Groups and Symmetry

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CHAPTER 1

Isometries in 2 dimensions

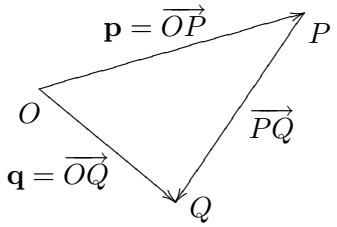
1. Some 2-dimensional vector geometry

We will denote a point in the plane by a letter such as P. The distance between points P and Q will be denoted $|PQ| = |QP|$. The (*undirected*) *line segment* joining P and Q will be denoted PQ, while the *directed line segment* joining P and Q will be denoted $\overrightarrow{PQ}$ (this is a *vector*). Of course,

$$\overrightarrow{QP} = -\overrightarrow{PQ}.$$

The origin O will be taken as the centre of a coordinate system based on the x and y-axes in the usual way. Given a point P, the *position vector* of P is the vector $\mathbf{p} = \overrightarrow{OP}$ which we think of as joining O to P. If the position vector of Q is $\mathbf{q} = \overrightarrow{OQ}$, then we have the diagram

(1.1)



from which we see that $\mathbf{p} + \overrightarrow{PQ} = \mathbf{q}$. Hence we have the formula

$$\overrightarrow{PQ} = \mathbf{q} - \mathbf{p}. \tag{1.2}$$

Each position vector $\mathbf{p}$ can be expressed in terms of its x and y coordinates p_1, p_2 and we will often write $\mathbf{p} = (p_1, p_2)$ or $\mathbf{p} = (x_P, y_P)$. Using this notation, Equation (1.2) expands to

$$\overrightarrow{PQ} = (q_1 - p_1, q_2 - p_2) = (x_Q - x_P, y_Q - y_P).$$

We will denote the set of all vectors (x, y) by $\mathbb{R}^2$, so

$$\mathbb{R}^2 = \{(x, y) : x, y \in \mathbb{R}\}.$$

This set will be identified with the plane by the correspondence

$$(x, y) \longleftrightarrow \text{the point with position vector } (x, y).$$

The distance between two points P and Q can be found using the formula

$$\begin{aligned} |PQ| &= \text{length of } \overrightarrow{PQ} \\ &= \text{length of } (\mathbf{q} - \mathbf{p}) \\ &= \sqrt{(q_1 - p_1)^2 + (q_2 - p_2)^2}. \end{aligned}$$

In particular, the length of the vector $\mathbf{p} = \overrightarrow{OP}$ is

$$|\mathbf{p}| = |OP| = \sqrt{p_1^2 + p_2^2}. \tag{1.3}$$

To find the angle θ between two non-zero vectors $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$ we can make use of the *dot* or *scalar product* which is defined to be

$$\mathbf{u} \cdot \mathbf{v} = (u_1, u_2) \cdot (v_1, v_2) = u_1 v_1 + u_2 v_2. \tag{1.4}$$

Notice that

$$|\mathbf{u}|^2 = \mathbf{u}\cdot\mathbf{u}, \quad |\mathbf{v}|^2 = \mathbf{v}\cdot\mathbf{v}.$$

Then

$$\theta = \cos^{-1}\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}|\,|\mathbf{v}|}.$$

The vectors $\mathbf{u}$ and $\mathbf{v}$ are *perpendicular*, *normal* or *orthogonal* if $\mathbf{u}\cdot\mathbf{v} = 0$ or equivalently if the angle between them is $\pi/2$.

If A, B, C are three distinct points then the angle between the lines AB and AC is given by

$$\angle BAC = \cos^{-1}\frac{\overrightarrow{AB}\cdot\overrightarrow{AC}}{|AB|\,|AC|}.$$

The lines AB and AC are *perpendicular*, *normal* or *orthogonal* if $\angle BAC = \pi/2$.

A line $\mathcal{L}$ can be specified in several different ways. First by using an *implicit equation* $ax + by = c$ with $(a, b) \neq (0, 0)$; this gives

$$\mathcal{L} = \{(x, y) \in \mathbb{R}^2 : ax + by = c\}. \tag{1.5a}$$

It is worth remarking that the vector (a, b) is perpendicular to $\mathcal{L}$. An alternative way to write the implicit equation is as $(a, b)\cdot(x, y) = c$, so we also have

$$\mathcal{L} = \{\mathbf{x} \in \mathbb{R}^2 : (a, b)\cdot\mathbf{x} = c\}. \tag{1.5b}$$

To determine c it suffices to know any point $\mathbf{x}_0$ on $\mathcal{L}$, then $c = (a, b)\cdot\mathbf{x}_0$.

Second, if we have a vector $\mathbf{u}$ parallel to $\mathcal{L}$ (and so perpendicular to (a, b)) then we can use the parametric equation $\mathbf{x} = t\mathbf{u} + \mathbf{x}_0$, where $t \in \mathbb{R}$ and $\mathbf{x}_0$ is some point on $\mathcal{L}$. It is usual to take $\mathbf{u}$ to be a unit vector, *i.e.*, $|\mathbf{u}| = 1$. Then

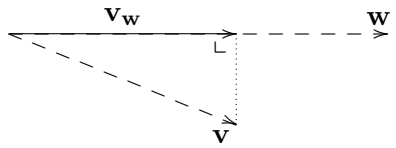
$$\mathcal{L} = \{t\mathbf{u} + \mathbf{x}_0 \in \mathbb{R}^2 : t \in \mathbb{R}\}. \tag{1.5c}$$

It is also useful to recall the idea of projecting a non-zero vector $\mathbf{v}$ onto another $\mathbf{w}$. To do this, we make use of the unit vector

$$\hat{\mathbf{w}} = \frac{1}{|\mathbf{w}|}\mathbf{w} = \frac{\mathbf{w}}{|\mathbf{w}|}.$$

Then the *component of* $\mathbf{v}$ *in the* $\mathbf{w}$*-direction*, or the *projection of* $\mathbf{v}$ *onto* $\mathbf{w}$ is the vector

$$\mathbf{v}_\mathbf{w} = (\mathbf{v}\cdot\hat{\mathbf{w}})\,\hat{\mathbf{w}} = \left(\frac{\mathbf{v}\cdot\mathbf{w}}{|\mathbf{w}|^2}\right)\mathbf{w}.$$



Then $\mathbf{v}_\mathbf{w}$ is parallel to $\mathbf{w}$ and

$$(\mathbf{v} - \mathbf{v}_\mathbf{w})\cdot\mathbf{w} = 0,$$

so the vector $(\mathbf{v} - \mathbf{v}_\mathbf{w})$ is perpendicular to $\mathbf{w}$.

We can also project a point P with position vector $\mathbf{p}$ onto a line $\mathcal{L}$ which does not contain P. To do this, we consider the line $\mathcal{L}'$ passing through P and perpendicular to $\mathcal{L}$,

$$\mathcal{L}' = \{s\mathbf{u}' + \mathbf{p} : s \in \mathbb{R}\},$$

where $\mathbf{u}'$ is any non-zero vector perpendicular to $\mathcal{L}$ (for example (a, b) or the unit vector in the same direction). Then the *projection of* P *onto* $\mathcal{L}$ is the point of intersection of $\mathcal{L}$ and $\mathcal{L}'$, whose position vector $\mathbf{p}' = s'\mathbf{u}' + \mathbf{p}$ can be determined by solving the following equation for s':

$$(a, b)\cdot(s'\mathbf{u}' + \mathbf{p}) = c.$$

2. Isometries of the plane

1.1. Definition. An *isometry* of the plane is a distance preserving function $F\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2$.

Here, *distance preserving* means that for points P and Q with position vectors $\mathbf{p}$ and $\mathbf{q}$,

$$|F(P)F(Q)| = |PQ|, \qquad \textit{i.e.}, \quad |F(\mathbf{p}) - F(\mathbf{q})| = |\mathbf{p} - \mathbf{q}|.$$

Before considering examples, we note the following important fact.

1.2. Proposition. *Let $F\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be an isometry which fixes the origin. Then F preserves scalar products and angles between vectors.*

Proof. Let $\mathbf{u}, \mathbf{v}$ be vectors and let U, V be the points with these as position vectors. Let $F(U)$ and $F(V)$ have position vectors $\mathbf{u}' = \overrightarrow{OF(U)}$ and $\mathbf{v}' = \overrightarrow{OF(V)}$. For every pair of points P, Q we have $|F(P)F(Q)| = |PQ|$, so

$$|\mathbf{u}' - \mathbf{v}'|^2 = |F(U)F(V)|^2 = |UV|^2 = |\mathbf{u} - \mathbf{v}|^2,$$

hence

$$|\mathbf{u}'|^2 + |\mathbf{v}'|^2 - 2\mathbf{u}' \cdot \mathbf{v}' = |\mathbf{u}|^2 + |\mathbf{v}|^2 - 2\mathbf{u} \cdot \mathbf{v}.$$

Since

$$|\mathbf{u}'| = |OF(U)| = |F(O)F(U)| = |OU| = |\mathbf{u}|, \qquad |\mathbf{v}'| = |OF(V)| = |F(O)F(V)| = |OV| = |\mathbf{v}|,$$

we obtain

$$\mathbf{u}' \cdot \mathbf{v}' = \mathbf{u} \cdot \mathbf{v},$$

which shows that the scalar product of two position vectors is unchanged by an isometry which fixes the origin. Similarly, angles are preserved since the angle between the vectors $\mathbf{u}', \mathbf{v}'$ is

$$\cos^{-1} \frac{\mathbf{u}' \cdot \mathbf{v}'}{|\mathbf{u}'|\,|\mathbf{v}'|} = \cos^{-1} \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}|\,|\mathbf{v}|}.$$

□

1.3. Corollary. *An isometry $F\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ preserves angles between lines.*

Proof. Consider the isometry $F_0\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ for which $F_0(P)$ has the position vector

$$\overrightarrow{OF_0(P)} = \overrightarrow{OF(P)} - \overrightarrow{OF(O)}.$$

Then $F_0(O) = O$. For any two points A, B we have

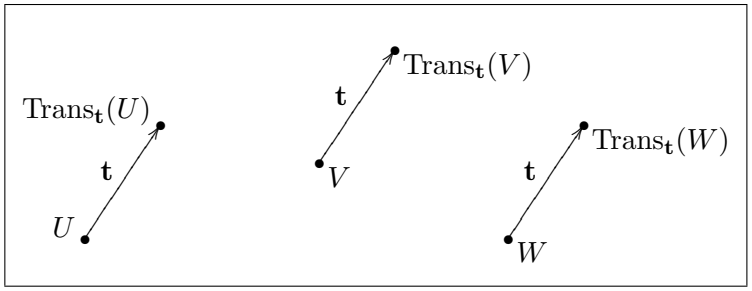
$$\overrightarrow{F(A)F(B)} = \overrightarrow{F_0(A)F_0(B)}$$

and the result follows from Proposition 1.2. □

Types of isometries. There are three basic types of isometries of the plane, *translations*, *reflections*, *rotations*. A fourth type, *glide reflections*, are built up as compositions of reflections and translations.

Translations. Let $\mathfrak{t} \in \mathbb{R}^2$. Then *translation by* $\mathfrak{t}$ is the function

$$\mathrm{Trans}_{\mathfrak{t}}\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2; \quad \mathrm{Trans}_{\mathfrak{t}}(\mathbf{x}) = \mathbf{x} + \mathbf{t}.$$



Notice that

$$|\,\mathrm{Trans}_{\mathbf{t}}(\mathbf{x}) - \mathrm{Trans}_{\mathbf{t}}(\mathbf{y})| = |(\mathbf{x} + \mathbf{t}) - (\mathbf{y} + \mathbf{t})| = |\mathbf{x} - \mathbf{y}|,$$

so $\mathrm{Trans}_{\mathbf{t}}$ is an isometry. If $\mathrm{Trans}_{\mathbf{s}}$ is a second such translation function, we have

$$\mathrm{Trans}_{\mathbf{t}} \circ \mathrm{Trans}_{\mathbf{s}}(\mathbf{x}) = \mathrm{Trans}_{\mathbf{t}}(\mathbf{x}+\mathbf{s}) = \mathbf{x}+\mathbf{s}+\mathbf{t} = \mathrm{Trans}_{\mathbf{s}+\mathbf{t}}(\mathbf{x}),$$

so

$$\mathrm{Trans}_{\mathbf{t}} \circ \mathrm{Trans}_{\mathbf{s}} = \mathrm{Trans}_{\mathbf{s}+\mathbf{t}}\,. \tag{1.6}$$

Since $\mathbf{s}+\mathbf{t} = \mathbf{t}+\mathbf{s}$, we also have

$$\mathrm{Trans}_{\mathbf{s}} \circ \mathrm{Trans}_{\mathbf{t}} = \mathrm{Trans}_{\mathbf{t}} \circ \mathrm{Trans}_{\mathbf{s}}\,. \tag{1.7}$$

So translations behave well with respect to composition. We also have

$$\mathrm{Trans}_{\mathbf{0}} = \mathrm{Id}_{\mathbb{R}^2}, \quad \mathrm{Trans}_{\mathbf{t}}^{-1} = \mathrm{Trans}_{-\mathbf{t}}\,.$$

Notice that when $\mathbf{t} \neq \mathbf{0}$, every point in the plane is moved by $\mathrm{Trans}_{\mathbf{t}}$, so such a transformation has no *fixed points*.

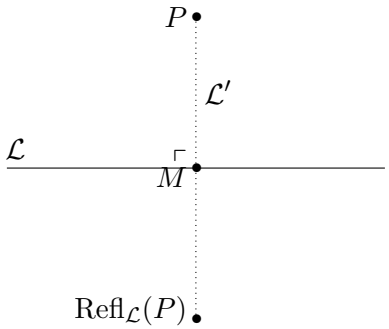
Reflections. The next type of isometry is a *reflection* in a line $\mathcal{L}$. Recall that a line in the plane has the form

$$\mathcal{L} = \{(x,y) \in \mathbb{R}^2 : ax+by = c\},$$

where $a, b, c \in \mathbb{R}$ with at least one of a and b non-zero. The *reflection in* $\mathcal{L}$ is the function

$$\mathrm{Refl}_{\mathcal{L}} \colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

which sends every point on $\mathcal{L}$ to itself and if P lies on a line $\mathcal{L}'$ perpendicular to $\mathcal{L}$ and intersecting it at M say, then $\mathrm{Refl}_{\mathcal{L}}(P)$ also lies on $\mathcal{L}'$ and satisfies $|M\,\mathrm{Refl}_{\mathcal{L}}(P)| = |MP|$.



This is equivalent to saying that if P and M have position vectors $\mathbf{p}$ and $\mathbf{m}$, then

$$\mathrm{Refl}_{\mathcal{L}}(\mathbf{p}) - \mathbf{p} = 2(\mathbf{m}-\mathbf{p})$$

or

$$\mathrm{Refl}_{\mathcal{L}}(\mathbf{p}) = 2\mathbf{m} - \mathbf{p}, \tag{1.8}$$

where $\mathrm{Refl}_{\mathcal{L}}(\mathbf{p}) - \mathbf{p}$ is perpendicular to $\mathcal{L}$.

In order to determine the effect of a reflection, recall that the vector (a,b) is perpendicular to $\mathcal{L}$. Consider the unit vector

$$\mathbf{u} = \frac{1}{\sqrt{a^2+b^2}}(a,b).$$

Then we can find the point M as follows. $\mathcal{L}'$ is the line given in parametric form by

$$\mathbf{x} = t\mathbf{u} + \mathbf{p} \quad (t \in \mathbb{R}),$$

and M is point on both $\mathcal{L}$ and $\mathcal{L}'$. So $\mathbf{m} = s\mathbf{u} + \mathbf{p}$, say, satisfies the linear equation in the unknown s,

$$\mathbf{u} \cdot \mathbf{m} = \frac{c}{\sqrt{a^2+b^2}}.$$

This expands to give

$$s + \mathbf{u} \cdot \mathbf{p} = \frac{c}{\sqrt{a^2+b^2}}.$$

Thus we have

$$\mathbf{m} = \left(\frac{c}{\sqrt{a^2+b^2}} - \mathbf{u}\cdot\mathbf{p} \right) \mathbf{u} + \mathbf{p}. \tag{1.9}$$

Substituting into Equation (1.8) we obtain

$$\mathrm{Refl}_{\mathcal{L}}(\mathbf{p}) = 2\left(\frac{c}{\sqrt{a^2+b^2}} - \mathbf{u}\cdot\mathbf{p} \right) \mathbf{u} + \mathbf{p}. \tag{1.10}$$

Performing a reflection twice gives the identity transformation,

$$(\mathrm{Refl}_{\mathcal{L}})^2 = \mathrm{Refl}_{\mathcal{L}} \circ \mathrm{Refl}_{\mathcal{L}} = \mathrm{Id}_{\mathbb{R}^2}\,. \tag{1.11}$$

Notice that points on the line $\mathcal{L}$ are fixed by $\mathrm{Refl}_{\mathcal{L}}$, while all other points are moved.

1.4. Example. Determine the effect of the reflection $\mathrm{Refl}_{\mathcal{L}}$, on the points $P(1,0)$, where

$$\mathcal{L} = \{(x,y) : x - y = 0\}.$$

Solution. First notice that the unit vector $\mathbf{u} = \dfrac{1}{\sqrt{2}}(1,-1)$ is perpendicular to $\mathcal{L}$. Using this, we resolve $\mathbf{p} = (1,0)$ into its components perpendicular and parallel to $\mathcal{L}$. These are the vectors

$$\mathbf{p}' = ((1,0)\cdot\mathbf{u})\mathbf{u} = \frac{1}{\sqrt{2}}\mathbf{u},$$
$$\mathbf{p}'' = (1,0) - \frac{1}{\sqrt{2}}\mathbf{u} = (1,0) - \frac{1}{2}(1,-1) = \frac{1}{2}(1,1).$$

Then we have

$$\mathrm{Refl}_{\mathcal{L}}(\mathbf{p}) = -\mathbf{p}' + \mathbf{p}'' = \frac{1}{2}(-1,1) + \frac{1}{2}(1,1) = (0,1). \qquad \square$$

1.5. Example. If $\theta \in [0,\pi)$ and

$$\mathcal{L}_\theta = \{(t\cos\theta, t\sin\theta) : t \in \mathbb{R}\},$$

find a formula for the effect of $\mathrm{Refl}_{\mathcal{L}_\theta}$ on $P(x,y) \neq (0,0)$.

