Georgescu, G. 2006, N-valued Logics and Lukasiewicz-Moisil Algebras, *Axiomathes*, **16** (1-2): 123-136.

N–VALUED LOGICS AND ŁUKASIEWICZ–MOISIL ALGEBRAS

G. GEORGESCU

ABSTRACT. Fundamental properties of N–valued logics are compared and eleven theorems are presented for their Logic Algebras, including Łukasiewicz–Moisil Logic Algebras represented in terms of categories and functors. For example, the Fundamental Logic Adjunction Theorem allows one to transfer certain universal, or global, properties of the Category of Boolean Algebras, $\mathfrak{B}$, (which are well-understood) to the more general category $\mathcal{L}M_n$ of Łukasiewicz–Moisil Algebras. Furthermore, the relationships of LM_n–algebras to other many-valued logical structures, such as the n–valued Post, MV and Heyting logic algebras, are investigated and several pertinent theorems are derived. Applications of Łukasiewicz–Moisil Algebras to biological problems, such as nonlinear dynamics of genetic networks–that were previously reported–are also briefly noted here, and finally, probabilities are precisely defined over LM_n–algebras with an eye to immediate, possible applications in biostatistics.

1. INTRODUCTION

Many–valued logics are non–classical logics. They differ significantly from classical logic by not restricting the number of truth values to only two, thus allowing for a larger class of *'degrees of truth'*. However, they share with classical logic the acceptance of *the principle of truth-functionality*, *i.e.* the 'truth' value of a compound sentence is determined by the 'truth' values of its component sentences. An overview of n–valued logics, and especially logic algebras, will be here presented. The relationships of LM_n–algebras to other many-valued logical

---

Date: October 5, 2005.
1991 *Mathematics Subject Classification.* 1.211.
Key words and phrases. Categories, N–Valued Logics and Łukasiewicz–Moisil Logic Algebras; Categories of Łukasiewicz–Moisil Algebras; The Fundamental Logic Adjunction Theorem; Equivalences between pairs of different categories of n–valued Logic Algebras; Colimits, Limits and Adjointness Relations in Biology; Category of Boolean Algebras; Post, MV and Heyting logic algebras; Universal, or global, properties of categories of logic algebras; full and faithful adjoint functors between certain categories of logic algebras and the Boolean logic category; the Logic(s) of Life Itself; Biological applications of Łukasiewicz–Moisil Algebras; defining probabilities over LM_n –algebras with potential applications in Biostatistics;
Acknowledgements: xx1.

structures are here investigated in the context of logic algebras, such as the n–valued Post, MV and Heyting logic algebras. Several pertinent theorems are also presented in this article and their proofs are briefly outlined. Functorial semantics of algebraic theories (Lawvere, 1963) are relevant in this context even though their pertinent axioms differ substantially from those defining the logic algebras considered here. Further developments may also include for example *higher order* categorical logic (Lambek and Scott, 1986) *algebras.*

The applications of many–valued logics to other areas of science such as biology, probability theory and statistics will also be considered here briefly. Since the original paper by Baianu (1977) on genetic network nonlinear dynamics, the first report of the 'non–standard' logic required by complete self-reproduction (Löfgren, 1968), and the earliest seminal papers on Category Theory applications in Relational Biology by Rosen (1958a,b), *n–valued logics* have attracted the increasing interest of mathematical biologists because of their important applications in theoretical genetics and neurosciences (see for example the contribution by Baianu *et al*, 2005 presented in this volume, as well as the very popular books by Rosen, 1991, 1999 aimed at defining the special logic entailed by *life itself*). On the other hand, Boolean logic has already had a long history of applications to modeling of neuronal networks, beginning with the early paper of McCulloch and Pitts (1943); its applications to automata and computer science are well–established and also quite numerous. Finally, we shall here consider the definition of probabilities over LM_n–algebras with an eye to immediate applications to 'fuzzy' systems, Biostatistics, and perhaps also other areas of Logical–Mathematical, or Relational, Biology.

