ŁUKASIEWICZ-MOISIL MANY-VALUED LOGIC ALGEBRA OF HIGHLY-COMPLEX SYSTEMS

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Abstract.

The fundamentals of Łukasiewicz-Moisil logic algebras and their applications to complex genetic network dynamics and highly complex systems are presented in the context of a categorical ontology theory of levels, Medical Bioinformatics and self-organizing, highly complex systems. Quantum automata were broadly defined in refs.[2] and [3] as generalized, probabilistic automata with quantum state spaces [1]. Their next-state functions operate through transitions between quantum states defined by the quantum equations of motions in the Schrödinger representation, with both initial and boundary conditions in space-time. A new theorem is proven which states that the *category of quantum automata and automata–homomorphisms has both limits and colimits.* Therefore, both categories of quantum automata and classical automata (sequential machines) are *bicomplete.* A second new theorem establishes that the standard automata category is a subcategory of the quantum automata category. The quantum automata category has a faithful representation in the category of Generalized (**M,R**)–Systems which are open, dynamic biosystem networks [4] with defined biological relations that represent physiological functions of primordial(s), single cells and the simpler organisms. A new *category of quantum computers* is also defined in terms of *reversible* quantum automata with quantum state spaces represented by topological groupoids that admit a local characterization through unique, quantum *Lie algebroids.* On the other hand, the category of n– Łukasiewicz algebras has a subcategory of *centered* n– Łukasiewicz algebras (as proven in ref. [15]) which can be employed to design and construct subcategories of quantum automata based on n–Łukasiewicz diagrams of existing VLSI. Furthermore, as shown in ref.[15]

the category of centered n–Łukasiewicz algebras and the category of Boolean algebras are naturally equivalent. A 'no-go' conjecture is also proposed which states that *generalized* (**M,R**)–systems complexity prevents their complete computability (as shown in refs. [5]–[6]) by either standard, or quantum, automata.

KEYWORDS: *LM-logic algebra, algebraic category of LM-logic algebras, fundamental theorems of LM-logic algebra, many-valued logics of highly complex systems and Categorical Ontology, quantum automata categories, limits and colimits, bicomplete categories, Quantum Relational Biology, generalized metabolic-replication (M,R)–systems, complex bio-networks; quantum computers, computability of complex biological systems; centered n–Łukasiewicz algebras categories of n–Łukasiewicz algebras, categories of Boolean algebras.*

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1. Algebraic Logic, Operational and Łukasiewicz Quantum Logic

As pointed out by Birkhoff and von Neumann as early as 1936, a logical foundation of quantum mechanics consistent with quantum algebra is essential for both the completeness and mathematical validity of the theory. The development of Quantum Mechanics from its very beginnings both inspired and required the consideration of specialized logics compatible with a new theory of measurements for microphysical systems. Such a specialized logic was initially formulated by Birkhoff and von Neumann in 1936, and called 'Quantum Logic' (QL). However, in recent QL research several approaches were developed involving several types of non-distributive lattice, and their corresponding algebras, for n–valued quantum logics. Thus, modifications of the Łukasiewicz logic algebras that were introduced in the context of algebraic categories [14] by Georgescu and Popescu [15]– also recently reviewed and expanded by Georgescu [16]– can provide an appropriate framework for representing quantum systems, or in their unmodified form, for describing the activities of complex networks in categories of Łukasiewicz logic algebras [5].

There is nevertheless a serious problem remaining which is caused by the logical inconsistency between any quantum algebra and the Heyting logic algebra which has been suggested as a candidate for quantum logic. Furthermore, quantum algebra and topological approaches that are ultimately based on set-theoretical concepts and differentiable spaces (manifolds) also encounter serious problems of internal inconsistency. Since it has been shown that standard set theory which is subject to the axiom of choice relies on Boolean logic there appears to exist a basic logical inconsistency between the quantum logic–which is not Boolean–and the Boolean logic underlying all differentiable manifold approaches that rely on continuous spaces of points, or certain specialized sets of elements. A possible solution to such inconsistencies is the definition of a generalized Topos concept, and more specifically, of an Extended Quantum Topos (EQT) concept which is consistent with both QL and Quantum Algebraic, Logic, thus being potentially suitable for the developing a framework that may unify quantum field theories with ultra-complex system modeling and Complex Systems Biology (CSB).

