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DECOMPOSITION OF REPRESENTATIONS INTO TENSOR PRODUCTS

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1. In this paper some generalizations of the classical theorem which classifies the irreducible representations of the direct product of two finite groups in terms of those of the factors will be discussed. The first generalization consists in expanding the class of groups considered.

THEOREM 1. *Let G_1, G_2 be locally compact totally disconnected groups and let $G = G_1 \times G_2$.*

(1.1) *If π_i is an admissible irreducible representation of G_i, $i = 1, 2$, then $\pi_1 \otimes \pi_2$ is an admissible irreducible representation of G.*

(1.2) *If π is an admissible irreducible representation of G, then there exist admissible irreducible representations π_i of G_i such that $\pi \simeq \pi_1 \otimes \pi_2$. The classes of the π_i are determined by that of π.*

We recall some notation. For a locally compact totally disconnected group G, the *Hecke algebra* $H(G)$ of G is the convolution algebra of locally constant complex valued functions on G with compact support. For a compact open subgroup K of G, let e_K be the function $(\text{meas } K)^{-1} \cdot \text{ch}_K$, where ch_K is the characteristic function of K and meas is the Haar measure on G which has been used to define convolution in $H(G)$. Then e_K is an idempotent of $H(G)$. The subalgebra $e_K H(G) e_K$ of $H(G)$ will be denoted $H(G, K)$. A smooth G-module W is in a natural way an $H(G)$-module, and for every compact open subgroup $K \subset G$ the space $W^K = e_K W$ is an $H(G, K)$-module.

Before proving Theorem 1 we state an *Irreducibility Criterion. A smooth G-module W is irreducible if and only if W^K is an irreducible $H(G, K)$-module for all compact open subgroups K of G.*

PROOF. This follows from the fact that if U is an $H(G, K)$-submodule of W^K, then $(H(G) \cdot U)^K = U$. □

Remark that in applying the irreducibility criterion it is sufficient to check that W^K is an irreducible $H(G, K)$-module for a set of K which forms a neighborhood base of the identity in G.

COROLLARY. *Let K be a compact open subgroup of G such that $H(G, K)$ is commutative, and let W be an admissible irreducible G-module. Then* $\dim W^K \leqq 1$.

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PROOF OF THEOREM 1. It is straightforward that

(i) $H(G_1 \times G_2) \simeq H(G_1) \otimes H(G_2)$,

(ii) $H(G_1 \times G_2, K_1 \times K_2) \simeq H(G_1, K_1) \otimes H(G_2, K_2)$ and

(iii) $(W_1 \otimes W_2)^{K_1 \times K_2} \simeq W_1^{K_1} \otimes W_2^{K_2}$

for every pair of compact open subgroups K_i of G_i and every pair of smooth G_i-modules W_i.

Assertion (1.1) follows from (iii) and the irreducibility criterion.

Conversely, let W be an admissible irreducible G-module. Let $K = K_1 \times K_2$, where K_i is a compact open subgroup of G_i, $i = 1, 2$, be such that $W^K \neq 0$. The space W^K is finite dimensional, so by the corollary on p. 94 of [2] there exist irreducible $H(G_i, K_i)$-modules $W_i^{K_i}$ and an $H(G, K)$ isomorphism a_K from W^K to $W_1^{K_1} \otimes W_2^{K_2}$. Similar remarks apply to every open subgroup $K' = K_1' \times K_2'$ of K. There exist $H(G_i, K_i)$-maps $b_i = b_i(K, K')$: $W_i^{K_i} \to W_i^{K_i'}$ such that the following diagram is commutative.

$$\begin{array}{ccc} W^K & \xrightarrow{a_K} & W_1^{K_1} \otimes W_2^{K_2} \\ \big\downarrow \text{incl.} & & \big\downarrow b_1 \otimes b_2 \\ W^{K'} & \xrightarrow{a_{K'}} & W_1^{K_1'} \otimes W_2^{K_2'} \end{array}$$

Moreover, the maps $b_i(K, K')$ can be chosen for every pair of compact open subgroups K, K' of this type in such a way as to form an inductive system. Then $W \simeq W_1 \otimes W_2$, where $W_i = \operatorname{ind} \lim_{K_i} W_i^{K_i}$, and W_i is an admissible irreducible representation of G_i, $i = 1, 2$.

