pi = \bigotimes_{\text{all primes}} \pi_p$, and for almost all primes $\pi_p \cong \mathrm{Ind}_B(\mu_1 \otimes \mu_2)$, where μ_1, μ_2 are unramified characters of $k_p^\times$. Such μ_1, μ_2 are described by $\mu_1(\tilde{p}), \mu_2(\tilde{p})$, where $\tilde{p}$ is the generator of the prime ideal p. Let n be an integer $n \geqq 1$. We can define local factors of the form

$$L^n(\pi_p, s) = (1 - \mu_1^n|\tilde{p}|^s)^{-1}(1 - \mu_1^{n-1}\mu_2|\tilde{p}|^s)^{-1} \ldots (1 - \mu_2^n|\tilde{p}|^s)^{-1}.$$

Here we are writing μ_1, μ_2 for $\mu_1(\tilde{p}), \mu_2(\tilde{p})$.

Conjecture. For π, n as above we can attach to *every* prime p a local factor $L^n(\pi_p, s)$, agreeing with the definition above at the unramified primes, such that the function $L^n(\pi, s) = \prod L^n(\pi_p, s)$ exists, and is "nice".

For $n = 1$ this conjecture is proved in the book by Jacquet-Langlands. For $n = 2$, it has been proved by Gelbart-Jacquet and Shimura. For $n = 3, 4$ it can be proved that $L^n(\pi, s)$ exists and has a meromorphic continuation. Beyond these cases, the situation is unresolved.

Notes of this talk were made by Lawrence Morris (I. H. E. S.) and Ben Seifert (U. C. Berkeley).

REFERENCES

1. E. Hecke, *Mathematische werke*, Vandenhoek and Ruprecht, Göttingen, 1959.

2. H. Jacquet and R.P. Langlands, *Automorphic forms on* GL(2), Lecture Notes in Math., vol. 114, Springer, 1970.

3. A. Borel, *Introduction aux groupes arithemétiques*, Hermann, Paris, 1969.

4. G. Harder, *Minkowskische Reduktionstheorie über Funktionen körpern*, Invent. Math. 7 (1969), 33–54.

5. A. Borel, *Automorphic L-functions*, these PROCEEDINGS, part 2, pp. 27–61.

UNIVERSITY OF TEL AVIV