2. N-valued Logics and Łukasiewicz–Moisil algebras

Definition 1. (Moisil, 1941, *cited in* Moisil, 1972) An *n–valued Łukasiewicz–Moisil algebra*, (*LM_n–algebra*) is a structure of the form $(L, \vee, \wedge, N, (\varphi_i)_{i\in\{1,\ldots,n-1\}}, 0, 1)$ such that:

(L1) $(L, \vee, \wedge, N, 0, 1)$ is a *de Morgan algebra*, that is, a bounded distributed lattice with a decreasing involution N satisfying the de Morgan property $N(x \vee y) = Nx \wedge Ny$;

(L2) For each $i \in \{1, \ldots, n-1\}$, $\varphi_i : L \longrightarrow L$ is a lattice endomorphism;*

*The φ_i's are called the *Chrysippian endomorphisms* of L.

(L3) For each $i \in \{1, \ldots, n-1\}, x \in L$, $\varphi_i(x) \vee N\varphi_i(x) = 1$ and $\varphi_i(x) \wedge N\varphi_i(x) = 0$;

(L4) For each $i, j \in \{1, \ldots, n-1\}$, $\varphi_i \circ \varphi_j = \varphi_k$ iff $(i + j) = k$;

(L5) For each $i, j \in \{1, \ldots, n-1\}$, $i \leqslant j$ implies $\varphi_i \leqslant \varphi_j$;

(L6) For each $i \in \{1, \ldots, n-1\}$ and $x \in L$, $\varphi_i(Nx) = N\varphi_{n-i}(x)$.

(L7) (Moisil's determination principle)

$[\forall i \in \{1, \ldots, n-1\},\ \varphi_i(x) = \varphi_i(y)]$ implies $x = y$.

Example 2.1. . Let $L_n = \{0, 1/(n-1), \ldots, (n-2)/(n-1), 1\}$. This set can be naturally endowed with an LM$_n$ –algebra structure as follows:

- the bounded lattice operations are those induced by the usual order on rational numbers;
- for each $j \in \{0, \ldots, n-1\}$, $N(j/(n-1)) = (n-j)/(n-1)$;
- for each $i \in \{1, \ldots, n-1\}$ and $j \in \{0, \ldots, n-1\}$, $\varphi_i(j/(n-1)) = 0$ if $j < i$ and $= 1$ otherwise.

Note that, for $n = 2$, $L_n = \{0, 1\}$, and there is only one Chrysippian endomorphism of L_n is φ_1, which is necessarily restricted by the determination principle to a bijection, thus making L_n a Boolean algebra (if we were also to disregard the redundant bijection φ_1). Hence, the 'overloaded' notation L_2, which is used for both the classical Boolean algebra and the two–element LM$_2$–algebra, remains consistent.

Example 2.2. Consider a Boolean algebra $(B, \vee, \wedge, ^-, 0, 1)$. Let $T(B) = \{(x_1, \ldots, x_n) \in B^{n-1} \mid x_1 \leqslant \ldots \leqslant x_{n-1}\}$. On the set $T(B)$, we define an LM$_n$-algebra structure as follows:

- the lattice operations, as well as 0 and 1, are defined component–wise from L_2;
- for each $(x_1, \ldots, x_{n-1}) \in T(B)$ and $i \in \{1, \ldots, n-1\}$ one has:

$N(x_1, \ldots x_{n-1}) = (\overline{x_{n-1}}, \ldots, \overline{x_1})$ and $\varphi_i(x_1, \ldots, x_n) = (x_i, \ldots, x_i)$.

The following result establishes the relationship between the structures L_2 and L_n.

Proposition 1. The LM$_n$-algebras L_n and $T(L_2)$ are isomorphic.

Remark 1. Łukasiewicz utilized two principal logical connectives, the implication $\rightarrow$ and the negation $\neg$, defined by $x \rightarrow y = \min(1, 1 - x + y)$ and $\neg x = 1 - x$. Based on these two connectives, one can also introduce all the other connectives needed to define the Łukasiewicz logics: $x \vee y = (x \to y) \to y = (y \to x) \to x$, and $x \wedge y = N(Nx \vee Ny)$.