Solution. The line $\mathcal{L}_\theta$ contains the origin O and the point $U(\cos\theta, \sin\theta)$. Also if $X(1,0)$ is the point on the x-axis, then $\angle XOU = \theta$. If $\angle XOP = \alpha$, then on setting $r = |OP| = \sqrt{x^2+y^2}$ we have

$$x = r\cos\alpha, \quad y = r\sin\alpha.$$

If $P' = \mathrm{Refl}_{\mathcal{L}_\theta}(P)$, with position vector (x', y'), we have

$$\angle XOP' = \theta - (\alpha - \theta) = 2\theta - \alpha,$$

hence

$$x' = r\cos(2\theta - \alpha), \quad y' = r\sin(2\theta - \alpha).$$

Recall that

$$\cos(\alpha+\theta) = \cos\alpha\cos\theta - \sin\alpha\sin\theta, \quad \sin(\alpha+\theta) = \cos\alpha\sin\theta + \sin\alpha\cos\theta.$$

Using these we obtain

$$x' = r(\cos 2\theta\, \cos\alpha + \sin 2\theta\, \sin\alpha), \quad y' = r(\sin 2\theta\, \cos\alpha - \cos 2\theta\, \sin\alpha),$$

which yield

$$x' = \cos 2\theta\, x + \sin 2\theta\, y, \tag{1.12a}$$

$$y' = \sin 2\theta\, x - \cos 2\theta\, y. \tag{1.12b}$$

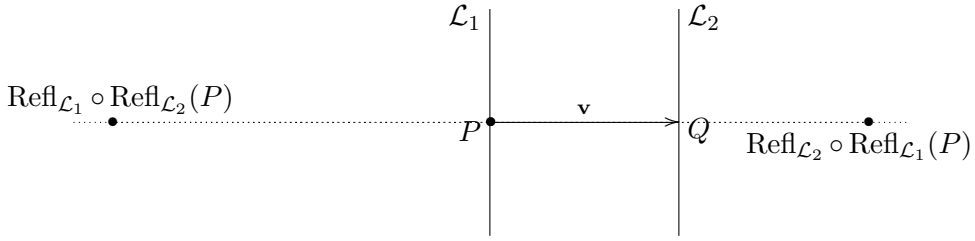
So applying $\mathrm{Refl}_{\mathcal{L}_\theta}$ to P we obtain the point

$$X'(\cos 2\theta\, x + \sin 2\theta\, y, \sin 2\theta\, x - \cos 2\theta\, y). \qquad \square$$

We can also describe the composition of two reflections in two distinct *parallel* lines.

1.6. PROPOSITION. *Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be distinct parallel lines. Then the two compositions* $\mathrm{Refl}_{\mathcal{L}_1} \circ \mathrm{Refl}_{\mathcal{L}_2}$ *and* $\mathrm{Refl}_{\mathcal{L}_2} \circ \mathrm{Refl}_{\mathcal{L}_1}$ *are translations.*

PROOF. Let $\mathbf{p}$ be the position vector of a point on $\mathcal{L}_1$ and let $\mathbf{v}$ be a vector perpendicular to $\mathcal{L}_1$ and chosen so that $\mathbf{q} = \mathbf{p} + \mathbf{v}$ is the position vector of a point Q on $\mathcal{L}_2$. Clearly $\mathbf{v}$ is independent of which point P on $\mathcal{L}_1$ we start with.



Then for any $t \in \mathbb{R}$,

$$\begin{aligned}\mathrm{Refl}_{\mathcal{L}_1} \circ \mathrm{Refl}_{\mathcal{L}_2}(\mathbf{p} + t\mathbf{v}) &= \mathrm{Refl}_{\mathcal{L}_1} \circ \mathrm{Refl}_{\mathcal{L}_2}(\mathbf{q} + (t-1)\mathbf{v}) \\ &= \mathrm{Refl}_{\mathcal{L}_1}(\mathbf{q} + (1-t)\mathbf{v}) \\ &= \mathrm{Refl}_{\mathcal{L}_1}(\mathbf{p} + (2-t)\mathbf{v}) \\ &= \mathbf{p} + (t-2)\mathbf{v} \\ &= (\mathbf{p} + t\mathbf{v}) - 2\mathbf{v}.\end{aligned}$$

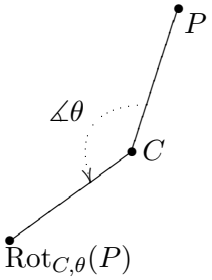
So

$$\mathrm{Refl}_{\mathcal{L}_1} \circ \mathrm{Refl}_{\mathcal{L}_2} = \mathrm{Trans}_{-2\mathbf{v}}\,.$$

Similarly we obtain

$$\mathrm{Refl}_{\mathcal{L}_2} \circ \mathrm{Refl}_{\mathcal{L}_1} = \mathrm{Trans}_{2\mathbf{v}}\,. \qquad \square$$

Rotations. Let C be a point with position vector $\mathbf{c}$. Then $\mathrm{Rot}_{C,\theta} \colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is the *rotation* of the plane around C *anti-clockwise* through the angle θ (measured in radians and taking the anti-clockwise direction to be positive).



Notice that C is fixed by $\mathrm{Rot}_{C,\theta}$ but unless $\theta = 2\pi k$ for some $k \in \mathbb{Z}$, no other point is fixed. For $k \in \mathbb{Z}$,

$$\mathrm{Rot}_{C,2\pi k} = \mathrm{Id}_{\mathbb{R}^2}, \quad \mathrm{Rot}_{C,\theta+2\pi k} = \mathrm{Rot}_{C,\theta}\,.$$

1.7. EXAMPLE. Find a formula for the effect of the $\mathrm{Rot}_{O,\theta}$ on the point $P(x, y)$.

SOLUTION. We assume that $P \neq O$ since the origin is fixed by this rotation. Recall that if $X(1, 0)$ is the point on the x-axis and $\angle XOP = \alpha$, then setting $r = |OP| = \sqrt{x^2 + y^2}$ we have

$$x = r\cos\alpha, \quad y = r\sin\alpha.$$

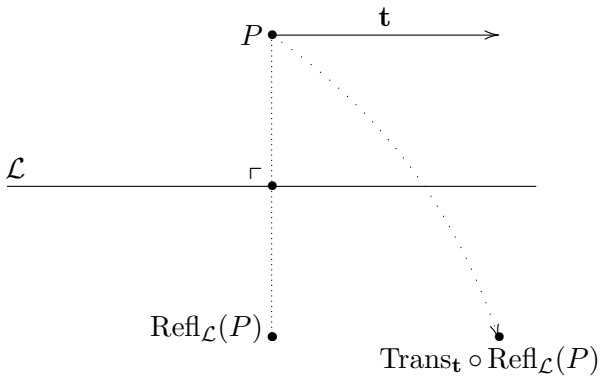
If $P' = \mathrm{Rot}_{O,\theta}(P)$, with position vector (x', y'), we have

$$x' = r\cos(\alpha + \theta), \quad y' = r\sin(\alpha + \theta).$$

Using the equations of (1.12) we obtain

$$\mathrm{Rot}_{O,\theta}(x,y) = (\cos\theta\, x - \sin\theta\, y, \sin\theta\, x + \cos\theta\, y). \tag{1.13}$$ □

Glide reflections. The composition of a reflection $\mathrm{Refl}_{\mathcal{L}}$ and a translation $\mathrm{Trans}_{\mathbf{t}}$ parallel to the line of reflection $\mathcal{L}$ (in either possible order) is called a *glide reflection*. We will study these in detail later. If the translation is not by $\mathbf{0}$ then such a glide reflection has no fixed points.



Composing isometries. We now record a useful fact about isometries that we have already seen for translations.

1.8. Proposition. *Let $F, G\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be two isometries. Then the two compositions $F \circ G, G \circ F\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ are isometries which are not necessarily equal.*

Proof. For any two points P, Q we have

$$|F \circ G(P) F \circ G(Q)| = |F(G(P))F(G(Q))| = |G(P)G(Q)| = |PQ|,$$
$$|G \circ F(P) G \circ F(Q)| = |G(F(P))G(F(Q))| = |F(P)F(Q)| = |PQ|,$$

hence $F \circ G$ and $G \circ F$ are isometries. The non-commutativity will be illustrated in examples. □

We also record a somewhat less obvious fact that will be proved in the next section.

1.9. Proposition. *Let $F\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be an isometry. Then F has an inverse which is also an isometry.*

Proof. See Corollary 1.12 below for a proof that an isometry is invertible. Assuming that F^{-1} exists, notice that for $\mathbf{x} \in \mathbb{R}^2$,

$$|F^{-1}(\mathbf{x})| = |F(F^{-1}(\mathbf{x}))| = |\mathbf{x}|,$$

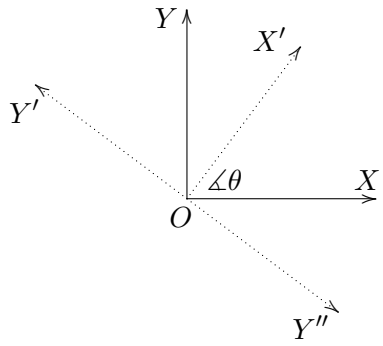
hence F^{-1} is also an isometry. □

3. Matrices and isometries

Consider an isometry $T\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ which fixes the origin O, *i.e.*, $T(O) = O$.

Suppose that $X(1,0)$ is sent to $X'(\cos\theta, \sin\theta)$ by T. Then $Y(0,1)$ must be sent to one of the two points $Y'(\cos(\theta+\pi/2), \sin(\theta+\pi/2))$ and $Y''(\cos(\theta-\pi/2), \sin(\theta-\pi/2))$ since these are

the only ones at unit distance from O making the angle $\pi/2$ with OX'.



If $P(x, y)$, then writing

$$x = r\cos\alpha, \quad y = r\sin\alpha,$$

where $r = \sqrt{x^2 + y^2} = |OP|$, we find that the point $P'(x', y')$ with $P' = T(P)$ has

$$x' = r\cos\alpha', \quad y' = r\sin\alpha',$$

for some α' since $|OP'| = |OP| = r$.

If $T(Y) = Y'$, then we must have $\alpha' = \alpha + \theta$, while if $T(Y) = Y''$, we must have $\alpha' = \theta - \alpha$. This means that

$$(x', y') = \begin{cases} r(\cos(\alpha+\theta), \sin(\alpha+\theta)) & \text{if } T(Y) = Y', \\ r(\cos(\theta-\alpha), \sin(\theta-\alpha)) & \text{if } T(Y) = Y'', \end{cases}$$

$$= \begin{cases} (\cos\theta\, x - \sin\theta\, y, \sin\theta\, x + \cos\theta\, y) & \text{if } T(Y) = Y', \\ (\cos\theta\, x + \sin\theta\, y, \sin\theta\, x - \cos\theta\, y) & \text{if } T(Y) = Y''. \end{cases}$$

The first case corresponds a rotation about the origin O through angle θ, while the second corresponds to a reflection in the line

$$\sin(\theta/2)x - \cos(\theta/2)y = 0.$$

through the origin. Notice that in either case, T is a *linear transformation* or *linear mapping* in that

$$T((x_1, y_1) + (x_2, y_2)) = T(x_1, y_1) + T(x_2, y_2), \tag{1.14a}$$

$$T(t(x, y)) = T(tx, ty) = tT(x, y). \tag{1.14b}$$

From now on, we will identify (x, y) with the column vector $\begin{bmatrix} x \\ y \end{bmatrix}$. This allows us to represent T by a matrix. Notice that

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{cases} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} & \text{if } T(Y) = Y', \\ \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} & \text{if } T(Y) = Y''. \end{cases}$$

So in each case we have $T(\mathbf{x}) = A\mathbf{x}$ for a suitable matrix A provided that we interpret a vector (x, y) as a 2×1 matrix. These matrices satisfy

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}^T \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = I_2,$$

$$\begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}^T \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix} = I_2,$$

so they are both *orthogonal matrices* in the sense of the following definition.

1.10. Definition. An $n \times n$ matrix A is *orthogonal* if $A^T A = I_n$ or equivalently if A is invertible with inverse $A^{-1} = A^T$.

It is easy to see that every $n \times n$ orthogonal matrix A has $\det A = \pm 1$. For the above matrices we have

$$\det \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = 1, \quad \det \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix} = -1.$$

It is also true that every 2×2 orthogonal matrix is of one or other of these two forms.

For a general isometry $F\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, on setting $\mathbf{t} = F(\mathbf{0})$ we can form the isometry $F_0 = \mathrm{Trans}_{-\mathbf{t}} \circ F\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ which fixes the origin and satisfies

$$F = \mathrm{Trans}_{\mathbf{t}} \circ F_0.$$

Combining all of these ingredients we obtain

1.11. Theorem. *Every isometry $F\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ can be expressed as a composition*

$$F = \mathrm{Trans}_{\mathbf{t}} \circ F_0,$$

where $F_0\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is an isometry that fixes O, hence there is an orthogonal matrix $[F_0]$ for which

$$F(\mathbf{x}) = [F_0]\mathbf{x} + \mathbf{t} \quad (\mathbf{x} \in \mathbb{R}^2).$$

1.12. Corollary. *Every isometry $F\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is invertible.*

Proof. Express F in matrix form,

$$F(\mathbf{x}) = [F_0]\mathbf{x} + \mathbf{t},$$

where $[F_0]$ is orthogonal and so has an inverse given by $[F_0]^{-1} = [F_0]^T$. Then the function $G\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ given by

$$G(\mathbf{x}) = [F_0]^{-1}(\mathbf{x} - \mathbf{t}) = [F_0]^{-1}\mathbf{x} - [F_0]^{-1}\mathbf{t}$$

satisfies

$$G \circ F = \mathrm{Id}_{\mathbb{R}^2} = F \circ G,$$

and so is the inverse of F. Therefore it is also an isometry (see the 'proof' of Proposition 1.9). □

Given an isometry $F\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, we can express it in the form

$$F(\mathbf{x}) = A\mathbf{x} + \mathbf{t},$$

for some orthogonal matrix A, and then use the *Seitz symbol* $(A \mid \mathbf{t})$ to describe it. We will use this notation freely from now on and often write

$$(A \mid \mathbf{t})\mathbf{x} = A\mathbf{x} + \mathbf{t} = F(\mathbf{x}).$$

For the composition we will write

$$(A_1 \mid \mathbf{t}_1)(A_2 \mid \mathbf{t}_2) = (A_1 \mid \mathbf{t}_1) \circ (A_2 \mid \mathbf{t}_2).$$

1.13. Proposition. *Suppose $(A \mid \mathbf{s})$ and $(B \mid \mathbf{t})$ represent the same isometry $\mathbb{R}^2 \longrightarrow \mathbb{R}^2$. Then $B = A$ and $\mathbf{t} = \mathbf{s}$.*

Proof. Since the functions $(A \mid \mathbf{s})$ and $(B \mid \mathbf{t})$ agree on every point, evaluating at any $\mathbf{x} \in \mathbb{R}^2$ gives

$$A\mathbf{x} + \mathbf{s} = B\mathbf{x} + \mathbf{t}.$$

In particular, taking $\mathbf{x} = \mathbf{0}$ we obtain $\mathbf{s} = \mathbf{t}$. In general this gives

$$A\mathbf{x} = B\mathbf{x}.$$

Now choosing $\mathbf{x} = \mathbf{e}_1, \mathbf{e}_2$, the standard basis vectors, we obtain $A = B$ since $A\mathbf{e}_i, B\mathbf{e}_i$ are the i-th columns of A, B. □

What happens when we compose two Seitz symbols or find the symbol of inverse function?

1.14. PROPOSITION. *We have the following algebraic rules for Seitz symbols of isometries.*

$$(A_1 \mid \mathbf{t}_1)(A_2 \mid \mathbf{t}_2) = (A_1A_2 \mid \mathbf{t}_1 + A_1\mathbf{t}_2),$$
$$(A \mid \mathbf{t})^{-1} = (A^{-1} \mid -A^{-1}\mathbf{t}) = (A^T \mid -A^T\mathbf{t}).$$

PROOF. The formula for the inverse was demonstrated earlier. For any $\mathbf{x} \in \mathbb{R}^2$,

$$\begin{aligned}(A_1 \mid \mathbf{t}_1)(A_2 \mid \mathbf{t}_2)\mathbf{x} &= (A_1 \mid \mathbf{t}_1)(A_2\mathbf{x} + \mathbf{t}_2)\\ &= A_1(A_2\mathbf{x} + \mathbf{t}_2) + \mathbf{t}_1\\ &= A_1A_2\mathbf{x} + A_1\mathbf{t}_2 + \mathbf{t}_1\\ &= (A_1A_2 \mid \mathbf{t}_1 + A_1\mathbf{t}_2)\mathbf{x}.\end{aligned}$$

□

We can now classify isometries of the plane in terms of their Seitz symbols. We will denote the 2×2 identity matrix by $I = I_2$.