2. Lattices and Von Neumann-Birkhoff (VNB) Quantum Logic: Definition and Some Logical Properties

We commence here by giving the *set-based Definition of a Lattice*. An *s–lattice* **L**, or a 'set-based' lattice, is defined as a *partially ordered set* that has all binary products (defined by the *s*–lattice operation "$\bigwedge$") and coproducts (defined by the *s*–lattice operation "$\bigvee$ "), with the "partial ordering" between two elements X and Y belonging to the *s*–lattice being written as "$X \preceq Y$". The partial order defined by $\preceq$ holds in **L** as $X \preceq Y$ if and only if $X = X \bigwedge Y$ (or equivalently, $Y = X \bigvee Y$ Eq.(3.1). A *lattice* can also be defined as a *category* (see, for example, ref. [9]) subject to all ETAC axioms, (but not subject, in general, to the Axiom of Choice usually encountered with sets relying on (distributive) Boolean Logic), that has all binary products and all binary coproducts, as well as partial ordering properties defined as follows:

(i) when unique arrows $X \to Y$ exist between objects X and Y in **L** such arrows will be labelled by "$\preceq$ ", as in " $X \preceq Y$";

(ii) *the coproduct* of X and Y , written as "$X \bigvee Y$" will be called the "*sup object*, or "*the least upper bound*", whereas the product of X and Y will be written as "$X \bigwedge Y$", and it will be called an *inf object*, or "*the greatest lower bound*";

(iii) *the partial order* defined by $\preceq$ holds in $\mathbf{L}$, as $X \preceq Y$ if and only if $X = X \bigwedge Y$ (or equivalently, $Y = X \bigvee Y$ (p. 49 of [23]).

If a lattice $\mathbf{L}$ has $\mathbf{0}$ and $\mathbf{1}$ as objects, such that $0 \to X \to 1$ (or equivalently, such that $0 \preceq X \preceq 1$) for all objects X in the lattice $\mathbf{L}$ viewed as a category, then $\mathbf{0}$ and $\mathbf{1}$ are the unique, initial, and respectively, terminal objects of this concrete category $\mathbf{L}$. Therefore, $\mathbf{L}$ has all finite limits and all finite colimits (p. 49 of ref. [23]), and is said to be *finitely complete and co-complete, or bicomplete*. Alternatively, the lattice 'operations' can be defined via functors in a 2-category (see, for example, refs. [9], [13] and [22]).

3. Quantum Logic (LQL), Łukasiewicz-Moisil (LM) and Operator Algebras

With all truth 'nuances' or assertions of the type << *system A* is excitable to the i-th level and system B is excitable to the j-th level >> one can define a special type of lattice that subject to the axioms introduced by Georgescu and Vraciu [15] becomes a *n-valued Łukasiewicz-Moisil, or LM -Algebra*. Further algebraic and logic details are provided in refs.[16] and [9].

In order to have the n-valued Łukasiewicz-Moisil logic (LML) algebra represent correctly the basic behavior of quantum systems [7], [17] –which is usually observed through measurements that involve a quantum system interactions with a macroscopic measuring instrument– several of these axioms have to be significantly changed so that the resulting lattice becomes non-distributive and also (possibly) non-associative. Several encouraging results in this direction were recently obtained by Dala Chiara and coworkers. With an appropriately defined quantum logic of events one can proceed to define Hilbert, or 'nuclear'/Frechet, spaces in order to be able to utilize the 'standard' procedures of quantum theories [17].

On the other hand, for classical systems, modeling with the unmodified Łukasiewicz logic algebra can also include both stochastic and fuzzy behaviors. For examples of such models the reader is referred to a previous report [5] where the activities of complex genetic networks are considered from a classical standpoint. One can also define as in [8] the 'centers' of certain types of Łukasiewicz n-logic algebras; then one has the following important theorem for such centered Łukasiewicz n-logic algebras which actually defines an equivalence relation.

The Logic Adjointness Theorem (Georgescu and Vraciu [15], Georgescu [16]): *There exists an Adjointness between the Category of Centered Łukasiewicz n-Logic Algebras,* ***CLuk***–*n, and the Category of Boolean Logic Algebras (**Bl**).*

Remarks

(1) The logic adjointness relation between **CLuk**–n and **Bl** is naturally defined by the left- and -right adjoint functors between these two categories of logic algebras.