Alan Rose (1956, 1962) noticed that for $n \geqslant 5$, $L'_n = \{0, 1/(n-1), (n-2)/(n-1), 1\}$ – which is an LM$_n$–subalgebra of L_n, is no longer closed under the Łukasiewicz's implication: $(n-2)/(n-1) \to 1/(n-1) = 2/(n-1) \notin L'_n$. This fact shows that for $n \geqslant 5$ the LM$_n$–algebras are no longer suitable algebraic models for Łukasiewicz's logics, although, as we shall soon discuss, they are suitable for $n \in \{3, 4\}$. Related studies were also carried out previously by Rose and Rosser (1958), and earlier still by Rosser and Turquette (1952). Subsequently, the logics represented by the LM$_n$–structures were called *Moisil logics* and were studied for their own sake, as an alternative many–valued extension of classical logic. Further formalisations of the N_0–valued Łukasiewicz logics were also carried out subsequently by Rose (1978) for the development of a suitable propositional calculus.

Proposition 2. The LM$_n$–algebras form a variety, the determination principle being replaceable by the following conditions:
- $x \wedge N\varphi_i(x) \wedge \varphi_{i+1}(y) \leqslant y$, for each $i \in \{1, \ldots, n-1\}$;
- $\varphi_1(x) \leqslant x \leqslant \varphi_{n-1}(x)$;
- $x \wedge \bigwedge_{i \in \{1,\ldots,n\}} N\varphi_i(x) \vee \varphi_i(y) \leqslant y$.

We now turn to a detailed consideration of the relationship between LM$_n$– and Boolean algebras. For each LM$_n$–algebra, L, we define its *Boolean center* $C(L)$ as the set of all complemented elements of L: $C(L) = \{x \in L \ / \ x \vee Nx = 1\}$. One can easily see that, for each $x \in L$, the following equivalencies hold:

$x \in C(L)$ iff $x \wedge Nx = 0$ iff $\exists i \in \{1, \ldots, n-1\},\ \varphi_i(x) = x$ iff $\forall i \in \{1, \ldots, n-1\},\ \varphi_i(x) = x$ iff $\exists i \in \{1, \ldots, n-1\}, \exists y \in L,\ \varphi_i(y) = x$.

Definition 2. An LM$_n$-algebra is called *centered* if there exist $d_2, \ldots, d_{n-1} \in L$ such that:

(1) $\varphi_i(d_j) = \begin{cases} 1 \ , & \text{if } j \leqslant i \\ 0 \ , & \text{if } j > i \end{cases}$, for each $i, j \in \{1, \ldots, n-1\}$.

(Note that, provided $d_2, \ldots, d_{n-1}$ exist, by the determination principle, it follows that $d_{n-1} \leqslant d_{n-2} \leqslant \ldots \leqslant d_2$.)

Let $\mathcal{L}M_n$, $\mathcal{C}LM_n$, $\mathcal{B}$ denote the categories of LM$_n$–, centered LM$_n$–, and Boolean algebras respectively, with usual algebra homomorphisms as arrows. The following result establishes an important link between the categories $\mathcal{L}M_n$ and $\mathcal{B}$, and points to the "categorical image" $\mathcal{C}LM_n$ of the extension from the Boolean to the Łukasiewicz–Moisil algebras.

Theorem 2.1. (*The Fundamental Logic Adjunction Theorem*)

Let $C : \mathcal{L}M_n \longrightarrow \mathcal{B}$ and $T : \mathcal{B} \longrightarrow \mathcal{L}M_n$ be the two functors defined by:

- for each LM$_n$–algebra L, $C(L)$ is the Boolean center of L (which is a Boolean algebra);
- for each LM$_n$–morphism $f : L \longrightarrow L'$, $C(f) : C(L) \longrightarrow C(L')$ is the restriction and co–restriction of f to $C(L)$ and $C(L')$;
- for each Boolean algebra B, $T(B)$ is the already defined LM$_n$–algebra;
- for each Boolean morphism $g : B \longrightarrow B'$, $T(g) : T(B) \longrightarrow T(B')$ is defined by $T(g)(u) = g \circ u$ for all $u \in T(B)$.

Then, the following statements are valid:

(1) C is faithful and T is *full and faithful*;

(2) C is a *left adjoint* (Kan, 1958) of T, where the unit η and the counit ε are given by: for each LM algebra L and Boolean algebra B,

- $\eta_L : L \longrightarrow TC(L)$, $\eta_L(x)(i) = \varphi_i(x)$, for all $x \in L$ and $i \in \{1, \ldots, n-1\}$.
- $\varepsilon_B : CT(B) \longrightarrow B$, $\varepsilon_B(u) = u(1)$ for all $u \in CT(B)$.