Translations. These have the form $(I \mid \mathbf{t})$. To compose two of them, we have the formula

$$(I \mid \mathbf{t}_1)(I \mid \mathbf{t}_2) = (I \mid \mathbf{t}_1 + \mathbf{t}_2).$$

Rotations. Consider a Seitz symbol $(A \mid \mathbf{t})$ where A is orthogonal with $\det A = 1$, hence it has the form

$$A = \begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}.$$

The equation $A\mathbf{x} + \mathbf{t} = \mathbf{x}$ is solvable if and only if $(I - A)\mathbf{x} = \mathbf{t}$ can be solved. Now

$$\begin{aligned}\det(I - A) &= \det\begin{bmatrix}1-\cos\theta & \sin\theta\\ -\sin\theta & 1-\cos\theta\end{bmatrix}\\ &= (1-\cos\theta)^2 + \sin^2\theta\\ &= 1 - 2\cos\theta + \cos^2\theta + \sin^2\theta\\ &= 2 - 2\cos\theta = 2(1-\cos\theta),\end{aligned}$$

so provided that $\cos\theta \neq 1$, $(I - A)$ is invertible. But $\cos\theta = 1$ if and only if $A \neq I$, so $(I - A)$ is invertible if and only if $A \neq I$.

So as long as $A \neq I$, we can find a vector $\mathbf{c} = (I - A)^{-1}\mathbf{t}$ for which $(A \mid \mathbf{t})\mathbf{c} = \mathbf{c}$. Then $(A \mid \mathbf{t})$ represents rotation about $\mathbf{c}$ through the angle θ. Notice that once we know A and $\mathbf{c}$ we can recover $\mathbf{t}$ using the formula $\mathbf{t} = (I - A)\mathbf{c}$.

If $A = I$, $(I \mid \mathbf{0})$ is a rotation through angle 0, while if $\mathbf{t} \neq \mathbf{0}$, $(I \mid \mathbf{t})$ is not a rotation.

1.15. REMARK. When working with rotations it is useful to recall the following formula for finding the inverse of a 2×2 matrix which is valid provided $ad - bc \neq 0$:

$$\begin{bmatrix}a & b\\ c & d\end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix}d & -b\\ -c & a\end{bmatrix} = \begin{bmatrix}d/(ad-bc) & -b/(ad-bc)\\ -c/(ad-bc) & a/(ad-bc)\end{bmatrix}. \tag{1.15}$$

In particular, provided $\cos\theta \neq 1$,

(1.16a)

$$\begin{bmatrix}1-\cos\theta & \sin\theta\\ -\sin\theta & 1-\cos\theta\end{bmatrix}^{-1} = \frac{1}{2(1-\cos\theta)}\begin{bmatrix}1-\cos\theta & -\sin\theta\\ \sin\theta & 1-\cos\theta\end{bmatrix} = \begin{bmatrix}\frac{1}{2} & \frac{-\sin\theta}{2(1-\cos\theta)}\\ \frac{\sin\theta}{2(1-\cos\theta)} & \frac{1}{2}\end{bmatrix}.$$

Using standard trigonmetric identities we also have

(1.16b)

$$\begin{bmatrix} 1-\cos\theta & \sin\theta \\ -\sin\theta & 1-\cos\theta \end{bmatrix}^{-1} = \begin{bmatrix} \dfrac{1}{2} & \dfrac{-\cos(\theta/2)}{2\sin(\theta/2))} \\ \dfrac{\cos(\theta/2)}{2\sin(\theta/2)} & \dfrac{1}{2} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & -\cot(\theta/2) \\ \cot(\theta/2) & 1 \end{bmatrix}.$$

Glide reflections. Consider a Seitz symbol $(A \mid \mathbf{t})$ where A is orthogonal with $\det A = -1$, hence it has the form

$$A = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}.$$

Recall that this matrix represents $\mathrm{Refl}_{\mathcal{L}_{\theta/2}}$, reflection in the line through the origin

$$\mathcal{L}_{\theta/2} = \{(x,y) \in \mathbb{R}^2 : \sin(\theta/2)x - \cos(\theta/2)y = 0\}.$$

We will see that $(A \mid \mathbf{t})$ represents a *glide reflection*, *i.e.*, the composition of a reflection in a line parallel to $\mathcal{L}_{\theta/2}$ and a translation by a vector parallel to $\mathcal{L}_{\theta/2}$.

Express $\mathbf{t}$ in the form $\mathbf{t} = \mathbf{u} + 2\mathbf{v}$, where $\mathbf{v}$ is perpendicular to the line $\mathcal{L}_{\theta/2}$ and $\mathbf{u}$ is parallel to it. To do this we may take the unit vectors

$$\mathbf{w}^{\parallel} = (\cos(\theta/2), \sin(\theta/2)), \quad \mathbf{w}^{\perp} = (\sin(\theta/2), -\cos(\theta/2))$$

which are parallel and perpendicular respectively to $\mathcal{L}_{\theta/2}$ and find the projections of $\mathbf{t}$ onto these unit vectors; then we have

$$\mathbf{u} = \mathbf{t}_{\mathbf{w}^{\parallel}}, \quad \mathbf{v} = \frac{1}{2}\mathbf{t}_{\mathbf{w}^{\perp}}.$$

From the proof of Proposition 1.6 we know that if $\mathcal{L}$ is the line parallel to $\mathcal{L}_{\theta/2}$ containing $\mathbf{v}$, then

$$\mathrm{Refl}_{\mathcal{L}} = \mathrm{Trans}_{2\mathbf{v}} \circ \mathrm{Refl}_{\mathcal{L}_{\theta/2}},$$

and so

$$\begin{aligned} \mathrm{Trans}_{\mathbf{u}} \circ \mathrm{Refl}_{\mathcal{L}} &= \mathrm{Trans}_{\mathbf{u}} \circ \mathrm{Trans}_{2\mathbf{v}} \circ \mathrm{Refl}_{\mathcal{L}_{\theta/2}} \\ &= \mathrm{Trans}_{\mathbf{u}+2\mathbf{v}} \circ \mathrm{Refl}_{\mathcal{L}_{\theta/2}} \\ &= \mathrm{Trans}_{\mathbf{t}} \circ \mathrm{Refl}_{\mathcal{L}_{\theta/2}} \\ &= (A \mid \mathbf{t}). \end{aligned}$$

This shows that $(A \mid \mathbf{t})$ represents reflection in $\mathcal{L}$ followed by translation by $\mathbf{u}$ parallel to $\mathcal{L}$; if we allow $\mathbf{u} = \mathbf{0}$ here, then a reflection can be interpreted as a special kind of glide reflection.

1.16. Remark. Here is another way to find the vectors $\mathbf{u}$ and $\mathbf{v}$ in the above situation. Notice that since $\mathbf{u}$ is parallel to $\mathcal{L}_{\theta/2}$ and $\mathbf{v}$ is perpendicular to it,

$$(A \mid \mathbf{0})\mathbf{t} = A(\mathbf{u} + 2\mathbf{v}) = A\mathbf{u} + 2A\mathbf{v} = \mathbf{u} - 2\mathbf{v}.$$

Hence we have

$$\mathbf{u} = \frac{1}{2}(\mathbf{t} + A\mathbf{t}), \quad \mathbf{v} = \frac{1}{4}(\mathbf{t} - A\mathbf{t}).$$

Summary of basic facts on Seitz symbols

Translations: $\mathrm{Trans}_{\mathbf{t}} = (I \mid \mathbf{t})$.
Rotations: $\mathrm{Rot}_{C,\theta} = (A \mid \mathbf{t})$, where

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \neq I, \quad \mathbf{t} = (I - A)\mathbf{c}, \quad \mathbf{c} = (I - A)^{-1}\mathbf{t}.$$

Glide reflections: $\mathrm{Trans}_{\mathbf{u}} \circ \mathrm{Trans}_{2\mathbf{v}} \circ \mathrm{Refl}_{\mathcal{L}_{\theta/2}} = (A \mid \mathbf{t})$, where

$$A = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix},$$

$\mathbf{v}$ is perpendicular to the line

$$\mathcal{L}_{\theta/2} = \{(x, y) \in \mathbb{R}^2 : \sin(\theta/2)x - \cos(\theta/2)y = 0\},$$

and $\mathbf{u}$ is parallel to it. This represents a glide reflection in the line parallel to $\mathcal{L}_{\theta/2}$ and containing the point with position vector $\mathbf{v}$; the translation is by $\mathbf{u}$. When $\mathbf{u} = \mathbf{0}$, this is a reflection.

Some examples. Using Seitz symbols and matrix algebra, compositions of isometries can be calculated effectively as illustrated in the following examples.

1.17. Example. Compose the rotation through $2\pi/3$ about $(1/2, \sqrt{3}/6)$ with reflection in the line $x = y$ in the two possible orders and give geometric interpretations of the results.

Solution. Let the Seitz symbols of these isometries be $(A \mid \mathbf{t})$ and $(B \mid \mathbf{0})$ (note that the line $x = y$ contains the origin). Then

$$A = \begin{bmatrix} \cos 2\pi/3 & -\sin 2\pi/3 \\ \sin 2\pi/3 & \cos 2\pi/3 \end{bmatrix} = \begin{bmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{bmatrix},$$
$$\mathbf{t} = (I - A)\begin{bmatrix} 1/2 \\ \sqrt{3}/6 \end{bmatrix} = \begin{bmatrix} 3/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & 3/2 \end{bmatrix}\begin{bmatrix} 1/2 \\ \sqrt{3}/6 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$
$$B = \begin{bmatrix} \cos\pi/2 & \sin\pi/2 \\ \sin\pi/2 & -\cos\pi/2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

On composing we obtain

$$(A \mid \mathbf{t})(B \mid \mathbf{0}) = (AB \mid \mathbf{t}), \quad (B \mid \mathbf{0})(A \mid \mathbf{t}) = (BA \mid B\mathbf{t}), \quad B\mathbf{t} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Evaluating the matrix products we obtain

$$AB = \begin{bmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -\sqrt{3}/2 & -1/2 \\ -1/2 & \sqrt{3}/2 \end{bmatrix} = \begin{bmatrix} \cos(-5\pi/6) & \sin(-5\pi/6) \\ \sin(-5\pi/6) & -\cos(-5\pi/6) \end{bmatrix},$$
$$BA = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{bmatrix} = \begin{bmatrix} \sqrt{3}/2 & -1/2 \\ -1/2 & -\sqrt{3}/2 \end{bmatrix} = \begin{bmatrix} \cos(-\pi/6) & \sin(-\pi/6) \\ \sin(-\pi/6) & -\cos(-\pi/6) \end{bmatrix}.$$

These matrices represent reflections in the lines

$$\mathcal{L}_1 \colon x\sin(-5\pi/12) - y\cos(-5\pi/12) = 0, \quad \mathcal{L}_2 \colon x\sin(\pi/12) + y\cos(\pi/12) = 0.$$

Now we need to resolve $\mathbf{t} = (1, 0)$ in parallel and normal directions with respect to the line $\mathcal{L}_1$. The vector $(\sin(-5\pi/12), -\cos(-5\pi/12))$ is a unit vector normal to $\mathcal{L}_1$, so we can take

$$\begin{aligned}
\mathbf{v}_1 &= \frac{1}{2}((\sin(-5\pi/12), -\cos(-5\pi/12)) \cdot (1, 0))(\sin(-5\pi/12), -\cos(-5\pi/12)) \\
&= \frac{\sin(-5\pi/12)}{2}(\sin(-5\pi/12), -\cos(-5\pi/12)) \\
&= \frac{1}{2}(\sin^2(-5\pi/12), -\sin(-5\pi/12)\cos(-5\pi/12)) \\
&= \frac{1}{4}(2\sin^2(-5\pi/12), -2\sin(-5\pi/12)\cos(-5\pi/12)) \\
&= \frac{1}{4}(1 - \cos(-5\pi/6), -\sin(-5\pi/6)) \\
&= \frac{1}{4}(1 + \sqrt{3}/2, 1/2) = \frac{1}{8}(2 + \sqrt{3}, 1),
\end{aligned}$$

which also gives

$$\mathbf{u}_1 = (1, 0) - 2\mathbf{v}_1 = \frac{1}{4}(4 - 2 - \sqrt{3}, -1) = \frac{1}{4}(2 - \sqrt{3}, -1).$$

Hence $(AB \mid \mathbf{t})$ represents reflection in the line

$$x\sin(-5\pi/12) - y\cos(-5\pi/12) = \frac{\sin(-5\pi/12)}{2},$$

followed by translation by the vector $\mathbf{u}_1 = \frac{1}{4}(2 - \sqrt{3}, -1)$ parallel to it.

We need to resolve $B\mathbf{t} = (0, 1)$ in directions parallel and perpedicular to the line $\mathcal{L}_2$. The vector $(\sin(\pi/12), \cos(\pi/12))$ is a unit vector perpedicular to $\mathcal{L}_2$, so we can take

$$\begin{aligned}
\mathbf{v}_2 &= \frac{1}{2}((\sin(\pi/12), \cos(\pi/12)) \cdot (0, 1))(\sin(\pi/12), \cos(\pi/12)) \\
&= \frac{\cos(\pi/12)}{2}(\sin(\pi/12), \cos(\pi/12)) \\
&= \frac{1}{2}(\cos(\pi/12)\sin(\pi/12), \cos^2(\pi/12)) \\
&= \frac{1}{4}(2\cos(\pi/12)\sin(\pi/12), 2\cos^2(\pi/12)) \\
&= \frac{1}{4}(\sin(\pi/6), 1 + \cos(\pi/6)) \\
&= \frac{1}{4}(1/2, 1 + \sqrt{3}/2) = \frac{1}{8}(1, 2 + \sqrt{3}),
\end{aligned}$$

which also gives

$$\mathbf{u}_2 = (0, 1) - 2\mathbf{v}_2 = \frac{1}{4}(-1, 4 - 2 - \sqrt{3}) = \frac{1}{4}(-1, 2 - \sqrt{3}).$$

So $(BA \mid B\mathbf{t})$ represents reflection in the line

$$x\sin(\pi/12) + y\cos(\pi/12) = \frac{\cos(\pi/12)}{2},$$

followed by translation by the vector $\mathbf{u}_2 = \frac{1}{4}(-1, 2 - \sqrt{3})$ parallel to it. □

1.18. Example. If $(A_1 \mid \mathbf{t}_1)$ and $(A_2 \mid \mathbf{t}_2)$ are glide reflections, show that their composition $(A_1 \mid \mathbf{t}_1)(A_2 \mid \mathbf{t}_2)$ is a rotation or a translation.

SOLUTION. We have

$$\begin{aligned}\det A_1 = -1 = \det A_2,\\ \det(A_1A_2) = \det A_1 \det A_2 = 1,\\ (A_1 \mid \mathbf{t}_1)(A_2 \mid \mathbf{t}_2) = (A_1A_2 \mid \mathbf{t}_1 + A_1\mathbf{t}_2).\end{aligned}$$

When $A_1A_2 = I$, the composition $(A_1 \mid \mathbf{t}_1)(A_2 \mid \mathbf{t}_2)$ is a translation (or a trivial rotation if $\mathbf{t}_1 + A_1\mathbf{t}_2 = \mathbf{0}$). When $A_1A_2 \neq I$, $(A_1 \mid \mathbf{t}_1)(A_2 \mid \mathbf{t}_2)$ is a rotation. □

1.19. EXAMPLE. For the matrices

$$A = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}, \quad B = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}, \quad \mathbf{t} = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

describe the geometric effect of the isometry represented by each of the Seitz symbols $(A \mid \mathbf{t})$ and $(B \mid \mathbf{0})$. Determine the composition $(A \mid \mathbf{t})(B \mid \mathbf{0})$.

SOLUTION. Since

$$A = \begin{bmatrix} \cos(\pi/4) & \sin(\pi/4) \\ \sin(\pi/4) & -\cos(\pi/4) \end{bmatrix} = \begin{bmatrix} \cos(2\pi/8) & \sin(2\pi/8) \\ \sin(2\pi/8) & -\cos(2\pi/8) \end{bmatrix}, \quad \det A = -1,$$

we see that $(A \mid \mathbf{0})$ represents reflection in the line

$$\mathcal{L} = \{(x, y) : \sin(\pi/8)\, x - \cos(\pi/8)\, y = 0\}.$$

Write $\mathbf{t} = \mathbf{u} + 2\mathbf{v}$ where $\mathbf{u}$ is parallel to $\mathcal{L}$ and $\mathbf{v}$ is perpendicular to $\mathcal{L}$. The vectors

$$\mathbf{w}^{\parallel} = (\cos(\pi/8), \sin(\pi/8)), \quad \mathbf{w}^{\perp} = (\sin(\pi/8), -\cos(\pi/8))$$

are unit vectors in these directions and

$$\mathbf{w}^{\perp} \cdot \mathbf{t} = (\sin(\pi/8), -\cos(\pi/8)) \cdot \mathbf{t} = \sin(\pi/8),$$

so we have

$$\begin{aligned}\mathbf{v} &= \frac{\sin(\pi/8)}{2}(\sin(\pi/8), -\cos(\pi/8)) \\ &= \frac{1}{2}(\sin^2(\pi/8), -\sin(\pi/8)\cos(\pi/8)) \\ &= \frac{1}{4}(2\sin^2(\pi/8), -2\sin(\pi/8)\cos(\pi/8)) \\ &= \frac{1}{4}(1 - \cos\pi/4, -\sin\pi/4) \\ &= \frac{1}{4}(1 - 1/\sqrt{2}, -1/\sqrt{2}) = \frac{1}{4\sqrt{2}}(\sqrt{2} - 1, -1).\end{aligned}$$

Hence

$$\mathbf{u} = \mathbf{t} - 2\mathbf{v} = (1, 0) - \frac{1}{2\sqrt{2}}(\sqrt{2} - 1, -1) = ((2 + \sqrt{2})/4, \sqrt{2}/4).$$

Since

$$\mathbf{w}^{\perp} \cdot \mathbf{v} = \frac{\sin(\pi/8)}{2}(\sin^2(\pi/8) + \cos^2(\pi/8)) = \frac{\sin(\pi/8)}{2},$$

we see that $(A \mid \mathbf{t})$ is a glide reflection consisting of reflection in the line

$$\mathcal{L}' = \{(x, y) : \sin(\pi/8)\, x - \cos(\pi/8)\, y = \sin(\pi/8)/2\},$$

which parallel to $\mathcal{L}$ and contains $\mathbf{v}$. Noting that

$$B = \begin{bmatrix} \cos 2(-\pi/4) & \sin 2(-\pi/4) \\ \sin 2(-\pi/4) & -\cos 2(-\pi/4) \end{bmatrix},$$

we see that $(B \mid \mathbf{0})$ represents reflection in the line

$$\begin{aligned}\{(x,y) : \sin(-\pi/4)\, x - \cos(-\pi/4)\, y = 0\} &= \{(x,y) : (-1/\sqrt{2})x - (1/\sqrt{2})\, y = 0\} \\ &= \{(x,y) : x + y = 0\}.\end{aligned}$$

Now we have

$$(A \mid \mathbf{t})(B \mid \mathbf{0}) = (AB \mid \mathbf{t}),$$

where

$$AB = \begin{bmatrix} -1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} \cos(3\pi/4) & -\sin(3\pi/4) \\ \sin(3\pi/4) & \cos(3\pi/4) \end{bmatrix}, \quad \det(AB) = 1.$$

Therefore this Seitz symbol represents a rotation through $3\pi/4$ with centre (obtained using Equation (1.16a))

$$\begin{aligned}\mathbf{c} = (I - AB)^{-1}\mathbf{t} &= \begin{bmatrix} 1 + 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1 + 1/\sqrt{2} \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} \dfrac{1}{2} & \dfrac{-1/\sqrt{2}}{2(1+1/\sqrt{2})} \\ \dfrac{1/\sqrt{2}}{2(1+1/\sqrt{2})} & \dfrac{1}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} \dfrac{1}{2} \\ \dfrac{1/\sqrt{2}}{2(1+1/\sqrt{2})} \end{bmatrix} = \begin{bmatrix} \dfrac{1}{2} \\ \dfrac{1}{2(\sqrt{2}+1)} \end{bmatrix} = \begin{bmatrix} \dfrac{1}{2} \\ \dfrac{\sqrt{2}-1}{2} \end{bmatrix}. \end{aligned}$$

□

Finally, here are some useful results on inverses, obtained by using Proposition 1.14.