The class of W_i is determined by that of W, for the restriction of W to G_i is W_i-isotypic. □

An analysis of the proof of Theorem 1 reveals that the groups G and G_i enter only through their Hecke algebras. This leads one to define an *idempotented algebra* (A, E) to be an algebra A with a directed family of idempotents E such that $A = \bigcup_{e \in E} eAe$. An *admissible module* W for (A, E) is an A-module W which is *nondegenerate* in the sense that $AW = W$ and is such that $\dim eW$ is finite for all $e \in E$. The tensor product of two idempotented algebras is naturally idempotented. The proof of Theorem 1 is readily adapted to establish a similar theorem about the admissible irreducible modules of the tensor product of two idempotented algebras.

2. The study of the representations of adelic groups, which are infinite restricted products of groups, requires the notion of restricted tensor product of vector spaces which was introduced in [4].

Let $\{W_v \mid v \in V\}$ be a family of vector spaces. Let V_0 be a finite subset of V. For each $v \in V \setminus V_0$, let x_v be a nonzero vector in W_v. For each finite subset S of V containing V_0, let $V_S = \bigotimes_{v \in S} W_v$; and if $S \subset S'$, let f_S: $W_S \to W_{S'}$ be defined by $\bigotimes_{v \in S} w_v \mapsto \bigotimes_{v \in S} w_v \bigotimes_{v \in S' \setminus S} x_v$. Then $W = \bigotimes_{x_v} W_v$, the *restricted tensor product* of the W_v with respect to the x_v, is defined by $W = \operatorname{ind} \lim_S W_S$. The space W is spanned by elements written in the form $w = \bigotimes w_v$, where $w_v = x_v$ for almost all $v \in V$.

The ordinary constructions with finite tensor products extend easily to restricted tensor products.

(1) Given linear maps B_v: $W_v \to W_v$ such that $B_v x_v = x_v$ for almost all $v \in V$, then one can define $B = \otimes B_v$: $W \to W$ by $B(\otimes w_v) = \otimes B_v w_v$.

(2) Given a family of algebras $\{A_v \mid v \in V\}$ and given nonzero idempotents $e_v \in A_v$ for almost all v, then $A = \otimes_{e_v} A_v$ is an algebra in the obvious way.

(3) If W_v is an A_v-module for each $v \in V$ such that $e_v \cdot x_v = x_v$ for almost all v, then $\otimes_{x_v} W_v$ is an A-module. The isomorphism class of W depends on $\{x_v\}$. However, if $\{x_v'\}$ is another collection of nonzero vectors such that x_v and x_v' lie on the same line in W_v for almost all v, then the A-modules $\otimes_{x_v} W_v$ and $\otimes_{x_v'} W_v$ are isomorphic.

EXAMPLE 1. The polynomial ring in an infinite number of variables $\boldsymbol{C}[X_1, X_2, \ldots]$ is isomorphic to $\otimes_{e_i} \boldsymbol{C}[X_i]$, where e_i is the identity element of $\boldsymbol{C}[X_i]$.

EXAMPLE 2. Let $G = \prod_{K_v}' G_v$ be the restricted product of locally compact totally disconnected groups G_v, restricted with respect to the compact open subgroups K_v. Then G itself is locally compact and totally disconnected, and $H(G)$ is isomorphic to $\otimes_{e_{K_v}} H(G_v)$.

For each $v \in V$ let W_v be an admissible G_v-module. Assume that $\dim W_v^{K_v} = 1$ for almost all v. Choosing for almost all v a nonzero vector $x_v \in W_v^{K_v}$, we may form the G-module $W = \otimes_{x_v} W_v$. The isomorphism class of W is in fact independent of the choice of $x_v \in W_v^{K_v}$ and will be called the tensor product of the representations W_v. One sees that W is admissible, and that it is irreducible if and only if each W_v is. The admissible irreducible representations of G isomorphic to ones constructed in this way are said to be *factorizable*.

THEOREM 2. *Suppose that $H(G_v, K_v)$ is commutative for almost all v. Then every admissible irreducible representation W of G is factorizable, $W \simeq \otimes W_v$. The isomorphism classes of the factors W_v are determined by that of W. For almost all v,* $\dim W_v^{K_v} = 1$.

PROOF. One first factorizes the finite dimensional spaces $W^{K'}$ for compact open subgroups $K' = \prod K_v'$ of G, then continues as in the proof of Theorem 1. □

3. Let G be a connected reductive algebraic group over a global field F. Let A be the adele ring of F, and let V be the set of places of F. The adelic group $G(A)$ is isomorphic to a restricted product $\prod_{K_v}' G(F_v)$, where the subgroups K_v are defined for all finite v and are certain maximal compact subgroups of $G(F_v)$. For almost all finite $v \in V$, $G(F_v)$ is a quasi-split group over F_v, and K_v is a special maximal compact subgroup. For these places v, $H(G(F_v), K_v)$ is commutative. See [5]. So the function field case of the following theorem, whose meaning has yet to be explained in the number field case, is a special case of Theorem 2.