(3) η_L is always a LM embedding and ε_B a Boolean isomorphism;

(4) For every Boolean algebra B, $T(B)$ is a centered LM algebra and

$$(\mathcal{C}LM_n \xrightarrow{C} \mathcal{B}, \mathcal{B} \xrightarrow{T} \mathcal{C}LM_n)$$

is an *equivalence* of categories.

This fundamental theorem allows one to transfer certain universal, or global, properties of $\mathcal{B}$ (which are well-understood) to the more general category $\mathcal{L}M_n$. The following result will illustrate the transfer procedure:

Theorem 2.2. (*Moisil's Representation Theorem*) For each LM$_n$–algebra L there exists a non–empty set X and an LM$_n$–monomorphism $L \longrightarrow {L_n}^X$.

Sketch of proof: The first step is to consider the Boolean *center* $C(A)$ of A, to which one can apply the Stone Representation Theorem in order to find both a non–empty set X and a Boolean monomorphism $d : C(L) \longrightarrow {L_2}^X$. The second step consists in applying the functor T to obtain the following natural sequence of monomorphisms and isomorphisms in $\mathcal{L}M_n$: $L \xrightarrow{\eta_L} T(C(L)) \xrightarrow{T(d)} T({L_2}^X) \simeq T(L_2)^X \simeq {L_n}^X$; this gives by composition an effective construction based on d of an LM$_n$–monomorphism between L and L_n^X.

Moisil's Representation Theorem plays an important role in the theory of LM$_n$–algebras, similar to that of Stone's Representation Theorem for the case of Boolean algebras. Namely, it reduces the problem of LM$_n$– algebraic calculus to the computation in L_n, which is readily programmable.

Remark 2. L_n is meant to represent the set of truth values for an n–valued logic. Therefore one can think of an LM$_n$–algebra as the "*set of propositions considered up to logical equivalence*", or more precisely, as the *Lindenbaum algebra* for such a logic. The algebraic operations are simply "*translations*" of the logical ones, and the corresponding logical notations are therefore also preserved. Whereas in the Łukasiewicz logic everything centers around implication, in the Moisil logic, the *lattice structure* serves as the site for the actions of the Chrysippian endomorphisms. Thus, to a single proposition p in the n–valued Moisil logic, there correspond $(n-1)$ propositions in Boolean logic, $\varphi_1(p), \ldots, \varphi_{n-1}(p)$, called the *nuances (shades of meaning)* of p. The principle of determination described in axiom (L7) shows that two propositions are *equivalent* if and only if their nuances are equivalent in the Boolean logic. Hence an n–valued proposition is perfectly determined by all its nuances. Notably, the nuances appear at two distinct levels:

(i) that of truth values;

(ii) that of logical inference (or, equivalently, that of algebraic calculus).

Moisil's Representation Theorem ensures the possibility of formulating axioms for a Moisil logic that is complete in the sense of representing L_n.

3. Relationships of LM_n–algebras to other many–valued logical structures

In this section we shall consider in some detail the relationships of LM_n–algebras to n–valued Post algebras, Heyting algebras (of intuitionistic, Brouwerian logic), and MV–algebras.

N–valued Post algebras. N–valued Post algebras were introduced by Rosenbloom (1962) as algebraic counterparts of n–valued Post logic. In a seminal 1941 publication (cited in Moisil, 1972), Moisil introduced the definition of *trivalent centered algebras for Łukasiewicz logic*, and these were extended by Georgescu and Vraciu (1970) to the already mentioned centered LM_n–algebras. Subsequently, it was shown by Cignoli (1977) that the latter are equivalent to the n–valued Post algebras.