1.20. Proposition. *Suppose that*

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}, \quad B = \begin{bmatrix} \cos\varphi & \sin\varphi \\ \sin\varphi & -\cos\varphi \end{bmatrix}.$$

The Seitz symbols of the inverses of the rotation $(A \mid \mathbf{s})$ *and the* (*glide*) *reflection* $(B \mid \mathbf{t})$ *are*

$$(A \mid \mathbf{s})^{-1} = (A^T \mid -A^T\mathbf{s}), \quad (B \mid \mathbf{t})^{-1} = (B \mid -B\mathbf{t}),$$

where

$$A^T = \begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix}.$$

4. Seitz matrices

For practical purposes, it is useful to encode a Seitz symbol $(A \mid \mathbf{t})$ as a 3×3 matrix. If

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad \mathbf{t} = \begin{bmatrix} u \\ v \end{bmatrix},$$

we introduce the *Seitz matrix*

$$\begin{bmatrix} A & \mathbf{t} \\ \mathbf{0} & 1 \end{bmatrix} = \begin{bmatrix} a & b & u \\ c & d & v \\ 0 & 0 & 1 \end{bmatrix},$$

which is *block form* or *partitioned matrix*. Given a vector $\mathbf{x} = (x, y) = \begin{bmatrix} x \\ y \end{bmatrix}$, for the 3×1 column vector $\begin{bmatrix} \mathbf{x} \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$ we find that

$$\begin{bmatrix} A & \mathbf{t} \\ \mathbf{0} & 1 \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ 1 \end{bmatrix} = \begin{bmatrix} A\mathbf{x} + \mathbf{t} \\ 1 \end{bmatrix} = \begin{bmatrix} (A \mid \mathbf{t})\mathbf{x} \\ 1 \end{bmatrix}. \tag{1.17}$$

We also

$$\begin{bmatrix} A_1 & \mathbf{t}_1 \\ \mathbf{0} & 1 \end{bmatrix} \begin{bmatrix} A_2 & \mathbf{t}_2 \\ \mathbf{0} & 1 \end{bmatrix} = \begin{bmatrix} A_1A_2 & \mathbf{t}_1 + A_1\mathbf{t}_2 \\ \mathbf{0} & 1 \end{bmatrix}. \tag{1.18}$$

By Proposition 1.14, this is the Seitz matrix of $(A_1 \mid \mathbf{t}_1)(A_2 \mid \mathbf{t}_2) = (A_1A_2 \mid \mathbf{t}_1 + A_1\mathbf{t}_2)$. Similarly, the Seitz matrix of $(A \mid \mathbf{t})^{-1} = (A^{-1} \mid -A^{-1}\mathbf{t})$ is

$$\begin{bmatrix} A & \mathbf{t} \\ \mathbf{0} & 1 \end{bmatrix}^{-1} = \begin{bmatrix} A^{-1} & -A^{-1}\mathbf{t} \\ \mathbf{0} & 1 \end{bmatrix}. \tag{1.19}$$

Thus calculations with isometries can be carried out with the aid of 3×3 Seitz matrices using matrix products to determine actions on vectors and compositions and inverses.

Exercises on Chapter 1

1.1. (a) Find a parametric equation for the line $\mathcal{L}_1$ with implicit equation $2x - 3y = 1$.
(b) Find an implicit equation for the line $\mathcal{L}_2$ which has parametric equation $\mathbf{x} = (t - 1, 3t + 1)$.
(c) Find parametric and implicit equations for the line $\mathcal{L}_3$ which contains the point $P(1, -1)$ and is parallel to the vector $(1, 1)$.
(d) Find the point of intersection of the lines $\mathcal{L}_1$ and $\mathcal{L}_3$ and the angle θ between them.

1.2. Let $\mathbf{u} = (5, 0)$ and $\mathbf{v} = (2, -1)$.
(a) Find the angle between $\mathbf{u}$ and $\mathbf{v}$.
(b) Find the projection of the vector $\mathbf{u}$ onto $\mathbf{v}$.
(c) Find the projection of the vector $\mathbf{v}$ onto $\mathbf{u}$.

1.3. Consider the lines

$$\mathcal{L}_1 = \{(x, y) : x + y = 2\}, \quad \mathcal{L}_2 = \{(x, y) : x - y = 2\}.$$

Find the effects on the point $P(1, 0)$ of the reflections $\mathrm{Refl}_{\mathcal{L}_1}$ and $\mathrm{Refl}_{\mathcal{L}_2}$.

1.4. Consider the lines

$$\mathcal{L}_1 = \{(x, y) : 2x + y = 0\}, \quad \mathcal{L}_2 = \{(x, y) : 2x + y = 2\}.$$

Express each of the isometries $\mathrm{Refl}_{\mathcal{L}_2} \circ \mathrm{Refl}_{\mathcal{L}_1}$ and $\mathrm{Refl}_{\mathcal{L}_1} \circ \mathrm{Refl}_{\mathcal{L}_2}$ as translations, *i.e.*, in the form $\mathrm{Trans}_{\mathbf{t}}$ for some $\mathbf{t} \in \mathbb{R}^2$.

1.5. Recall the standard identification of the pair (x, y) with the column vector $\begin{bmatrix} x \\ y \end{bmatrix}$.
(a) Give a matrix interpretation of the dot product $(x_1, y_1) \cdot (x_2, y_2)$.
(b) Let $\mathbf{u} \in \mathbb{R}^2$ be a unit vector. Show that the 2×2 matrix $U = \mathbf{u}\mathbf{u}^T$ satisfies

$$U\mathbf{x} = \begin{cases} \mathbf{0} & \text{if } \mathbf{u} \cdot \mathbf{x} = 0, \\ \mathbf{x} & \text{if } \mathbf{x} = t\mathbf{u} \text{ for some } t \in \mathbb{R}. \end{cases}$$

(c) Deduce that the matrix $U' = I_2 - 2U$ has the same effect on vectors as reflection in the line

$$\mathcal{L} = \{\mathbf{x} \in \mathbb{R}^2 : \mathbf{u} \cdot \mathbf{x} = 0\}.$$

1.6. For each of the following cases, describe geometrically the effect of the isometry $(A \mid \mathbf{t})$.

(a) $A = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$, $\mathbf{t} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, (b) $A = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$, $\mathbf{t} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$,

(c) $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, $\mathbf{t} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

In each case, determine the Seitz symbol of $(A \mid \mathbf{t})^2 = (A \mid \mathbf{t})(A \mid \mathbf{t})$ and describe the effect of the corresponding isometry.

1.7. (a) Show that an $n \times n$ orthogonal matrix A has determinant $\det A = \pm 1$.

[*If you don't know about determinants for arbitrary sized square matrices, do this for* $n = 2, 3$.]

(b) Show that a 2×2 real orthogonal matrix A with determinant $\det A = 1$ has the form $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$.

[*Write down a system of equations for the four entries of* A*, then solve it using the fact that when a pair of real numbers* x, y *satisfies* $x^2 + y^2 = 1$ *there is a real number* φ *such that* $x = \cos\varphi$*,* $y = \sin\varphi$.]

(c) Show that a 2×2 real orthogonal matrix B with determinant $\det B = -1$ has the form $B = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$ for some $\theta \in \mathbb{R}$.

[*Observe that* $C = B \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ *is orthogonal and satisfies* $\det C = 1$*, then apply* (b).]

(d) If P, Q are $n \times n$ orthogonal matrices, show that their product PQ is also orthogonal.

1.8. Show that for the Seitz symbol $(A \mid \mathbf{t})$ of an isometry, the Seitz symbol of the inverse isometry is $(A^T \mid -A^T\mathbf{t})$.

1.9. Let $F\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be an isometry that fixes a point P with position vector $\mathbf{p}$.
(a) Show that the composition

$$G = \mathrm{Trans}_{-\mathbf{p}} \circ F \circ \mathrm{Trans}_{\mathbf{p}}$$

fixes the origin and describe the effect this isometry geometrically in terms of that of F.
(b) If Q is a second point with position vector $\mathbf{q}$ show that the composition

$$H = \mathrm{Trans}_{(\mathbf{q}-\mathbf{p})} \circ F \circ \mathrm{Trans}_{(\mathbf{p}-\mathbf{q})}$$

fixes Q and describe the effect of this isometry geometrically in terms of that of F.

1.10. Let $(A \mid \mathbf{t})$ be the Seitz symbol of an isometry $\mathbb{R}^2 \longrightarrow \mathbb{R}^2$.
(a) If $s \in \mathbb{R}$ and $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$, show that

$$(A \mid \mathbf{t})(s\mathbf{x} + (1-s)\mathbf{y}) = s(A \mid \mathbf{t})\mathbf{x} + (1-s)(A \mid \mathbf{t})\mathbf{y}.$$

(b) For $n \geqslant 2$, show that if $s_1, \ldots, s_n \in \mathbb{R}$ satisfy $s_1 + \cdots + s_n = 1$ and $\mathbf{x}_1, \ldots, \mathbf{x}_n \in \mathbb{R}^2$, then

$$(A \mid \mathbf{t})(s_1\mathbf{x}_1 + \cdots + s_n\mathbf{x}_n) = s_1(A \mid \mathbf{t})\mathbf{x}_1 + \cdots + s_n(A \mid \mathbf{t})\mathbf{x}_n.$$

CHAPTER 2

Groups and symmetry

1. Groups and subgroups

Let G be set and $*$ a *binary operation* which combines each pair of elements $x, y \in G$ to give another element $x * y \in G$. Then $(G, *)$ is a *group* if it satisfies the following conditions.

Gp1: for all elements $x, y, z \in G$, $(x * y) * z = x * (y * z)$;
Gp2: there is an element $\iota \in G$ such that for every $x \in G$, $\iota * x = x = x * \iota$;
Gp3: for every $x \in G$, there is a unique element $y \in G$ such that $x * y = \iota = y * x$.

Gp1 is usually called the associativity law. ι is usually called the identity element of $(G, *)$. In Gp3, the unique element y associated to x is called the inverse of x and denoted x^{-1}.

2.1. Example. For each of the following cases, $(G, *)$ is a group.

(1) $G = \mathbb{Z}$, $* = +$, $\iota = 0$ and $x^{-1} = -x$.
(2) $G = \mathbb{Q}$, $* = +$, $\iota = 0$ and $x^{-1} = -x$.
(3) $G = \mathbb{R}$, $* = +$, $\iota = 0$ and $x^{-1} = -x$.

2.2. Example. Let

$$\begin{aligned} \mathrm{GL}_2(\mathbb{R}) &= \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in R,\ ad - bc \neq 0 \right\}, \\ * &= \text{multiplication of matrices}, \\ \iota &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2, \\ \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} &= \begin{bmatrix} \dfrac{d}{ad-bc} & -\dfrac{b}{ad-bc} \\ -\dfrac{c}{ad-bc} & \dfrac{a}{ad-bc} \end{bmatrix}. \end{aligned}$$

2.3. Example. Let X be a finite set and let $\mathrm{Perm}(X)$ be the set of all bijections $f\colon X \longrightarrow X$ (also known as *permutations*). Then $(\mathrm{Perm}(X), \circ)$ is a group where

$$\begin{aligned} \circ &= \text{composition of functions}, \\ \iota &= \mathrm{Id}_X = \text{the identity function on } X, \\ f^{-1} &= \text{the inverse function of } f. \end{aligned}$$

$(\mathrm{Perm}(X), \circ)$ is called the *permutation group* of X. We will study these and other examples in more detail.

When discussing a group $(G, *)$, we will often write xy for the product $x * y$ if no confusion seems likely to arise. For example, when dealing with a permutation group $(\mathrm{Perm}(X), \circ)$ we will write $\alpha\beta$ for $\alpha \circ \beta$.

2.4. Example. Let

$$\begin{aligned} \mathrm{Euc}(2) &= \text{set of all isometries } \mathbb{R}^2 \longrightarrow \mathbb{R}^2, \\ * &= \circ, \\ \iota &= (I_2 \mid \mathbf{0}). \end{aligned}$$

$(\mathrm{Euc}(2), \circ)$ forms a group known as the *Euclidean group* of $\mathbb{R}^2$.

If a group $(G, *)$ has a finite underlying set G, then the number of elements in the G is called the *order* of G and is denoted $|G|$. If $|G|$ is not finite, G is said to be *infinite*.

A group G is *commutative* or *abelian* if for every pair of elements $x, y \in G$, $x * y = y * x$. Most groups are *not* commutative.

Let $(G, *)$ be a group and $H \subseteq G$ a subset. Then H is a *subgroup* of G if $(H, *)$ is a group. In detail this means

- for $x, y \in H$, $x * y \in H$;
- $\iota \in H$;
- if $z \in H$ then $z^{-1} \in H$.

We don't need to check associativity since Gp1 holds for all elements of G and so in particular for elements of H.

We write $H \leqslant G$ whenever H is a subgroup of G and $H < G$ if $H \neq G$, *i.e.*, H is a *proper subgroup* of G.

If $(G, *)$ is a group, then for any $g \in G$ we can consider the subset

$$\langle g \rangle = \{g^n : n \in \mathbb{Z}\} \subseteq G,$$

where

$$g^n = \begin{cases} \overbrace{g * g * \cdots * g}^{n \text{ factors}} & \text{if } n > 0, \\ (g^{-1})^{-n} & \text{if } n < 0, \\ \iota & \text{if } n = 0. \end{cases}$$

It is easy to see that $\langle g \rangle \leqslant G$, and it is known as the *cyclic subgroup generated by* g. If for some $c \in G$ we have $G = \langle c \rangle$ then G is called a *cyclic group*.

For $g \in G$, if there is an $n > 0$ such that $g^n = \iota$ then g is said to have *finite order*, otherwise g has *infinite order*.

2.5. Proposition. *Let $(G, *)$ be a group and $g \in G$.*
a) *If g has infinite order then all the integer powers of g are distinct and so G is infinite. In particular,*

$$\langle g \rangle = \{\ldots, g^{-2}, g^{-1}, \iota, g^1, g^2, \ldots\}.$$

b) *If g has finite order then there is a smallest positive exponent n_0 for which $g^{n_0} = \iota$ and the distinct powers of g are $g, g^2, \ldots, g^{n_0-1}, g^{n_0} = \iota$, so*

$$\langle g \rangle = \{g, g^2, \ldots, g^{n_0-1}, \iota\}.$$

If g has finite order the number n_0 is called the order of g, and is denoted $|g|$. Sometimes we write $|g| = \infty$ if g has infinite order and $|g| < \infty$ when it has finite order.

2. Permutation groups

We will follow the ideas of Example 2.3 and consider the *standard set with n elements*

$$\mathbf{n} = \{1, 2, \ldots, n\}.$$

If we write $\mathrm{S}_n = \mathrm{Perm}(\mathbf{n})$, the group $(\mathrm{S}_n, \circ)$ is called the *symmetric group on n objects* or the *symmetric group of degree n* or the *permutation group on n objects*.

2.6. Theorem. *S_n has order $|\mathrm{S}_n| = n!$.*

Proof. Defining an element $\sigma \in \mathrm{S}_n$ is equivalent to specifying the list

$$\sigma(1), \sigma(2), \ldots, \sigma(n)$$

consisting of the n numbers $1, 2, \ldots, n$ taken in some order with no repetitions. To do this we have

- n choices for $\sigma(1)$,
- $n - 1$ choices for $\sigma(2)$ (taken from the remaining $n - 1$ elements),
- and so on.