THEOREM 3. *Every admissible irreducible representation of $G(A)$ is factorizable. The factors are unique up to equivalence.*

Let F be a number field. Then the class of admissible representations of $G(A)$ has yet to be defined. For each archimedean place $v \in V$, let K_v be a maximal compact subgroup of $G(F_v)$, and let $\mathfrak{g}_v$ be the real Lie algebra of $G(F_v)$. Let $K_\infty = \prod_{\text{arch } v} K_v$, $K = \prod_{\text{all } v} K_v$, and $G_\infty = \prod_{\text{arch } v} G(F_v)$. Let $\mathfrak{g}_\infty$ be the real Lie algebra of G_∞. Let A_f be the ring of finite adeles of F.

DEFINITION. An *admissible* $G(A)$-module W is a vector space W which is both a $(\mathfrak{g}_\infty, K_\infty)$-module and a smooth $G(A_f)$-module such that

(1) the action of $G(A_f)$ commutes with the action of $\mathfrak{g}_\infty$ and K_∞, and

(2) for each isomorphism class γ of continuous irreducible representations of K, the γ-isotypic component of W has finite dimension.

In Theorem 3, the factors at the archimedean places v are to be admissible $(\mathfrak{g}_v, K_v)$-modules. The proof when F is a number field is the same as that when it is a function field once an idempotented algebra is found for each archimedean place v whose admissible modules are the same as admissible $(\mathfrak{g}_v, K_v)$-modules.

Let G be a Lie group, and let K be a compact subgroup. Let $\mathfrak{g}$ and $\mathfrak{k}$ be the real Lie algebras of G and K. Let $U(\mathfrak{g}_C)$ and $U(\mathfrak{k}_C)$ be the universal enveloping algebras of the complexified Lie algebras. Define the *Hecke algebra* $H(\mathfrak{g}, K)$ of $(\mathfrak{g}, K)$ to be the algebra of left and right K-finite distributions on G with support in K. It contains the algebra A_K of K-finite measures on K viewed as distributions on G. The map $(X, \mu) \to X * \mu$ from $U(\mathfrak{g}_C) \times A_K$ to the space of distributions on G induces a vector space isomorphism of $U(\mathfrak{g}_C) \otimes_{U(\mathfrak{k}_C)} A_K$ with $H(\mathfrak{g}, K)$. With the set E of central idempotents of A_K, $H(\mathfrak{g}, K)$ is an idempotented algebra.

Let (π, W) be a $(\mathfrak{g}, K)$-module. By means of the formula $X \otimes \mu \cdot w = \pi(X)\pi(\mu)w$ for $X \in U(\mathfrak{g}_C)$, $\mu \in A_K$, and $w \in W$, the space W becomes a nondegenerate $H(\mathfrak{g}, K)$-module. Moreover, it is not difficult to verify that this construction establishes an isomorphism between the categories of admissible $(\mathfrak{g}, K)$-modules and of admissible $(H(\mathfrak{g}, K), E)$-modules.

4. In practice, a more analytic theory than that described above is needed as well.

Let $\{H_v \mid v \in V\}$ be a family of Hilbert spaces. For almost all $v \in V$, let x_v be a unit vector in H_v. The *Hilbert restricted product* $H = \widehat{\otimes}_{x_v} H_v$ is a Hilbert space which can be conveniently described by giving an orthonormal basis. Let P_v be an orthonormal basis for H_v for each $v \in V$ which extends $\{x_v\}$ for almost all v. The set of symbols $\widehat{\otimes} h_v$ such that $h_v \in P_v$ for all v and $h_v = x_v$ for almost all v is an orthonormal basis for H. Constructions analogous to those described above with reference to the ordinary restricted tensor product are available in the Hilbert space context.

For the following theorem see [3] and [1].

THEOREM 4. *Let $G(A)$ be as in Theorem* 3. *Let π be a continuous irreducible unitary Hilbert space representation of $G(A)$. Then*

(1) *There exist continuous irreducible unitary Hilbert space representations π_v of $G(F_v)$, almost all of which are unramified, such that $\pi \simeq \widehat{\otimes}\pi_v$. The factors π_v are unique up to isomorphism.*

(2) *For each isomorphism class γ of continuous irreducible representations of K, the γ-isotypic component of π has finite dimension.*

(3) *The space of K-finite vectors of π is in a natural way an admissible irreducible $G(A)$-module π^K. Let $\pi^K \simeq \otimes\pi_v^K$ be the factorization of π^K given by Theorem* 3. *Then π_v^K is isomorphic as an admissible $G(F_v)$-module to the space of K_v-finite vectors of π_v.*

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