Definition 3. An *n–valued Post algebra* is a structure of the form $(L, \vee, \wedge, N, 0, 1, c_1, \ldots, c_{n-2})$, where $c_1, \ldots, c_{n-2}$ are constants, such that:
(1) $(L, \vee, \wedge, N, 0, 1)$ is a de Morgan algebra;
(2) $0 \leqslant c_1 \leqslant \ldots \leqslant c_{n-2} \leqslant 1$;
(3) every $x \in L$ can be uniquely represented in the form $x = (b_1 \wedge c_1) \vee \ldots \vee (b_{n-2} \wedge c_{n-2}) \vee b_{n-1}$, with $b_1, \ldots, b_{n-1} \in C(L)$, $b_{n-1} \leqslant \ldots \leqslant b_1$ (where $C(L)$ is defined as for LM algebras, being in fact the Boolean center of the de Morgan algebra L).[†]

Let $\mathcal{P}ost_n$, denote the category of n–valued Post algebras with the usual algebra homomorphisms as arrows.

[†]Of course, the coefficients b_i depend on x.

Proposition 3. (1) Any centered LM_n-algebra becomes an n–valued Post algebra with the same underlying set if we define, for each $i \in \{1, \ldots, n-2\}$, $c_i = d_{n-i}$. (Then, for each element x and $i \in \{1, \ldots, n-1\}$, the coefficient b_i is $\varphi_{n-i}(x)$.)

(2) Any n–valued Post algebra becomes a centered LM_n–algebra with the same underlying set if we define, for each $i \in \{2, \ldots, n-2\}$, $d_i = c_{n-i}$.

(3) The above two mappings, taken together with the identity morphisms, yield an isomorphism of categories between $\mathcal{P}ost_n$ and $\mathcal{C}LM_n$.

Theorem 3.1. (1) $\mathcal{P}ost_n$ is a subcategory of $\mathcal{L}M_n$, and the inclusion functor $\mathcal{P}ost_n \longrightarrow \mathcal{L}M_n$ has a left adjoint;

(2) $\mathcal{P}ost_n$ and $\mathcal{B}$ are *equivalent* categories.

Remark 3. The process of modeling a logical system algebraically occurs at two distinct levels:

(i) a *local* level, based on the assumption that one can use the *internal structure* of the corresponding logic algebras that are employed to derive logical inferences;

(ii) a *global* level, which is developed by taking into account the *higher dimensional algebra* represented by the category of all the initially defined logic algebras.

Theorem 3.1 shows that the behavior of a logical system is not completely determined by its properties at the *global* level, even though $\mathcal{B}$ and $\mathcal{P}ost_n$ are *equivalent in the categorical sense.* The actual behavior of n–valued Post logic is clearly distinct to, and richer than, that of Boolean logic. Therefore, one concludes that a complete study of logical systems must include *both* a global and a local analysis of their properties.

MV–algebras. MV–algebras were introduced by Chang in 1958 as algebraic models for the Łukasiewicz–Tarski infinitely–valued logic.

Definition 4. An *MV–algebra* is a structure $(A, \oplus, ^-, 0)$, where $\oplus$ is a binary operation, $^-$ is unitary, and 0, 1 are constants, satisfying the following axioms:

(M1) $(A, \oplus, 0)$ is a *commutative monoid* ;

(M2) $\overline{\overline{x}} = x$;

(M3) $x \oplus \overline{0} = \overline{0}$;

(M4) $\overline{\overline{x} \oplus y} \oplus y = \overline{\overline{y} \oplus x} \oplus x$.

We let 1 denote $\overline{0}$ and $x \odot y$ denote $\overline{\overline{x} \oplus \overline{y}}$. Then the axioms (M3) and (M4) can be rewritten as $x \oplus 1 = 1$ and $(x \odot \overline{y}) \oplus y = (y \odot \overline{x}) \oplus x$. Moreover, if we further denote $x \vee y = (x \odot \overline{y}) \oplus y$ and $x \wedge y = (x \oplus \overline{y}) \odot y$, we obtain the following:

Proposition 4. The structure $(A, \vee, \wedge, 0, 1)$ is a bounded distributive lattice.

Example 3.1. The standard MV–algebra is $L_\infty = ([0,1], \oplus, {}^-, 0)$, where $\oplus$ and ${}^-$ are defined by: $x \oplus y = \min(x+y, 1)$ and $\overline{x} = 1 - x$ for all $x, y \in [0,1]$.

Theorem 3.2. (Chang's Representation Theorem, 1958) Any MV–algebra is isomorphic to a subdirect product of MV–chains (*i.e.*, totally–ordered MV–algebras).