(2) The natural equivalence logic classes defined by the adjointness relationships in the above Adjointness Theorem define a fundamental, *'logical groupoid'* structure.

(3) In order to adapt the standard Łukasiewicz-Moisil logic algebra to the appropriate Quantum Łukasiewicz-Moisil logic algebra, LQL, a few axioms of LM-algebra need modifications, such as : $N(N(X)) = Y \neq X$ (instead of the restrictive identity $N(N(X)) = X$, whenever the context, 'reference frame for the measurements', or 'measurement preparation' interaction conditions for quantum systems are incompatible with the standard 'negation' operation N of the Łukasiewicz-Moisil logic algebra; the latter remains however valid for classical or semi–classical systems, such as various complex networks with n-states (cf. [5]). Further algebraic and conceptual details were provided in a rigorous review by George Georgescu [16], and also in the recently published reports, [5]–[6].

4. Quantum Automata and Quantum Computation

Quantum computation and quantum 'machines' (or nanobots) were much publicized in the early 1980' s by Richard Feynman (Nobel laureate in Physics for Quantum Electrodynamics (QED)), and subsequently a very large number of papers- too many to cite all of them here- were published on this topic by a rapidly growing number of quantum theoreticians and some applied mathematicians.

Two such specific definitions are briefly considered next. *Quantum automata* were defined in refs. [2] and [3] as generalized, probabilistic automata with quantum state spaces. Their next-state functions operate through transitions between quantum states defined by the quantum equations of motions in the Schrödinger representation, with both initial and boundary conditions in space-time.

A new theorem was proven which states that the *category of quantum automata and automata homomorphisms has both limits and colimits.*

Therefore, both categories of quantum automata and classical automata (sequential machines) are *bicomplete.* A second new theorem established that the standard automata category is a subcategory of the quantum automata category.

Definition 1. One obtains a simple definition of *quantum automaton* by considering instead of the transition function of a classical sequential machine, the (quantum) transitions in a finite quantum system with definite probabilities determined by quantum dynamics. The *quantum state space* of a *quantum automaton* is thus defined as a quantum groupoid over a bundle of Hilbert spaces, or over rigged Hilbert spaces.

Formally, whereas a sequential machine A, or state machine with state space S, input set I and output set O, is defined as a quintuple: $(S, I, O, \delta : S \times S \rightarrow S, \lambda : S \times I \rightarrow O)$, a *quantum automaton* is defined by a triple $(H, \Delta : H \rightarrow H, \mu)$, where H is either a Hilbert space or a rigged Hilbert space of quantum states and operators acting on H, and μ is a measure related to the quantum logic, LM, and (quantum) transition probabilities of this quantum system.

Remark.

Quantum computation becomes possible only when macroscopic blocks of quantum states can be controlled *via* quantum preparation and subsequent, classical observation. Obstructions to actually building, or constructing quantum computers are known to exist in dimensions greater than 2 as a result of the standard **K-S** theorem. Subsequent definitions of quantum computers reflect attempts to either avoid or surmount such difficulties often without seeking solutions through quantum operator algebras and their representations related to extended quantum symmetries which define fundamental invariants that are key to actual constructions of this type of quantum computers.

Definition 2. Alternatively, a*quantum automaton* is defined as a quantum algebraic topology object– the triplet $(G_d, H - R_{G_d}, Aut(G))$, where G_d is a locally compact *quantum groupoid*, $H - R_{G_d}$ are the unitary representations of G_d on rigged Hilbert spaces R_{G_d} of quantum states and quantum operators on H, and $Aut(G_d)$ is the transformation, or automorphism, groupoid of quantum transitions.

Remark. Other definitions of quantum automata and quantum computations have also been reported that are closely related to recent experimental attempts at constructing quantum computing devices.

5. Applications of Quantum Automata to Modeling Complex Systems

The quantum automata category has a faithful representation in the category of *generalized* **(M,R)** -systems which are open, dynamic bio-networks [6] with defined biological relations that represent physiological functions of primordial(s), single cells and the simpler organisms. A new *category of quantum computers* is also defined in terms of *reversible* quantum automata with quantum state spaces represented by topological groupoids that admit a local characterization through unique 'quantum' *Lie algebroids*. On the other hand, the category of n-Łukasiewicz algebras has a subcategory of *centered* n- Łukasiewicz algebras [15] (which can be employed to design and construct subcategories of quantum automata based on n-Łukasiewicz diagrams of existing VLSI. Furthermore, as shown in ref. [15] the category of centered n-Łukasiewicz algebras and the category of Boolean algebras are naturally equivalent.