In all, this gives $n \times (n-1) \times \cdots \times 2 \times 1 = n!$ choices for σ, so $|\mathrm{S}_n| = n!$ as claimed. We will often use the notation

$$\sigma = \begin{pmatrix} 1 & 2 & \dots & n \\ \sigma(1) & \sigma(2) & \dots & \sigma(n) \end{pmatrix}. \qquad \square$$

2.7. Example. The elements of S_3 are the following:

$$\iota = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}.$$

We can calculate the composition $\tau \circ \sigma$ of two permutations $\tau, \sigma \in \mathrm{S}_n$, where $\tau\sigma(k) = \tau(\sigma(k))$. Notice that we apply σ to k first then apply τ to the result $\sigma(k)$. For example,

$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} = \iota.$$

In particular,

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}^{-1}.$$

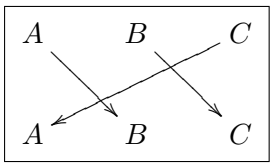
Let X be a set with exactly n elements which we list in some order, $x_1, x_2, \dots, x_n$. Then there is an *action* of S_n on X given by

$$\sigma \cdot x_k = x_{\sigma(k)} \quad (\sigma \in \mathrm{S}_n,\ k = 1, 2, \dots, n).$$

For example, if $X = \{A, B, C\}$ we can take $x_1 = A, x_2 = B, x_3 = C$ and so

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \cdot A = B, \quad \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \cdot B = C, \quad \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \cdot C = A.$$

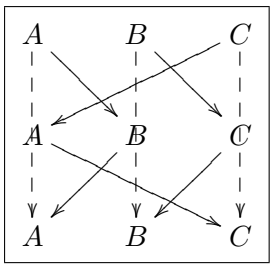
Often it is useful to display the effect of a permutation $\sigma \colon X \longrightarrow X$ by indicating where each element is sent by σ with the aid of arrows. To do this we display the elements of X in two similar rows with an arrow joining x_i in the first row to $\sigma(x_i)$ in the second. For example, the action of the permutation $\sigma = \begin{pmatrix} A & B & C \\ B & C & A \end{pmatrix}$ on $X = \{A, B, C\}$ can be displayed as



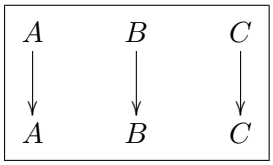
We can compose permutations by composing the arrows. Thus

$$\begin{pmatrix} A & B & C \\ C & A & B \end{pmatrix} \begin{pmatrix} A & B & C \\ B & C & A \end{pmatrix}$$

can be determined from the diagram



which gives the identity function whose diagram is



Let $\sigma \in \mathrm{S}_n$ and consider the arrow diagram of σ as above. Let c_σ be the number of crossings of arrows. The *sign* of σ is the number

$$\operatorname{sgn}\sigma = (-1)^{c_\sigma} = \begin{cases} +1 & \text{if } c_\sigma \text{ is even,} \\ -1 & \text{if } c_\sigma \text{ is odd.} \end{cases}$$

Then $\operatorname{sgn}\colon \mathrm{S}_n \longrightarrow \{+1, -1\}$. Notice that $\{+1, -1\}$ is a group under multiplication.

2.8. Proposition. *The function* $\operatorname{sgn}\colon \mathrm{S}_n \longrightarrow \{+1, -1\}$ *satisfies*

$$\operatorname{sgn}(\tau\sigma) = \operatorname{sgn}(\tau)\operatorname{sgn}(\sigma) \quad (\tau, \sigma \in \mathrm{S}_n).$$

Proof. By considering the arrow diagram for $\tau\sigma$ obtained by joining the diagrams for σ and τ, we see that the total number of crossings is $c_\sigma + c_\tau$. If we straighten out the paths starting at each number in the top row, so that we change the total number of crossings by 2 each time, hence $(-1)^{c_\sigma + c_\tau} = (-1)^{c_{\tau\sigma}}$. □

A permutation σ is called *even* if $\operatorname{sgn}\sigma = 1$, otherwise it is *odd*. The set of all even permutations in S_n is denoted by A_n. Notice that $\iota \in \mathrm{A}_n$ and in fact the following result is true.

2.9. Proposition. *The set* A_n *is a subgroup of* S_n, $\mathrm{A}_n \leqslant \mathrm{S}_n$.

Proof. By Proposition 2.8, for $\sigma, \tau \in \mathrm{A}_n$,

$$\operatorname{sgn}(\tau\sigma) = \operatorname{sgn}(\tau)\operatorname{sgn}(\sigma) = 1.$$

Note also that $\iota \in \mathrm{A}_n$.

The arrow diagram for σ^{-1} is obtained from that for σ by interchanging the rows and reversing all the arrows, so $\operatorname{sgn}\sigma^{-1} = \operatorname{sgn}\sigma$. Thus if $\sigma \in \mathrm{A}_n$, we have $\operatorname{sgn}\sigma^{-1} = 1$.

Hence, A_n is a subgroup of S_n. □

A_n is called the *n-th alternating group*.

2.10. Example. The elements of A_3 are

$$\iota = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}.$$

2.11. Proposition. *For* $n \geqslant 2$, A_n *has order* $|\mathrm{A}_n| = |\mathrm{S}_n|/2 = n!/2$.

Proof. Let $\sigma \in \mathrm{S}_n$ and let $\tau \in \mathrm{S}_n$ be the permutation which has the effect

$$\tau(j) = \begin{cases} 2 & \text{if } j = 1, \\ 1 & \text{if } j = 2, \\ j & \text{otherwise.} \end{cases}$$

Then either $\sigma \in \mathrm{A}_n$ or $(1\ 2)\sigma \in \mathrm{A}_n$. Furthermore, if $\sigma' \in \mathrm{S}_n$ and $(1\ 2)\sigma' = (1\ 2)\sigma$ then $\sigma' = \sigma$, so we can write $\sigma \in \mathrm{S}_n$ uniquely in one of the forms $\sigma \in \mathrm{A}_n$ or $\sigma = (1\ 2)\theta$ with $\theta \in \mathrm{A}_n$. This shows that $|\mathrm{S}_n| = 2|\mathrm{A}_n|$. □

Suppose $\sigma \in \mathrm{S}_n$. Now carry out the following steps.

- Form the sequence
$$1 \to \sigma(1) \to \sigma^2(1) \to \cdots \to \sigma^{r_1 - 1}(1) \to \sigma^{r_1}(1) = 1$$
where $\sigma^k(j) = \sigma(\sigma^{k-1}(j))$ and r_1 is the smallest positive power for which this is true.
- Take the smallest number $k_2 = 1, 2, \ldots, n$ for which $k_2 \neq \sigma^t(k_1)$ for every t. Form the sequence
$$k_2 \to \sigma(k_2) \to \sigma^2(k_2) \to \cdots \to \sigma^{r_2 - 1}(k_2) \to \sigma^{r_2}(k_2) = k_2$$
where r_2 is the smallest positive power for which this is true.
- Repeat this with the smallest number $k_3 = 1, 2, \ldots, n$ such that $k_3 \neq \sigma^t(k_2)$ for every t.
- ⋮

Writing $k_1 = 1$, we obtain a collection of *disjoint cycles*

$$\begin{aligned} k_1 &\to \sigma(k_1) \to \sigma^2(k_1) \to \cdots \to \sigma^{r_1-1}(k_1) \to \sigma^{r_1}(k_1) = k_1 \\ k_2 &\to \sigma(k_2) \to \sigma^2(k_2) \to \cdots \to \sigma^{r_2-1}(k_2) \to \sigma^{r_2}(k_2) = k_2 \\ &\vdots \\ k_d &\to \sigma(k_d) \to \sigma^2(k_d) \to \cdots \to \sigma^{r_d-1}(k_d) \to \sigma^{r_d}(k_d) = k_d \end{aligned}$$

in which every number $k = 1, 2, \ldots, n$ occurs in exactly one row.

The s-th one of these cycles can be viewed as corresponding to the permutation of $\mathbf{n}$ which behaves according to the action of σ on the elements that appear as $\sigma^t(k_s)$ and fix every other element. We indicate this permutation using the *cycle notation*

$$(k_s\ \sigma(k_s)\ \cdots\ \sigma^{r_s-1}(k_s)).$$

Then we have

$$\sigma = (k_1\ \sigma(k_1)\ \cdots\ \sigma^{r_1-1}(k_1)) \cdots (k_d\ \sigma(k_d)\ \cdots\ \sigma^{r_d-1}(k_d)),$$

which is the *disjoint cycle decomposition of* σ. It is unique apart from the order of the factors and the order in which the numbers within each cycle occur.

For example, in S_4,

$$\begin{aligned} &(1\ 2)(3\ 4) = (2\ 1)(4\ 3) = (3\ 4)(1\ 2) = (4\ 3)(2\ 1), \\ &(1\ 2\ 3)(4) = (3\ 1\ 2)(4) = (2\ 3\ 1)(4) = (4)(1\ 2\ 3) = (4)(3\ 1\ 2) = (4)(2\ 3\ 1). \end{aligned}$$

We usually leave out cycles of length 1, so for example $(1\ 2\ 3)(4) = (1\ 2\ 3) = (4)(1\ 2\ 3)$.

A permutation $\tau \in \mathrm{S}_n$ which interchanges two elements of $\mathbf{n}$ and leaves the rest fixed is called a *transposition*.

2.12. Proposition. *For $\sigma \in \mathrm{S}_n$, there are transpositions $\tau_1, \ldots, \tau_k$ such that $\sigma = \tau_1 \cdots \tau_k$.*

One way to decompose a permutation σ into transpositions is to first decompose it into disjoint cycles then use the easily checked formula

$$(i_1\ i_2\ \ldots\ i_r) = (i_1\ i_r) \cdots (i_1\ i_3)(i_1\ i_2). \tag{2.1}$$

2.13. Example. Decompose

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 5 & 3 & 1 & 4 \end{pmatrix} \in S_5$$

into a product of transpositions.

Solution. We have

$$\sigma = (3)(1\ 2\ 5\ 4) = (1\ 2\ 5\ 4) = (1\ 4)(1\ 5)(1\ 2).$$

Some alternative decompositions are

$$\sigma = (2\ 1)(2\ 4)(2\ 5) = (5\ 2)(5\ 1)(5\ 4). \qquad \square$$

2.14. Example. In S_6, compose the permutations $\alpha = (1\ 2\ 3\ 4)$ and $\beta = (1\ 3\ 5)(2\ 4)$.

Solution. We will determine $\alpha\beta = (1\ 2\ 3\ 4)(1\ 3\ 5)(2\ 4)$ by building up its cycles.

Beginning with 1, we see that

$$1 \xrightarrow{(2\ 4)} 1 \xrightarrow{(1\ 3\ 5)} 3 \xrightarrow{(1\ 2\ 3\ 4)} 4,$$

so $\alpha\beta(1) = 4$. Now repeat this with 4,

$$4 \xrightarrow{(2\ 4)} 2 \xrightarrow{(1\ 3\ 5)} 2 \xrightarrow{(1\ 2\ 3\ 4)} 3,$$

so $\alpha\beta(4) = 3$. Repeating with 3 we obtain

$$3 \xrightarrow{(2\ 4)} 3 \xrightarrow{(1\ 3\ 5)} 5 \xrightarrow{(1\ 2\ 3\ 4)} 5,$$

so $\alpha\beta(3) = 5$. Repeating with 5 we obtain

$$5 \xrightarrow{(2\ 4)} 5 \xrightarrow{(1\ 3\ 5)} 1 \xrightarrow{(1\ 2\ 3\ 4)} 2,$$

so $\alpha\beta(5) = 2$. Repeating with 2 we obtain

$$2 \xrightarrow{(2\ 4)} 4 \xrightarrow{(1\ 3\ 5)} 4 \xrightarrow{(1\ 2\ 3\ 4)} 2,$$

so $\alpha\beta(2) = 1$. This shows that $\alpha\beta$ contains the 5-cycle (1 4 3 5 2),

$$1 \longrightarrow 4 \longrightarrow 3 \longrightarrow 5 \longrightarrow 2 \longrightarrow 1.$$

Applying $\alpha\beta$ to 6 we find that $\alpha\beta(6) = 6$, so $\alpha\beta$ also contains the 1-cycle (6). Hence

$$\alpha\beta = (1\ 4\ 3\ 5\ 2)(6) = (6)(1\ 4\ 3\ 5\ 2) = (1\ 4\ 3\ 5\ 2).$$

Similarly we find that

$$\beta\alpha = (1\ 3\ 5)(2\ 4)(1\ 2\ 3\ 4) = (1\ 4\ 3\ 2\ 5).$$

It is worth noting that $\beta\alpha \neq \alpha\beta$, which shows that S_n is not a commutative group in general. □

3. Groups of isometries

From Example 2.4 we have the Euclidean group $(\mathrm{Euc}(2), \circ)$, which is clearly infinite.

2.15. Example. Consider the set of translations in Euc(2),

$$\mathrm{Trans}(2) = \{(I_2 \mid \mathbf{t}) \in \mathrm{Euc}(2) : \mathbf{t} \in \mathbb{R}^2\}.$$

Then $\mathrm{Trans}(2) \leqslant \mathrm{Euc}(2)$.

Proof. In Equation (1.6) we have seen that Trans(2) is closed under composition. We also know that $(I_2 \mid \mathbf{0}) \in \mathrm{Trans}(2)$ and for $\mathbf{t} \in \mathbb{R}^2$,

$$(I_2 \mid \mathbf{t})^{-1} = (I_2 \mid -\mathbf{t}) \in \mathrm{Trans}(2).$$

So $\mathrm{Trans}(2) \leqslant \mathrm{Euc}(2)$. □

Trans(2) is called the *translation subgroup* of Euc(2).

2.16. Example. Let

$$\mathrm{O}(2) = \{(A \mid \mathbf{0}) \in \mathrm{Euc}(2) : A \text{ is orthogonal}\}.$$

Then $\mathrm{O}(2) \leqslant \mathrm{Euc}(2)$.

Proof. For $(A \mid \mathbf{0}), (B \mid \mathbf{0}) \in \mathrm{O}(2)$ we have

$$(A \mid \mathbf{0})(B \mid \mathbf{0}) = (AB \mid \mathbf{0})$$

and

$$(AB)^T(AB) = (B^TA^T)(AB) = B^T(A^TA)B = B^TI_2B = B^TB = I_2.$$

So $(A \mid \mathbf{0})(B \mid \mathbf{0}) \in \mathrm{O}(2)$. Also, $(I_2 \mid \mathbf{0}) \in \mathrm{O}(2)$ and

$$(A \mid \mathbf{0})^{-1} = (A^{-1} \mid \mathbf{0}) \in \mathrm{O}(2)$$

since $A^{-1} = A^T$ and

$$(A^T)^T(A^T) = AA^T = AA^{-1} = I_2,$$

hence A^{-1} is orthogonal. □

O(2) is the *orthogonal subgroup* of Euc(2). It consists of all the isometries of $\mathbb{R}^2$ which fix the origin.

2.17. Example. Let

$$\mathrm{SO}(2) = \{(A \mid \mathbf{0}) \in \mathrm{Euc}(2) : A \text{ is orthogonal and } \det A = 1\}.$$

Then $\mathrm{SO}(2) \leqslant \mathrm{O}(2) \leqslant \mathrm{Euc}(2)$.

PROOF. If $(A \mid \mathbf{0}), (B \mid \mathbf{0}) \in \mathrm{SO}(2)$, then $(A \mid \mathbf{0})(B \mid \mathbf{0}) = (AB \mid \mathbf{0})$ and

$$\det(AB) = \det A \det B = 1,$$

so $(A \mid \mathbf{0})(B \mid \mathbf{0}) \in \mathrm{SO}(2)$. Checking the remaining points is left as an exercise. □

SO(2) is called the *special orthogonal subgroup* of Euc(2) and consists of all rotations about the origin. Elements of Euc(2) of the form $(A \mid \mathbf{t})$ with $A \in \mathrm{SO}(2)$ are called *direct isometries*, while those with $A \notin \mathrm{SO}(2)$ are called *indirect isometries*. We denote the subset of direct isometries by $\mathrm{Euc}^+(2)$ and the subset of indirect isometries by $\mathrm{Euc}^-(2)$.

2.18. EXAMPLE. The direct isometries form a subgroup of Euc(2), *i.e.*, $\mathrm{Euc}^+(2) \leqslant \mathrm{Euc}(2)$.

PROOF. If $(A_1 \mid \mathbf{t}_1), (A_2 \mid \mathbf{t}_2) \in \mathrm{Euc}^+(2)$, then

$$(A_1 \mid \mathbf{t}_1)(A_2 \mid \mathbf{t}_2) = (A_1A_2 \mid \mathbf{t}_1 + A_1\mathbf{t}_2)$$

with $A_1A_2 \in \mathrm{SO}(2)$, so this product is in $\mathrm{Euc}^+(2)$. □

4. Symmetry groups of plane figures

If $S \subseteq \mathbb{R}^2$ is a non-empty subset, we can consider the subset

$$\mathrm{Euc}(2)_S = \{\alpha \in \mathrm{Euc}(2) : \alpha S = S\} \subseteq \mathrm{Euc}(2).$$

2.19. PROPOSITION. $\mathrm{Euc}(2)_S$ *is a subgroup of* Euc(2), $\mathrm{Euc}(2)_S \leqslant \mathrm{Euc}(2)$

PROOF. By definition, for $\alpha \in \mathrm{Euc}(2)$,

$$\alpha S = \{\alpha(s) : s \in S\}.$$

So $\alpha S = S$ if and only if

- for every $s \in S$, $\alpha(s) \in S$;
- every $s \in S$ has the form $s = \alpha(s')$ for some $s' \in S$.

Since an isometry is injective, this really says that each $\alpha \in \mathrm{Euc}(2)_S$ acts by permuting the elements of S and preserving distances between them.

If $\alpha, \beta \in \mathrm{Euc}(2)_S$ then for $s \in S$,

$$\alpha\beta(s) = \alpha(\beta(s)) \in \alpha S = S.$$

Also, there is an $s' \in S$ such that $s = \alpha(s')$ and similarly an $s'' \in S$ such that $s' = \beta(s'')$; hence

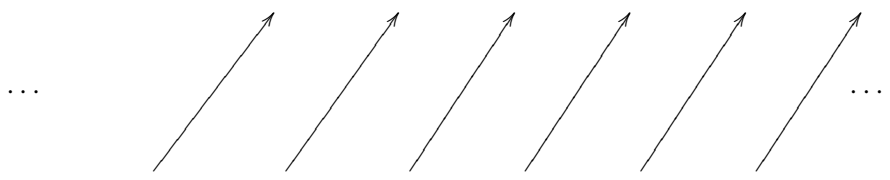
$$s = \alpha(s') = \alpha(\beta(s'')) = \alpha\beta(s'').$$

It is easy to see that $\mathrm{Id}_{\mathbb{R}^2} \in \mathrm{Euc}(2)_S$. Finally, if $\alpha \in \mathrm{Euc}(2)_S$ then $\alpha^{-1} \in \mathrm{Euc}(2)_S$ since

$$\alpha^{-1}S = \alpha^{-1}(\alpha S) = (\alpha^{-1}\alpha)S = S.$$ □

$\mathrm{Euc}(2)_S$ is called the *symmetry subgroup* of S and is often referred to as the *symmetry group* of S as a subset of $\mathbb{R}^2$.