The above result reduces the algebraic calculus of MV–algebras to the much simpler one of MV–chains. In fact, one can prove that the class of MV–algebras is generated both as a variety and as a quasi–variety by L_∞; hence, the computation reduction is even more substantial in the algebraic calculus of L_∞.

Theorem 3.3. (Mundici, 1986) The category of MV–algebras is equivalent to the category of lattice–ordered Abelian groups with a strong unit.

Definition 5. (Grigolia, 1978). An MV$_n$–algebra is an MV–algebra A subject to the following additional requirements:

(M5) $(n-1)x = nx$;

(M6) $(jx^{j-1})^n = nx^j$ for each $j \in \{2, \ldots, (n-2)\}$ such that j does not divide $(n-1)$.

It was shown that MV$_n$–algebras are *equivalent* – in the categorical sense – to a subclass of LM$_n$–algebras which was introduced by R. Cignoli in 1977, the subclass of *proper* LM$_n$–algebras.

One also has the following important result:

Theorem 3.4. For $n \in \{3, 4\}$, the category $\mathcal{L}M_n$ is isomorphic to the category of MV_n–algebras.

This categorical result shows the fact that, for the "multi–value degrees" 3 and 4, the Moisil logic actually coincides with the corresponding Łukasiewicz logic. A very interesting, and rather unexpected, result is that the class of n–valued Post algebras can be seen as an *infinitely axiomatizable subvariety* of the class of MV–algebras.

Theorem 3.5. The n–valued Post algebras are *polynomially equivalent* to the MV_n–algebras.

Heyting algebras. We already noted that the Łukasiewicz implication cannot be defined in an LM_n–algebra when $n \geqslant 5$. On the other hand, one can always define on a LM_n–algebra L the following implication operator, called the *Heyting implication*: $x \Rightarrow y = y \vee \bigwedge_{i=1}^{(n-1)} N\varphi_i(x) \vee \varphi_i(y)$.

Theorem 3.6. The structure $(L, \vee, \wedge, \Rightarrow, 0, 1)$ is a Heyting algebra (for Brouwerian intutionistic logic).

In the light of the above result, and from a purely logical standpoint, the Moisil logics could also be developed within the framework of Brower's intuitionistic logic.

Proposition 5. The "forgetful" functor between the categories of LM_n–algebras and of Heyting algebras, given by: $(L, \vee, \wedge, N, (\varphi_i)_{i\in\{1,\ldots,n-1\}}, 0, 1) \mapsto (L, \vee, \wedge, \Rightarrow, 0, 1)$, is full and faithful, and it also has a left adjoint.

4. Defining probabilities over LM_n–algebras

The classical theory of probabilities began by developing two fundamental concepts:

(i) the *field of events* associated to random experiments;

(ii) the definition of *probability* based on Kolmogorov axiomatics.

The set of events is classically assumed to have the structure of a Boolean algebra when the random experiments follow the laws of classical logic, as this is the case for classical statistical mechanics or relativistic theories of classical stochastic processes. On the other hand, we can consider situations, as encountered in quantum theories, when the random experiments seem to be governed by many–valued logics. In such situations, the set of events could be structured by an associated Łukasiewicz–Moisil algebra. The natural next step in many–valued probability theory is then to provide an axiomatic basis, by defining the notion of *probability on a Łukasiewicz–Moisil algebra*.

Definition 6. A probability over a LM_n–algebra L (the latter representing the "set of events") is a function $s : L \longrightarrow [0,1]$ such that the following properties are valid:
(P1) $s(x \vee y) + s(x \wedge y) = s(x) + s(y)$ for all $x, y \in L$;
(P2) $s(x) = 1/(n-1)[s(\varphi_1 x) + \ldots + s(\varphi_{n-1} x)]$ for all $x \in L$;
(P3) $s(0) = 0$ and $s(1) = 1$.

Remark 4. $s \mid C(L)$– the restriction of s to the Boolean center of L– is a classical probability over $C(L)$.

Theorem 4.1. Any classical probability $m : C(L) \longrightarrow [0,1]$ over the Boolean center of an LM_n–algebra L has a unique extension to a probability $s : L \longrightarrow [0,1]$ defined over L.

The above theorem shows that any probability defined over an LM_n–algebra is determined by the restriction to its Boolean center.