Variable machines with a varying transition function were previously discussed informally by Norbert Wiener as a possible model for complex biological systems although how this might be achieved in *Biocybernetics* has not been specifcally, or mathematically presented by Wiener.

A *'no-go' conjecture* was also proposed which states that *generalized* **(M,R)**–systems complexity prevents their complete computability by either standard or quantum automata. Note however that simple **(M,R)**-systems [19],[20] are representable as standard automata [21].

The concepts of quantum automata and quantum computation were initially studied and are also currently further investigated in the contexts of quantum genetics, genetic networks with nonlinear dynamics, and bioinformatics. In a previous publication [2]– after introducing the formal concept of quantum automaton–the possible implications of this concept for correctly modeling genetic and metabolic activities in living cells and organisms were also considered. This was followed by a formal report on quantum and abstract, symbolic computation based on the theory of categories, functors and natural transformations [3]. The notions of topological semigroup, quantum automaton,or quantum computer, were then suggested with a view to their potential applications to the analogous simulation of biological systems, and especially genetic activities and nonlinear dynamics in genetic networks. Further, detailed studies of nonlinear dynamics in genetic networks were carried out in categories of n-valued, Łukasiewicz Logic Algebras that showed significant dissimilarities [6] from the widespread Boolean models of human neural networks that may have begun with the early publication of [18]. Molecular models in terms of categories, functors and natural transformations were then formulated for uni-molecular chemical transformations, multi-molecular chemical and biochemical transformations [7]. Previous applications of computer modeling, classical automata theory, and relational biology to molecular biology, oncogenesis and medicine were extensively reviewed and several important conclusions were reached regarding both the potential and limitations of the computation-assisted modeling of biological systems, and especially complex organisms such as *Homo sapiens sapiens* [8], [9], [10]. Novel approaches to solving the realization problems of Relational Biology models in Complex System Biology are introduced in terms of natural transformations between functors of such molecular categories. Several applications of such natural transformations of functors were then presented to protein biosynthesis, embryogenesis and nuclear transplant experiments. Other possible realizations in Molecular Biology and Relational Biology of organisms were then suggested in terms of quantum automata models of Quantum Genetics and Interactomics. Future developments of this novel approach are likely to also include: fuzzy relations in Biology and Epigenomics, Relational Biology modeling of Complex Immunological and Hormonal regulatory systems, n-categories and *generalized* LM–Topoi of Łukasiewicz Logic Algebras and intuitionistic logic (Heyting) algebras for modeling nonlinear dynamics and cognitive processes in complex neural networks that are present in the human brain, as well as stochastic modeling of genetic networks in Łukasiewicz Logic Algebras (LLA).

Remark. Previous applications of computer modeling, classical automata theory, and relational biology to molecular biology, neural networks, oncogenesis and medicine were extensively reviewed in a previous monograph and several important conclusions were reached regarding both the potential and the severe limitations of the algorithm driven, recursive computation-assisted modeling of complex biological systems [6].

6. Conclusions

Non-distributive varieties of many-valued, LM-logic algebras that are also non-commutative open new possibilities for formal treatments of both complex quantum systems and highly complex biological networks, such as genetic nets, metabolic-replication systems (see for example refs. [19]–[21]), the interactome and neural networks [6]. This novel approach that involves both Algebraic Logic and Category Theory, provides an important framework for understanding the complexity inherent in intelligent systems and their flexible, adaptive behaviors. A consequence of the Logical Adjointness Theorem– which defines categorically the natural equivalence between the category of centered LM-logic algebras and that of Boolean logic algebras– is that one may be able to define Artificial Intelligence analogs of neural networks based on centered LM-logic algebras. In this process, higher dimensional algebra (HDA; [12]-[13]) and categorical models of human brain dynamics (refs. [8]–[11]) were predicted to play a central role. These new approaches are also relevant for resolving the tug-of-war between nature-vs.-nurture theories of human development and the 'natural' emergence through co-evolution of intelligence in the first *H. sapiens sapiens* societies.

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