2.20. EXAMPLE. Let $S \subseteq \mathbb{R}^2$ be the following pattern.



A_{-2} A_{-1} A_0 A_1 A_2 A_3

Find the symmetry subgroup of S.

SOLUTION. It is clear that there are translations which move each arrow a fixed number of steps right or left by distance equal to the horizontal distance between these arrows. If the translation sending A_n to A_{n+1} is $\mathrm{Trans}_{\mathbf{t}}$ then each of the translations $\mathrm{Trans}_{k\mathbf{t}}$ with $k \in \mathbb{Z}$ is in $\mathrm{Euc}(2)_S$. So

$$\mathrm{Euc}(2)_S = \{\mathrm{Trans}_{k\mathbf{t}} : k \in \mathbb{Z}\},$$

since it is also clear that there are no further isometries of $\mathbb{R}^2$ that map S into itself.

Note that this pattern is essentially the same as Frieze Pattern 1 in Figure 2.1. □

In this case we see that all symmetries of S are translations and indeed all powers of a fixed one, since if $k \leqslant 1$,

$$\mathrm{Trans}_{k\mathbf{t}} = (\mathrm{Trans}_{\mathbf{t}})^k = \underbrace{\mathrm{Trans}_{\mathbf{t}} \circ \cdots \circ \mathrm{Trans}_{\mathbf{t}}}_{k \text{ factors}},$$

while if $k \leqslant -1$,

$$\mathrm{Trans}_{k\mathbf{t}} = (\mathrm{Trans}_{-\mathbf{t}})^{-k} = \underbrace{\mathrm{Trans}_{-\mathbf{t}} \circ \cdots \circ \mathrm{Trans}_{-\mathbf{t}}}_{-k \text{ factors}}$$

where $(\mathrm{Trans}_{\mathbf{t}})^{-1} = \mathrm{Trans}_{-\mathbf{t}}$. So the symmetry subgroup of this plane figure is cyclic with generator $\mathrm{Trans}_{\mathbf{t}}$.

For any subset $S \subseteq \mathbb{R}^2$ we can consider the set of translational symmetries of S,

$$\mathrm{Trans}(2)_S = \mathrm{Trans}(2) \cap \mathrm{Euc}(2)_S \subseteq \mathrm{Euc}(2)_S.$$

The following is easy to prove.

2.21. LEMMA. $\mathrm{Trans}(2)_S \leqslant \mathrm{Euc}(2)_S$.

A *frieze pattern* in the plane is a subset $S \subseteq \mathbb{R}^2$ for which the subset of translational symmetries $\mathrm{Trans}(2)_S$ is an infinite cyclic group. This means that there is a translation vector $\mathbf{t}$ for which

$$\mathrm{Trans}(2)_S = \langle \mathrm{Trans}_{\mathbf{t}} \rangle = \{\mathrm{Trans}_{k\mathbf{t}} : k \in \mathbb{Z}\}.$$

The examples in Figure 2.1 illustrate all the possible symmetry groups for frieze patterns that can occur. Notice that Pattern 1 is essentially equivalent to that of Example 2.20 since it only has translational symmetries.

2.22. EXAMPLE. Find the symmetry subgroup of Pattern 2.

SOLUTION. Let $S \subseteq \mathbb{R}^2$ be this pattern. Suppose that this lies along the x-axis with the origin midway up an edge. Then if $\mathbf{t}$ is the vector pointing in the positive x-direction with length equal to the width of a block, $\mathrm{Euc}(2)_S$ contains the cyclic subgroup generated by $\mathrm{Trans}_{2\mathbf{t}}$,

$$\langle \mathrm{Trans}_{2\mathbf{t}} \rangle = \{\mathrm{Trans}_{2k\mathbf{t}} : k \in \mathbb{Z}\} \leqslant \mathrm{Euc}(2)_S.$$

The glide reflection that reflects in the x-axis then moves each block by $\mathbf{t}$ has Seitz symbol $(R_x \mid \mathbf{t})$ where

$$R_x = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Clearly every symmetry is either a translation $\mathrm{Trans}_{2k\mathbf{t}} = (I \mid 2k\mathbf{t})$ for some $k \in \mathbb{Z}$, or a glide reflection of the form $(R_x \mid (2k+1)\mathbf{t})$ for some $k \in \mathbb{Z}$, where

$$(R_x \mid (2k+1)\mathbf{t}) = (I \mid 2k\mathbf{t})(R_x \mid \mathbf{t}).$$

So we have

$$\mathrm{Euc}(2)_S = \langle \mathrm{Trans}_{2\mathbf{t}} \rangle \cup \{(R_x \mid (2k+1)\mathbf{t}) : k \in \mathbb{Z}\}.$$ □

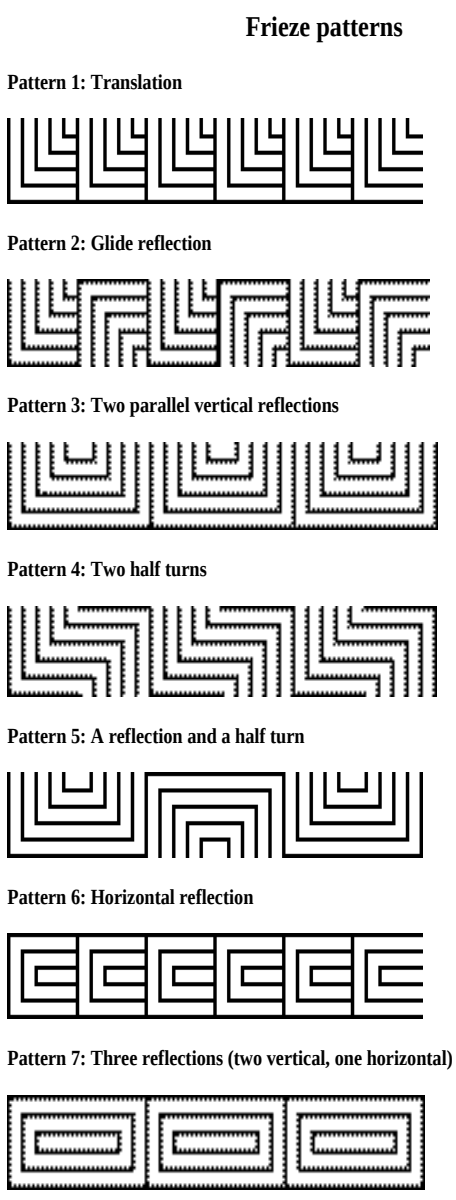
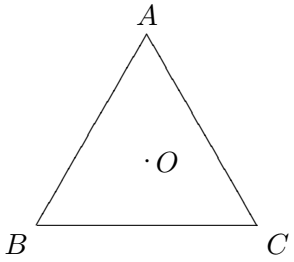


FIGURE 2.1. Frieze Patterns

2.23. Example. Let $\triangle \subseteq \mathbb{R}^2$ be an equilateral triangle with vertices A, B, C.

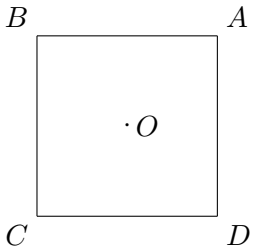


A symmetry of $\triangle$ is defined once we know where the vertices go, hence there are as many symmetries as permutations of the set $\{A, B, C\}$. Each symmetry can be described using permutation notation and we obtain the six distinct symmetries

$$\begin{pmatrix} A & B & C \\ A & B & C \end{pmatrix} = \iota, \qquad \begin{pmatrix} A & B & C \\ B & C & A \end{pmatrix} = (A\ B\ C), \qquad \begin{pmatrix} A & B & C \\ C & A & B \end{pmatrix} = (A\ C\ B),$$
$$\begin{pmatrix} A & B & C \\ A & C & B \end{pmatrix} = (B\ C), \qquad \begin{pmatrix} A & B & C \\ C & B & A \end{pmatrix} = (A\ C), \qquad \begin{pmatrix} A & B & C \\ B & A & C \end{pmatrix} = (A\ B).$$

Therefore we have $|\operatorname{Euc}(2)_{\triangle}| = 6$. Notice that the identity and the two 3-cycles represent rotations about O, while each of the three transpositions represents a reflection in lines through O and a vertex.

2.24. Example. Let $\square \subseteq \mathbb{R}^2$ be the square centred at the origin O and whose vertices are at the points $A(1, 1)$, $B(-1, 1)$, $C(-1, -1)$, $D(1, -1)$.



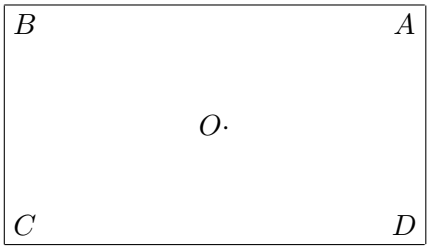
Then a symmetry is defined by sending A to any one of the 4 vertices then choosing how to send B to one of the 2 adjacent vertices. This gives a total of $4 \times 2 = 8$ such symmetries, therefore $|\operatorname{Euc}(2)_{\square}| = 8$.

Again we can describe symmetries in terms of their effect on the vertices. Here are the eight elements of $\operatorname{Euc}(2)_{\square}$ described in permutation notation.

$$\begin{pmatrix} A & B & C & D \\ A & B & C & D \end{pmatrix} = \iota, \qquad \begin{pmatrix} A & B & C & D \\ B & C & D & A \end{pmatrix} = (A\ B\ C\ D),$$
$$\begin{pmatrix} A & B & C & D \\ C & D & A & B \end{pmatrix} = (A\ C)(B\ D), \qquad \begin{pmatrix} A & B & C & D \\ D & A & B & C \end{pmatrix} = (A\ D\ C\ B),$$
$$\begin{pmatrix} A & B & C & D \\ A & D & C & B \end{pmatrix} = (B\ D), \qquad \begin{pmatrix} A & B & C & D \\ D & C & B & A \end{pmatrix} = (A\ D)(B\ C),$$
$$\begin{pmatrix} A & B & C & D \\ C & B & A & D \end{pmatrix} = (A\ C), \qquad \begin{pmatrix} A & B & C & D \\ B & A & D & C \end{pmatrix} = (A\ B)(C\ D).$$

Each of the two 4-cycles represents a rotation through a quarter turn about O, while $(A\ C)(B\ D)$ represents a half turn. The transpositions $(B\ D)$ and $(A\ C)$ represent reflections in the diagonals while $(A\ D)(B\ C)$ and $(A\ B)(C\ D)$ represent reflections in the lines joining opposite midpoints of edges.

2.25. Example. Let $R \subseteq \mathbb{R}^2$ be the rectangle centred at the origin O with vertices at $A(2,1)$, $B(-2,1)$, $C(-2,-1)$, $D(2,-1)$.



A symmetry can send A to any of the vertices, and then the long edge AB must go to the longer of the adjacent edges. This gives a total of 4 such symmetries, thus $|\operatorname{Euc}(2)_R| = 4$.

Again we can describe symmetries in terms of their effect on the vertices. Here are the four elements of $\operatorname{Euc}(2)_R$ described using permutation notation.

$$\begin{pmatrix} A & B & C & D \\ A & B & C & D \end{pmatrix} = \iota, \qquad \begin{pmatrix} A & B & C & D \\ B & A & D & C \end{pmatrix} = (A\ B)(C\ D),$$

$$\begin{pmatrix} A & B & C & D \\ C & D & A & B \end{pmatrix} = (A\ C)(B\ D), \qquad \begin{pmatrix} A & B & C & D \\ D & C & B & A \end{pmatrix} = (A\ D)(B\ C).$$

$(A\ C)(B\ D)$ represents a half turn about O while $(A\ B)(C\ D)$ and $(A\ D)(B\ C)$ represent reflections in lines joining opposite midpoints of edges.

2.26. Example. Given a regular n-gon (*i.e.*, a regular polygon with n sides all of the same length and n vertices $V_1, V_2, \ldots, V_n$), the symmetry group is a *dihedral group of order* $2n$, with elements

$$\iota, \alpha, \alpha^2, \ldots, \alpha^{n-1}, \beta, \alpha\beta, \alpha^2\beta, \ldots, \alpha^{n-1}\beta,$$

where α^k is an anticlockwise rotation through $2\pi k/n$ about the centre and β is a reflection in the line through V_1 and the centre. In fact each of the elements $\alpha^2\beta$ is a reflection in a line through the centre. Moreover we have

$$|\alpha| = n,\ |\beta| = 2,\ \beta\alpha\beta = \alpha^{n-1} = \alpha^{-1}.$$

In permutation notation this becomes the n-cycle

$$\alpha = (V_1\ V_2\ \cdots\ V_n),$$

but β is more complicated to describe since it depends on whether n is even or odd.

For example, if $n = 6$ we have

$$\alpha = (V_1\ V_2\ V_3\ V_4\ V_5\ V_6), \quad \beta = (V_2\ V_6)(V_3\ V_5),$$

while if $n = 7$

$$\alpha = (V_1\ V_2\ V_3\ V_4\ V_5\ V_6\ V_7), \quad \beta = (V_2\ V_7)(V_3\ V_6)(V_4\ V_5).$$

We have seen that when $n = 3$, $\operatorname{Euc}(2)_\triangle$ is the permutation group of the vertices and so D_6 is essentially the same group as S_6.

If we take the regular n-gon centred at the origin with the first vertex V_1 at $(1,0)$, the generators α and β can be represented as $(A \mid \mathbf{0})$ and $(B \mid \mathbf{0})$ using the matrices

$$A = \begin{bmatrix} \cos 2\pi/n & -\sin 2\pi/n \\ \sin 2\pi/n & \cos 2\pi/n \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

In this case the symmetry group is *the* dihedral group of order $2n$,

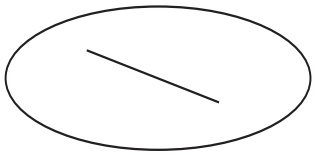
$$\mathrm{D}_{2n} = \{\iota, \alpha, \alpha^2, \ldots, \alpha^{n-1}, \beta, \alpha\beta, \alpha^2\beta, \ldots, \alpha^{n-1}\beta\} \leqslant \mathrm{O}(2).$$

Notice that the subgroup of direct symmetries is

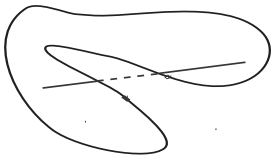
$$\mathrm{D}^+_{2n} = \{\iota, \alpha, \alpha^2, \ldots, \alpha^{n-1}\} \leqslant \mathrm{SO}(2).$$

More generally we have the following Theorem. A *convex* region of is a subset $S \subseteq \mathbb{R}^2$ in which for each pair of points $\mathbf{x}, \mathbf{y} \in S$, the line segment joining them lies in S, *i.e.*,

$$\{t\mathbf{x} + (1-t)\mathbf{y} : 0 \leqslant t \leqslant 1\} \subseteq S.$$



Convex



Non-convex

2.27. THEOREM. *If $V_1, \ldots, V_n$ are the vertices in order of a polygon which bounds a convex region $\mathcal{P}$ of $\mathbb{R}^2$ containing a point not on the boundary, then* $\mathrm{Euc}(2)_{\mathcal{P}}$ *can be identified with a subgroup of the permutation group* $\mathrm{Perm}_{\{V_1,\ldots,V_n\}}$ *of the vertices.*

5. Similarity of isometries and subgroups of the Euclidean group

It is often the case that two subsets of the plane have the 'same' symmetry subgroups. For example, any two frieze patterns which only have translational symmetries are the same in this sense. We need to make this notion more precise.

A *dilation* or *scaling* of the plane is a function $H\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ which has the form

$$H(\mathbf{x}) = \delta(\mathbf{x} - \mathbf{c}) + \mathbf{c},$$

where $\delta > 0$ is the *dilation factor* and $\mathbf{c}$ is the *centre of the dilation*. It is easy to see that $H(\mathbf{c}) = \mathbf{c}$ and

$$|H(\mathbf{x}) - \mathbf{c}| = \delta|\mathbf{x} - \mathbf{c}|,$$

so the effect of H is the expand or contract the plane radially from the point $\mathbf{c}$. We can rewrite the above formula to give

$$H(\mathbf{x}) = \delta\mathbf{x} + (1-\delta)\mathbf{c},$$

so we can express H as the Seitz symbol $(\delta I \mid (1-\delta)\mathbf{c})$. Of course, if $\delta = 1$ then this is just the identity function, otherwise it is not an isometry.

We can compose a dilation $(\delta I \mid (1-\delta)\mathbf{c})$ with an isometry $(A \mid \mathbf{t})$ to give a new function $\mathbb{R}^2 \longrightarrow \mathbb{R}^2$ with Seitz symbol

$$(\delta I \mid (1-\delta)\mathbf{c})(A \mid \mathbf{t}) = (\delta A \mid \delta\mathbf{t} + (1-\delta)\mathbf{c}).$$

This has the form $(\delta A \mid \mathbf{s})$ for some vector $\mathbf{s}$ and orthogonal matrix A. Provided that $\delta \neq 1$, the matrix $(I - \delta A)$ is invertible (this uses knowledge of the *eigenvalues* of A) and so the vector $\mathbf{s}_0 = (I - \delta A)^{-1}\mathbf{s}$ is the unique fixed point of this transformation. Indeed, this transformation amounts to a rotation or reflection about $\mathbf{s}_0$ followed by a dilation by δ centred at this point. We will call such a transformation a *similarity transformation* of the plane centred at $\mathbf{s}_0$.