Let s be a probability over an LM_n–algebra L and $a \in L$ such that $s(a) > 0$. Then we define the functions $s(\cdot \,|\, a) : L \longrightarrow [0,1]$ by setting:

$$s(x \,|\, a) = \frac{1}{(n-1)s(a)} \sum_{i=1}^{(n-1)} s(x \wedge \varphi_i a) = 1/[(n-1)s(a)] \sum_{i=1}^{n-1} s(\varphi_i x \wedge a) \ .$$

The function $s(\cdot \,|\, a)$ thus defined is called a *conditional probability*.

Remark 5. If $a \in C(L)$, then $s(x \,|\, a) = s(x \wedge a)/s(a)$ for all $x \in C(L)$, thus one recovers the conditional probability defined classically over a Boolean algebra.

Proposition 6. If s is a probability over L and $a \in L$ such that $s(a) > 0$, then $s(\cdot \,|\, a)$ is also a probability over L.

Definition 7. An LM_n–algebra L is said to be σ–*complete* if it is so as a lattice, i.e., all countable sequences in L have suprema and infima in L. For each subset L' of L, we let $S(L')$ denote the σ–complete sub–LM_n-algebra of L generated by L'.

Definition 8. A probability s over a σ–complete LM_n–algebra L is called:

- a σ-probability, if $s(\bigvee_{k=1}^{\infty} x_k) = \sum_{k=1}^{\infty} s(x_k)$ for each sequence $(x_k)_k \subseteq L$ such that $x_k \wedge x_l = 0$ whenever $k \neq l$;

- continuous, if $\bigvee_{k=1}^{\infty} x_k = x$ implies $\lim_{k\to\infty} s(x_k) = s(x)$ for each increasing sequence $(x_k)_k \subseteq L$ and each $x \in L$.

Theorem 4.2. Let L be a σ–complete LM_n–algebra and L' a sub-LM_n-algebra of L. Then any continuous state on L' can be uniquely extended to a σ-state on $S(L')$.

Theorem 4.3. Let s be a probability (σ–probability) over an LM_n–algebra L. Then there is a unique probability (σ–probability) $s^{\sim}$ over $T(C(L))$ such that $s(x) = s^{\sim}(\varphi_1 x, \ldots, \varphi_{n-1} x)$ for all $x \in L$.

$\cong\equiv \tilde{w}$

References

Baianu, I. C., R. Brown , G. Georgescu and J. F. Glazebrook: 2005, ‘Complex Nonlinear Biodynamics in Categories, Higher Dimensional Algebra and Łukasiewicz–Moisil Topos: Transformations of Neuronal, Genetic and Neoplastic Networks.’, *Axiomathes, (current volume).*

Baianu, I. C.: 1977, ‘A Logical Model of Genetic Activities in Łukasiewicz Algebras: The Non–linear Theory’, *Bull. of Math. Biol.* **39**, 249–258.

Birkhoff, G.: 1948, *Lattice Theory*, Amer. Math. Soc., New York.

Boicescu, V., A. Filipoiu, G. Georgescu, and S. Rudeanu.: 1991, *Łukasiewicz–Moisil Algebras*, North-Holland, Amsterdam.

Chang, C. C.: 1958, ‘Algebraic analysis of many valued logics’. *Trans. Amer. Math. Soc.*, **88**, 467–490.

Chang, C. C.: 1959, ‘A new proof of the completeness of the Łukasiewicz axioms’, *Transactions American Mathematical Society* **93**, 74-80.

Cignoli, R., Esteva, F., Godo, L. and Torrens, A. : 2000, ‘Basic Fuzzy Logic is the logic of continuous t-norms and their residua’, *Soft Computing* **4**, 106-112.

Cignoli, R.: 'Moisil algebras', *Notas de Logica Matematica*, Inst. Mat., Univ. Nacional del Sur, Bahia-Blanca, No. 27.

Bourbaki, N. : 1964. *Eléments de Mathématique, Livre II, Algébre*, **4**, Hermann, Editor, Paris.

Carnap, R.: 1938, *The Logical Syntax of Language*, Harcourt, Brace and Co., New York.

Ehresmann, C.: 1965, *Catégories et Structures*, Dunod, Paris.

Eilenberg, S. and S. MacLane: 1945, 'The General Theory of Natural Equivalences', *Trans. Amer. Math. Soc.* **58**, 231–294.

Georgescu, G. and D. Popescu: 1968, 'On Algebraic Categories', *Rev. Roum. Math. Pures et Appl.* **13**, 337–342.

Georgescu, G., and C. Vraciu.: 1970. 'On the characterization of centered Łukasiewicz algebras'. *J. Algebra* **16**, 486–495.

Georgescu, G., and I. Leuştean.: 2000. 'Towards a probability theory based on Moisil logic', *Soft Computing* **4**, 19–26.

Grigolia, R.S.: 1977. 'Algebraic analysis of Łukasiewicz-Tarski's logical systems', in Wójcicki, R., Malinowski, G. (Eds), *Selected Papers on Łukasiewicz Sentential Calculi*, Osolineum, Wroclaw, pp. 81-92.

Hilbert, D. and W. Ackerman: 1927, *Grunduge der Theoretischen Logik*, Springer, Berlin.

Kan, D.M.: 1958, 'Adjoint Functors', *Trans Amer. Math. Soc.* **87**, 294–329.

Lambek J. and P. J. Scott: 1986, *Introduction to Higher Order Categorical Logic*, Cambridge University Press, Cambridge, UK, 1986.

Lawvere, F.W.: 1963, 'Functorial Semantics of Algebraic Theories', *Proc. Natl. Acad. Sci. USA.* **50**, 869–872.

Löfgren, L.: 1968, 'An Axiomatic Explanation of Complete Self-Reproduction', *Bull. Math. Biophys.* **30**, 317–348.

Łukasiewicz, J.: 1970, *Selected Works*, (ed.: L. Borkowski), North-Holland Publ. Co., Amsterdam and PWN, Warsaw.

MacLane, S. and I. Moerdijk: 1992, *Sheaves in Geometry and Logic - A first Introduction to Topos Theory*, Springer Verlag, New York.

McCulloch, W. and W. Pitts: 1943, 'A Logical Calculus of Ideas Immanent in Nervous Activity', *Bull. Math. Biophys.* **5**, 115–133.

McNaughton, R.: 1951, 'A theorem about infinite-valued sentential logic', *Journal Symbolic Logic* **16**, 1-13.

Moisil, Gr. C.: 1972, *Essai sur les logiques non-chrysippiennes.* Ed. Academiei, Bucharest.

Mundici, D.: 1986, 'Interpretation of AF C*-algebras in Łukasiewicz sentential calculus', *J. Functional Analysis* **65**, 15-63.

Rose, A.: 1956, 'Formalisation du calcul propositionnel implicatif à $\aleph_0$ valeurs de Łukasiewicz., *C. R. Acad. Sci. Paris* **243**, 11831185.

Rose, A. and Rosser, J.B.: 1958, 'Fragments of many-valued statement calculi', *Transactions American Mathematical Society* **87**, 1-53.

Rose, A.: 1962, 'Extensions of Some Theorems of Anderson and Belnap', *J. Symbolic Logic*, **27**, (4), 423–425.

Rose, A.: 1978, 'Formalisations of Further N_0–Valued Łukasiewicz Propositional Calculi'. *J. Symbolic Logic*, **43**(2): 207-210

Rosen, R.: 1958a, 'A Relational Theory of Biological Systems', *Bull. Math. Biophys.* **20**, 245–260.

Rosen, R.: 1958b, 'The Representation of Biological Systems from the Standpoint of the Theory of Categories', *Bull. Math. Biophys.* **20**, 317–341.

Rosen, R.: 1991, *Life Itself*, Columbia University Press, New York.

Rosen, R.: 1999, *Essays on Life Itself*, Columbia University Press, New York.

Rosenbloom, Paul.: 1950, *The Elements of Mathematical Logic*, Dover, New York.

Rosenbloom, Paul.: 1962, *ibid.*, Prentice–Hall, Englewood Cliffs, N.J.

Rosser, J.B. and Turquette, A.R.: 1952, *Many-Valued Logics*. North-Holland Publ. Co., Amsterdam.

Department of Mathematics, University of Bucharest, Bucharest, Romania