Now suppose that $F_1, F_2\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ are two isometries of the plane. If $H\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is a similarity transformation, then F_2 is *H-similar to F_1* if

$$F_2 = H \circ F_1 \circ H^{-1}.$$

We will sometimes use the notation

$$H_*F = H \circ F \circ H^{-1}.$$

We also say that F_2 is *similar to F_1* if there is some similarity transformation H for which F_2 is H-similar to F_1.

If $\Gamma \subseteq \mathrm{Euc}(2)$ then we set

$$H_*\Gamma = \{H_*F : F \in \Gamma\} \subseteq \mathrm{Euc}(2).$$

2.28. LEMMA. *If $\Gamma \leqslant \mathrm{Euc}(2)$ then $H_*\Gamma \leqslant \mathrm{Euc}(2)$.*

PROOF. For example, if $F_1, F_2 \in \Gamma$ then

$$(H_*F_1) \circ (H_*F_2) = H \circ F_1 \circ H^{-1} \circ H \circ F_2 \circ H^{-1} = H \circ (F_1 \circ F_2) \circ H^{-1} = H_*(F_1 \circ F_2). \quad \square$$

For two subgroups $\Gamma_1 \leqslant \mathrm{Euc}(2)$ and $\Gamma_2 \leqslant \mathrm{Euc}(2)$, Γ_2 is *H-similar to* Γ_1 if

$$\Gamma_2 = H_*\Gamma_1.$$

We also say that Γ_2 is *similar to* Γ_1 if there is some similarity transformation H for which Γ_2 is H-similar to Γ_1.

2.29. EXAMPLE. Consider any two rotations $\mathrm{Rot}_{P,\theta}$ and $\mathrm{Rot}_{Q,\theta}$ through the same angle θ. Then $\mathrm{Rot}_{Q,\theta}$ is similar to $\mathrm{Rot}_{P,\theta}$.

PROOF. Let $\mathbf{t} = \overrightarrow{PQ} = \mathbf{q} - \mathbf{p}$. Then $\mathrm{Trans}_{\mathbf{t}} \circ \mathrm{Rot}_{P,\theta} \circ \mathrm{Trans}_{-\mathbf{t}}$ is a rotation through the angle θ which has the following effect on the point Q,

$$\begin{aligned}\mathrm{Trans}_{\mathbf{t}} \circ \mathrm{Rot}_{P,\theta} \circ \mathrm{Trans}_{-\mathbf{t}}(\mathbf{q}) &= (\mathbf{q} - \mathbf{p}) + \mathrm{Rot}_{P,\theta}(\mathbf{q} - (\mathbf{q} - \mathbf{p})) \\ &= (\mathbf{q} - \mathbf{p}) + \mathrm{Rot}_{P,\theta}(\mathbf{p}) \\ &= (\mathbf{q} - \mathbf{p}) + \mathbf{p} = \mathbf{q}.\end{aligned}$$

Hence this rotation fixes Q, which must be its centre. Therefore

$$\mathrm{Trans}_{\mathbf{t}} \circ \mathrm{Rot}_{P,\theta} \circ \mathrm{Trans}_{-\mathbf{t}} = \mathrm{Rot}_{Q,\theta},$$

and $\mathrm{Rot}_{Q,\theta}$ is similar to $\mathrm{Rot}_{P,\theta}$. $\square$

2.30. EXAMPLE. Let $\mathbf{u}$ and $\mathbf{v}$ be any two non-zero vectors. Then $\mathrm{Trans}_{\mathbf{v}}$ is similar to $\mathrm{Trans}_{\mathbf{u}}$.

SOLUTION. Let $\delta = |\mathbf{v}|/|\mathbf{u}|$. If the angle between $\mathbf{u}$ and $\mathbf{v}$ is θ, then

$$\mathbf{v} = \delta\, \mathrm{Rot}_{O,\theta}(\mathbf{u}).$$

Now consider the similarity transformation obtained by composing a rotation with a dilation,

$$H = \delta\, \mathrm{Rot}_{O,\theta},$$

and with inverse

$$H^{-1} = (1/\delta)\, \mathrm{Rot}_{O,-\theta}\,.$$

Then we have

$$\begin{aligned}H \circ \mathrm{Trans}_{\mathbf{u}} \circ H^{-1}(\mathbf{x}) &= H(\mathbf{u} + (1/\delta)\, \mathrm{Rot}_{O,-\theta}\, \mathbf{x}) \\ &= \delta\, \mathrm{Rot}_{O,\theta}(\mathbf{u} + (1/\delta)\, \mathrm{Rot}_{O,-\theta}\, \mathbf{x}) \\ &= \delta\, \mathrm{Rot}_{O,\theta}(\mathbf{u}) + \delta\, \mathrm{Rot}_{O,\theta}((1/\delta)\, \mathrm{Rot}_{O,-\theta}\, \mathbf{x}) \\ &= \mathbf{v} + \mathrm{Rot}_{O,\theta} \circ \mathrm{Rot}_{O,-\theta}(\mathbf{x}) \\ &= \mathbf{v} + \mathbf{x} = \mathrm{Trans}_{\mathbf{v}}(\mathbf{x}).\end{aligned}$$

This show that

$$\mathrm{Trans}_{\mathbf{v}} = H \circ \mathrm{Trans}_{\mathbf{u}} \circ H^{-1},$$

so $\mathrm{Trans}_{\mathbf{v}}$ is similar to $\mathrm{Trans}_{\mathbf{u}}$. $\square$

Using this result it is easy to deduce that $\langle \mathrm{Trans}_{\mathbf{v}} \rangle$ is similar to $\langle \mathrm{Trans}_{\mathbf{u}} \rangle$.

2.31. EXAMPLE. Let Δ_1 and Δ_2 be two equilateral triangles. Show that the symmetry subgroups $\Gamma_1 = \mathrm{Euc}(2)_{\Delta_1}$ and $\Gamma_2 = \mathrm{Euc}(2)_{\Delta_2}$ are similar.

SOLUTION. Let the centres be at $C_1\mathbf{c}_1$ and $C_2\mathbf{c}_2$ and let the sides be of lengths ℓ_1 and ℓ_2. Let the vertices of the triangles be U_1, U_2, U_3 and V_1, V_2, V_3 taken in order in the anti-clockwise direction.

Setting $\delta = \ell_2/\ell_1$, we construct the following sequence of transformations $\mathbb{R}^2 \longrightarrow \mathbb{R}^2$.

- H_1 is the translation which moves C_1 to C_2.
- H_2 is the dilation by δ centred at C_1.

- H_3 is the rotation about C_1 which sends each U_i to $H_1^{-1} \circ H_2^{-1}(V_i)$.

Now take $H = H_3 \circ H_2 \circ H_1$.

Now let $F \in \Gamma_1$. Then for each vertex V_i, $F(V_i)$ is a vertex and the effect of F on the vertices determines F. Then for each V_i,

$$H_*F(V_i) = H \circ F \circ H^{-1}(V_i) = H_3 \circ H_2 \circ H_1 \circ F(U_i),$$

which is a vertex of Δ_2, so H_*F is indeed a symmetry of Δ_2. It is easy to see that every symmetry of Δ_2 arises as H_*F for some F. Hence $\Gamma_2 = H_*\Gamma_1$ and so Γ_2 is similar to Γ_1. □

6. Finite subgroups of the Euclidean group of the plane

In this section we will describe all finite subgroups of Euc(2), up to similarity transformations. First we will show that every finite subgroup has a *fixed point*, *i.e.*, a point fixed by *every* element of Γ.

2.32. Theorem. *Let $\Gamma \leqslant \mathrm{Euc}(2)$ be a finite subgroup. Then there is a point of $\mathbb{R}^2$ which is fixed by every element of Γ.*

Proof. Let the distinct elements of Γ be $F_1, \ldots, F_n$, where $n = |\Gamma|$. Let $\mathbf{p} \in \mathbb{R}^2$ be (the position vector of) any point. Define

$$\mathbf{p}_0 = \frac{1}{n}F_1(\mathbf{p}) + \cdots + \frac{1}{n}F_n(\mathbf{p}).$$

For any $k = 1, \ldots, n$, by a result contained in Exercise 1.10, we have

$$F_k(\mathbf{p}_0) = \frac{1}{n}F_kF_1(\mathbf{p}) + \cdots + \frac{1}{n}F_kF_n(\mathbf{p}).$$

Now if $F_kF_i = F_kF_j$, then $F_k^{-1}F_kF_i = F_k^{-1}F_kF_j$ and so $F_i = F_j$. Also, every F_r can be written as $F_r = F_k(F_k^{-1}F_r)$ where $F_k^{-1}F_r \in \Gamma$ has the form $F_k^{-1}F_r = F_s$ for some s and therefore $F_r = F_kF_s$. So in the above expression for $F_k(\mathbf{p}_0)$, the terms are the same as those in the formula for $\mathbf{p}_0$ apart from the order in which they appear. This shows that $F_k(\mathbf{p}_0) = \mathbf{p}_0$. □

2.33. Remark. In the proof we can of course take $\mathbf{p} = \mathbf{0}$, but any initial value will do. Also, if Γ contains a non-trivial rotation then it has exactly one fixed point, so it doesn't matter what we choose for $\mathbf{p}$ since $\mathbf{p}_0$ will be this unique fixed point.

2.34. Corollary. *Let $\Gamma \leqslant \mathrm{Euc}(2)$ be a finite subgroup. Then Γ is similar to a finite subgroup of the orthogonal group* O(2).

Proof. Let $\mathbf{q}$ be any fixed point of Γ. Setting $H = \mathrm{Trans}_{-\mathbf{q}}$, for $F \in \Gamma$ we have

$$H_*F(\mathbf{0}) = \mathrm{Trans}_{-\mathbf{q}} F \,\mathrm{Trans}_{\mathbf{q}}(\mathbf{0}) = \mathrm{Trans}_{-\mathbf{q}} F(\mathbf{q}+\mathbf{0}) = F(\mathbf{q}) - \mathbf{q} = \mathbf{q} - \mathbf{q} = \mathbf{0}.$$

Thus $H_*\Gamma \leqslant \mathrm{O}(2)$. □

Let us now consider finite subgroups of O(2). First we describe all commutative subgroups, beginning with finite groups of rotations. Recall that

$$\mathrm{SO}(2) = \{(A \mid \mathbf{0}) \in \mathrm{O}(2) : \det A = 1\} \leqslant \mathrm{O}(2) \leqslant \mathrm{Euc}(2).$$

Furthermore, if $(A \mid \mathbf{0}) \in \mathrm{SO}(2)$ then

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

for some angle $\theta \in \mathbb{R}$. Notice that for all $\theta, \varphi \in \mathbb{R}$,

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix} = \begin{bmatrix} \cos(\theta+\varphi) & -\sin(\theta+\varphi) \\ \sin(\theta+\varphi) & \cos(\theta+\varphi) \end{bmatrix}$$
$$= \begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix},$$

which shows that SO(2) is in fact commutative.

2.35. Proposition. *Let $\Gamma \leqslant \mathrm{SO}(2)$ be a finite subgroup. Then Γ is cyclic with a generator of the form $\langle (A \mid \mathbf{0}) \rangle$ for some matrix*

$$A = \begin{bmatrix} \cos 2\pi/d & -\sin 2\pi/d \\ \sin 2\pi/d & \cos 2\pi/d \end{bmatrix}$$

where $d = 1, 2, \ldots$.

Proof. We won't prove this here, but note that proofs can be found in many textbooks or in 3H Algebra.

The element $(A \mid \mathbf{0}) \in \Gamma$ can be chosen so that

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix},$$

where the angle $\theta \in [0, 2\pi)$ is as small as possible. Since Γ is a finite group, every element has finite order, and so $\theta = 2k\pi/d$ for some $d = 1, 2, \ldots$ and $k = 0, 1, \ldots, (d-1)$. It is always possible to arrange things so that $k = 1$. Notice that $\langle (A \mid \mathbf{0}) \rangle$ represents rotation about the origin through the angle $2\pi/d$. □

2.36. Proposition. *Let $\Gamma \leqslant \mathrm{O}(2)$ be a finite subgroup. Then either $\Gamma \leqslant \mathrm{SO}(2)$ or Γ is similar to a dihedral group D_{2n} for some $n \geqslant 1$.*

Proof. Again we will not give a proof. □

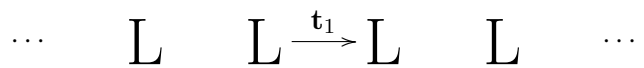
7. Frieze patterns and their symmetry groups

In this section we will see how frieze patterns can be understood by classifying their symmetry groups up to similarity. Recall that the group of translational symmetries of a frieze pattern is infinite and cyclic. We will discuss in detail the seven distinct patterns shown in the handout on Frieze Patterns.

Pattern 1. This only has translational symmetries. There is a smallest translation vector $\mathbf{t}_1$ say, with

$$\mathrm{Euc}(2)_{\text{Pattern 1}} = \langle \mathrm{Trans}_{\mathbf{t}_1} \rangle .$$

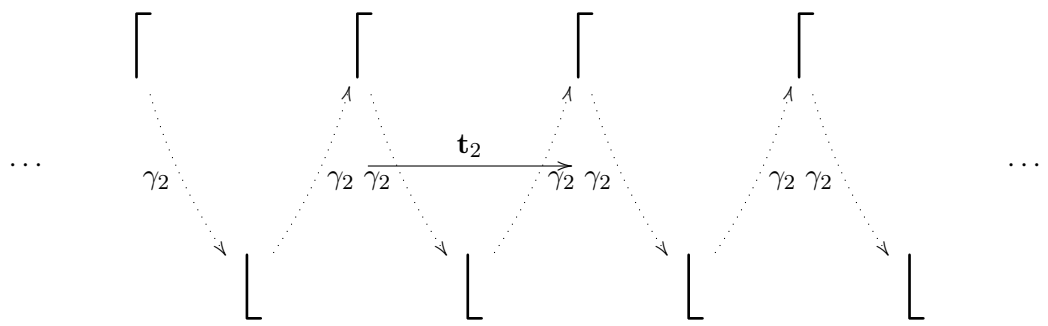
Notice that there are no points simultaneously fixed by all the elements of this symmetry group, hence there can be no non-trivial finite subgroups.



Pattern 2. This has a glide reflection γ_2 whose square is a translation by $\mathbf{t}_2$, say. The symmetry group is infinite and cyclic,

$$\mathrm{Euc}(2)_{\text{Pattern 2}} = \langle \gamma_2 \rangle .$$

Again there are no points simultaneously fixed by all the elements of this symmetry group, so there are no non-trivial finite subgroups.



Pattern 3. This has a smallest translation vector $\mathbf{t}_3$ say, but also some reflections in lines perpendicular to $\mathbf{t}_3$. If we choose such a reflection σ_3 in a line $\mathcal{L}$, say, then any other such reflection is either in a parallel line $\mathcal{L}_{k\mathbf{t}_3}$ obtained by translating $\mathcal{L}$ by some integer multiple $k\mathbf{t}_3$, or reflection in a line $\mathcal{L}_{(\ell+1/2)\mathbf{t}_3}$ obtained by translating by some multiple $(\ell+1/2)\mathbf{t}_3$ for some $\ell \in \mathbb{Z}$. However, from the proof of Proposition 1.6, we know that

$$\begin{aligned}\mathrm{Refl}_{\mathcal{L}_{k\mathbf{t}_3}} &= \mathrm{Trans}_{2k\mathbf{t}_3} \circ \mathrm{Refl}_{\mathcal{L}} = \mathrm{Trans}_{2k\mathbf{t}_3} \circ \sigma_3,\\ \mathrm{Refl}_{\mathcal{L}_{(\ell+1/2)\mathbf{t}_3}} &= \mathrm{Trans}_{(2\ell+1)\mathbf{t}_3} \circ \mathrm{Refl}_{\mathcal{L}} = \mathrm{Trans}_{(2\ell+1)\mathbf{t}_3} \circ \sigma_3.\end{aligned}$$

This group is not commutative, *e.g.*, $\sigma_3 \circ \mathrm{Trans}_{\mathbf{t}_3} = \mathrm{Trans}_{-\mathbf{t}_3} \circ \sigma_3$. Writing

$$\alpha = \mathrm{Trans}_{\mathbf{t}_3}, \quad \beta = \sigma_3,$$

then we see that abstractly this group has the following form,

$$\{\alpha^r : r \in \mathbb{Z}\} \cup \{\alpha^r\beta : r \in \mathbb{Z}\},$$

where α, β satisfy the following *relations*:

$$\beta^2 = \iota, \quad \beta\alpha\beta = \alpha^{-1}.$$

It is easy to see that for $r \in \mathbb{Z}$,

$$\beta\alpha^r\beta = \alpha^{-r}.$$

This is an infinite version of the dihedral groups D_{2n}, and it is often referred to as D_∞. We will use the notation

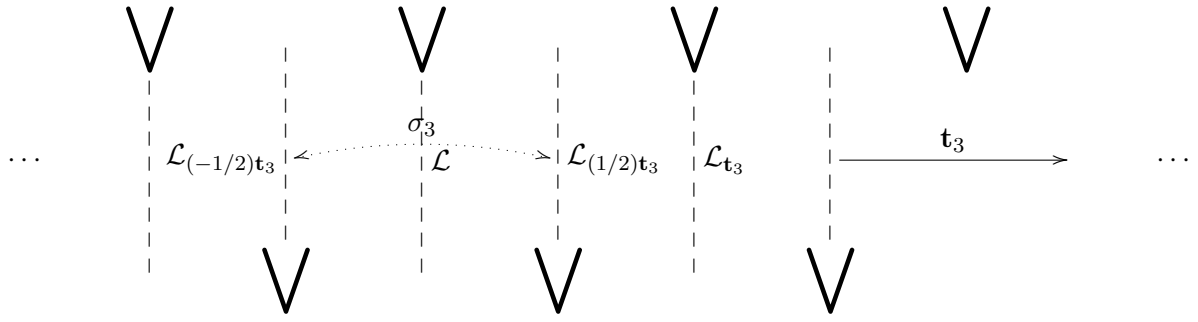
$$\mathrm{Euc}(2)_{\text{Pattern 3}} = \mathrm{D}_{\infty,3}.$$

Notice that every such symmetry for this frieze is obtained by combining a power of $\mathrm{Trans}_{\mathbf{t}_3}$ with the zeroth or first power of σ_3.

There are points fixed simultaneously by all the elements of this symmetry group, and indeed there are some non-trivial finite subgroups of $\mathrm{Euc}(2)_{\text{Pattern 3}}$. These are obtained by taking one of the vertical reflections $\mathrm{Refl}_{\mathcal{L}_{k\mathbf{t}_3}}$, $\mathrm{Refl}_{\mathcal{L}_{(\ell+1/2)\mathbf{t}_3}}$ and considering the cyclic subgroups (each with two elements)

$$\left\langle \mathrm{Refl}_{\mathcal{L}_{k\mathbf{t}_3}} \right\rangle, \quad \left\langle \mathrm{Refl}_{\mathcal{L}_{(\ell+1/2)\mathbf{t}_3}} \right\rangle.$$

Each of these subgroups fixes all the points on the corresponding line of reflection and no others.



Pattern 4. This time there is a shortest translation vector $\mathbf{t}_4$ say. There are also two types of rotations, namely half rotations about points marked $\bullet$ and $\times$. If we choose one of these to be ρ_4 centred at $\mathbf{c}$ say, then one of the other type is obtained by composing it with translation by $\mathbf{t}_4$, to give a half rotation about the point $\mathbf{c} + (1/2)\mathbf{t}_4$, *i.e.*,

$$\rho_4' = \mathrm{Trans}_{\mathbf{t}_4} \circ \rho_4.$$

More generally, the half rotation symmetries are of the form

$$\mathrm{Trans}_{2k\mathbf{t}_4} \circ \rho_4, \ \mathrm{Trans}_{(2k+1)\mathbf{t}_4} \circ \rho_4 \quad (k \in \mathbb{Z}).$$

Each of these generates a cyclic subgroup of order 2,

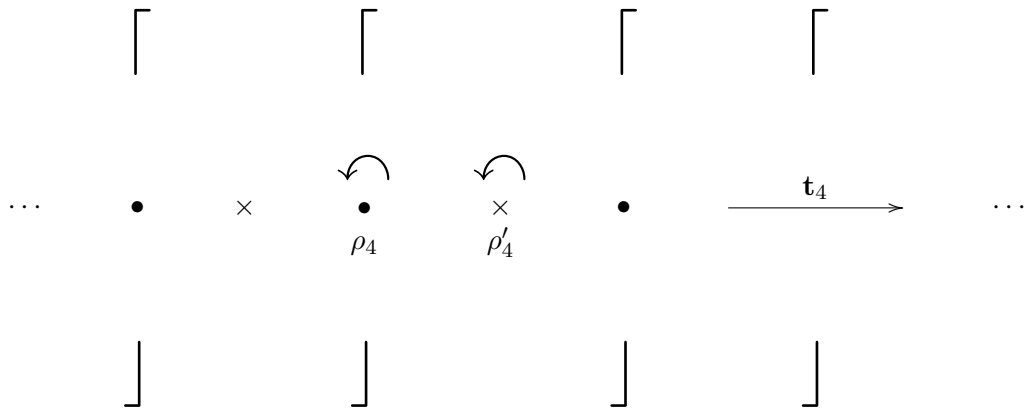
$$\left\langle \mathrm{Trans}_{2k\mathbf{t}_4} \circ \rho_4 \right\rangle, \quad \left\langle \mathrm{Trans}_{(2k+1)\mathbf{t}_4} \circ \rho_4 \right\rangle.$$

We also have

$$\rho_4 \circ \mathrm{Trans}_{\mathbf{t}_4} \circ \rho_4 = \mathrm{Trans}_{-\mathbf{t}_4},$$

so the symmetry group here is again a dihedral group with generators

$$\alpha = \mathrm{Trans}_{\mathbf{t}_4}, \quad \beta = \rho_4.$$



Pattern 5. Let $\mathbf{t}_5$ be a smallest translation vector. There is a glide reflection γ_5 whose square is $\gamma_5 \circ \gamma_5 = \mathrm{Trans}_{\mathbf{t}_5}$. There is also a reflection symmetry σ_5 in the vertical line $\mathcal{L}$. Let $\mathcal{L}_{\mathbf{v}}$ denote the line obtained by translation of $\mathcal{L}$ by a vector $\mathbf{v}$. Then the half rotation ρ_5 about one of the points marked $\bullet$ at $(-1/4)\mathbf{t}_5$ away from $\mathcal{L}$ agrees with the composition

$$\rho_5 = \gamma_5 \circ \sigma_5.$$

The following equations also hold:

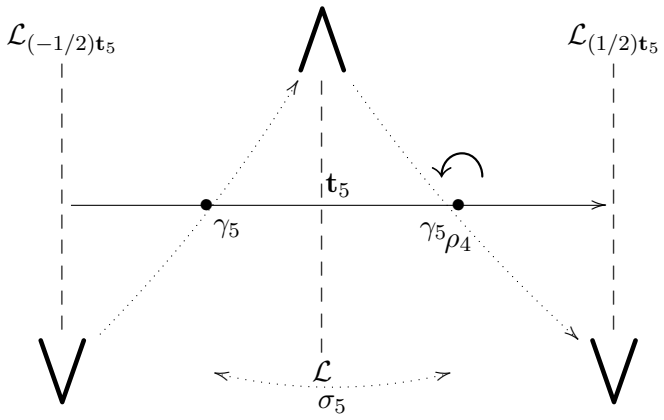
$$\sigma_5 \circ \gamma_5 \circ \sigma_5 = {\gamma_5}^{-1}, \quad \sigma_5 \circ \mathrm{Trans}_{\mathbf{t}_5} \circ \sigma_5 = \mathrm{Trans}_{-\mathbf{t}_5} = {\mathrm{Trans}_{\mathbf{t}_5}}^{-1}$$

The remaining rotations and glide reflections have the form

$$\mathrm{Trans}_{k\mathbf{t}_5} \circ \rho_5, \quad \mathrm{Trans}_{k\mathbf{t}_5} \circ \sigma_5$$

for $k \in \mathbb{Z}$. The symmetry group is again a dihedral group with generators

$$\alpha = \gamma_5, \quad \beta = \sigma_5.$$



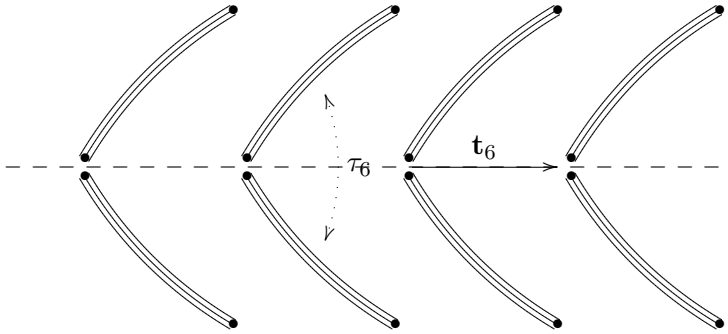
Pattern 6. This time there is a shortest translation vector $\mathrm{Trans}_{\mathbf{t}_6}$ and a horizontal reflection τ_6, where

$$\tau_6 \circ \mathrm{Trans}_{\mathbf{t}_6} = \mathrm{Trans}_{\mathbf{t}_6} \circ \tau_6.$$

There are also glide reflections of the form

$$\tau_6 \circ \mathrm{Trans}_{k\mathbf{t}_6} = \mathrm{Trans}_{k\mathbf{t}_6} \circ \tau_6$$

for some $k \in \mathbb{Z}$. This group is commutative.



Pattern 7. There is a shortest translation vector $\mathrm{Trans}_{\mathbf{t}_7}$, a horizontal reflection τ_7 and a vertical reflection σ_7 in lines that meet at a point marked below with $\bullet$. The $\bullet$ points are whole multiples $k\mathbf{t}_7$ apart, while the $\times$ points are translated by $(k+1/2)\mathbf{t}_7$ from them for $k \in \mathbb{Z}$. The composition

$$\rho_7 = \tau_7 \circ \sigma_7 = \sigma_7 \circ \tau_7.$$

is a half rotation.

There are rotations about the points marked $\bullet, \times$

$$\mathrm{Trans}_{k\mathbf{t}_7} \circ \rho_7 \quad (k \in \mathbb{Z}),$$

and reflections in vertical lines through the points $\bullet, \times$

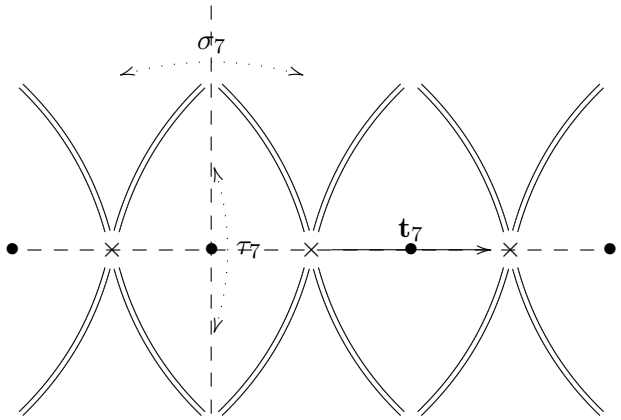
$$\mathrm{Trans}_{k\mathbf{t}_7} \circ \sigma_7 = \mathrm{Trans}_{(k+1)\mathbf{t}_7} \circ \rho_7 \quad (k \in \mathbb{Z}),$$

There are also glide reflections

$$\tau_7 \circ \mathrm{Trans}_{\ell\mathbf{t}_7} = \mathrm{Trans}_{\ell\mathbf{t}_7} \circ \tau_7 \quad (\ell \in \mathbb{Z}).$$

τ_7 commutes with every other symmetry. However, the symmetry group here is not commutative, and indeed contains a dihedral group generated by

$$\alpha = \mathrm{Trans}_{\mathbf{t}_7}, \quad \beta = \rho_7.$$



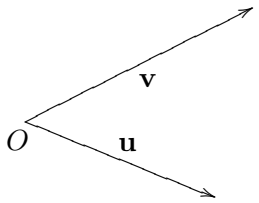
8. Wallpaper patterns and their symmetry groups

A *wallpaper pattern* in the plane is a subset $\mathcal{W} \subseteq \mathbb{R}^2$ whose group of translational symmetries $\mathrm{Trans}(2)_W$ has two *linearly independent* generators $\mathbf{u}, \mathbf{v}$, so

$$\mathrm{Trans}(2)_{\mathcal{W}} = \{\mathrm{Trans}_{m\mathbf{u}+n\mathbf{v}} : m, n \in \mathbb{Z}\} = \langle \mathrm{Trans}_{\mathbf{u}}, \mathrm{Trans}_{\mathbf{v}} \rangle, \tag{2.2}$$

with neither of $\mathbf{u}$ and $\mathbf{v}$ being a multiple of the other. This means that the angle between them satisfies the pair of inequalities

$$0 < \cos^{-1} \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}|\,|\mathbf{v}|} < \pi$$



or equivalently

$$-1 < \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}|\,|\mathbf{v}|} < 1.$$

The set of points

$$\mathrm{span}_{\mathbb{Z}}(\mathbf{u}, \mathbf{v}) = \{m\mathbf{u} + n\mathbf{v} : m, n \in \mathbb{Z}\}$$

is called the *lattice spanned by the vectors* $\mathbf{u}, \mathbf{v}$ and it forms a commutative group under addition. In the situation where Equation (2.2) holds we have

$$\mathrm{Trans}(2)_{\mathcal{W}} = \{\mathrm{Trans}_{\mathbf{t}} : \mathbf{t} \in \mathrm{span}_{\mathbb{Z}}(\mathbf{u}, \mathbf{v})\}$$

and call $\mathrm{span}_{\mathbb{Z}}(\mathbf{u}, \mathbf{v})$ the *translation lattice* of $\mathcal{W}$. Examples of all the basic types of lattices that occur are shown in Figures 2.2–2.6.

We will discuss plane patterns making use of their translation lattices. First we remark that if the symmetry group of a plane pattern $\mathcal{W}$ contains the group of translations

$$\langle \mathrm{Trans}_{\mathbf{u}}, \mathrm{Trans}_{\mathbf{v}} \rangle = \{(m\mathbf{u} + n\mathbf{v} \mid \mathbf{0}) : m, n \in \mathbb{Z}\}.$$

then given any point W with position vector $\mathbf{w} \in \mathcal{W}$, each of the points

$$\mathbf{w} + m\mathbf{u} + n\mathbf{v} \quad (m, n \in \mathbb{Z})$$

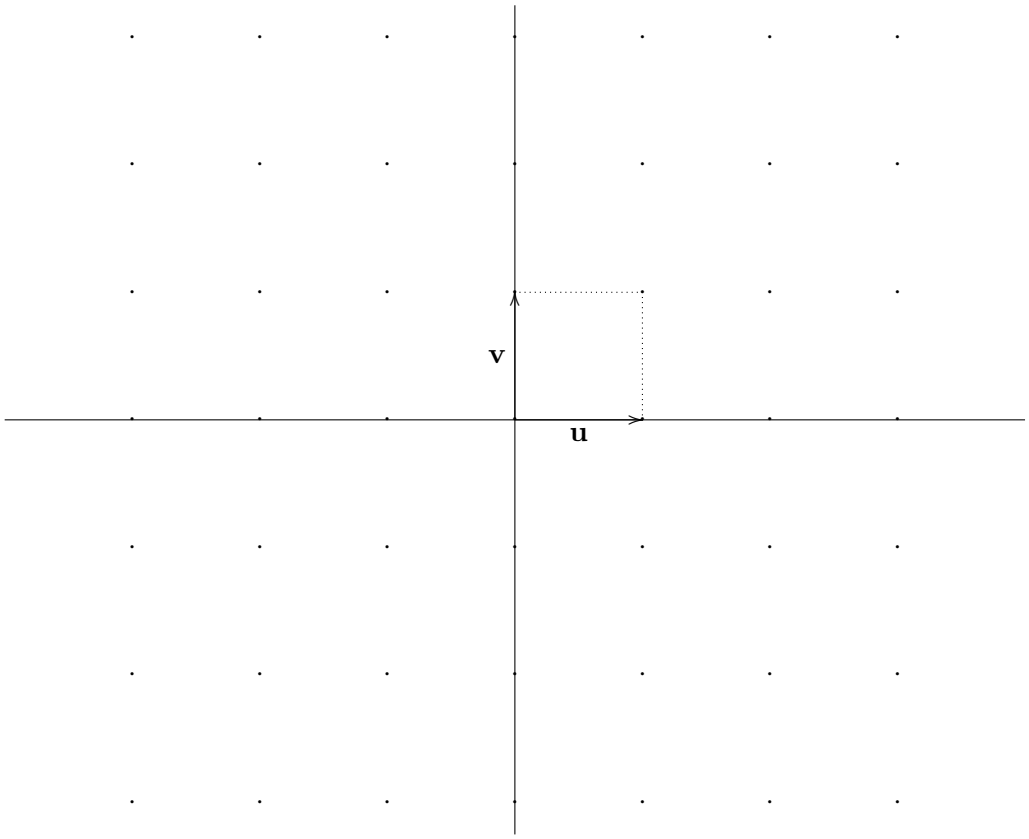


FIGURE 2.2. Square lattice

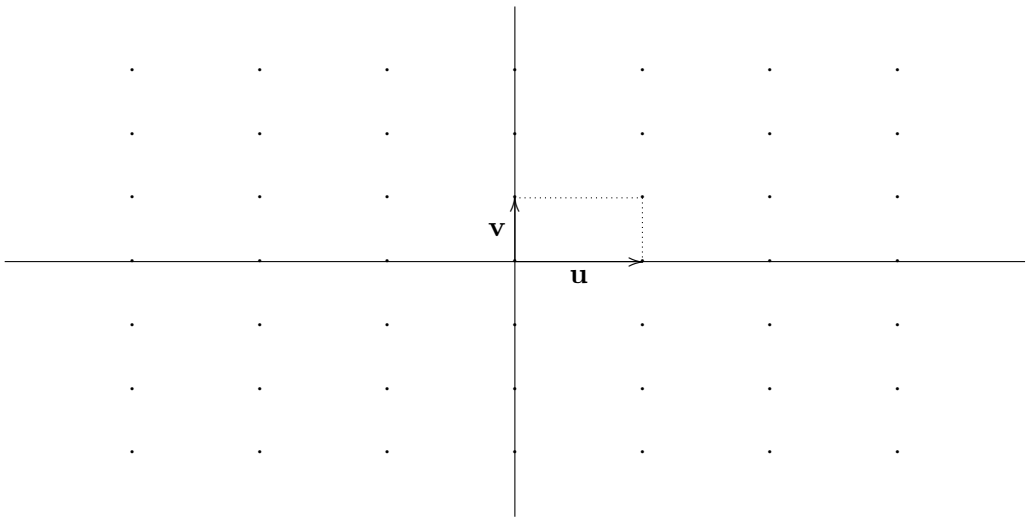


FIGURE 2.3. Rectangular lattice

is in $\mathcal{W}$. If $\mathrm{Euc}(2)_{\mathcal{W},\mathbf{w}} \leqslant \mathrm{Euc}(2)_{\mathcal{W}}$ is the symmetry subgroup fixing the point W and if $W_{m,n}$ is the point with position vector $\mathbf{w} + m\mathbf{u} + m\mathbf{v}$ for some $m, n \in \mathbb{Z}$, then

$$\mathrm{Euc}(2)_{\mathcal{W},W_{m,n}} = H_* \, \mathrm{Euc}(2)_{\mathcal{W},\mathbf{w}},$$

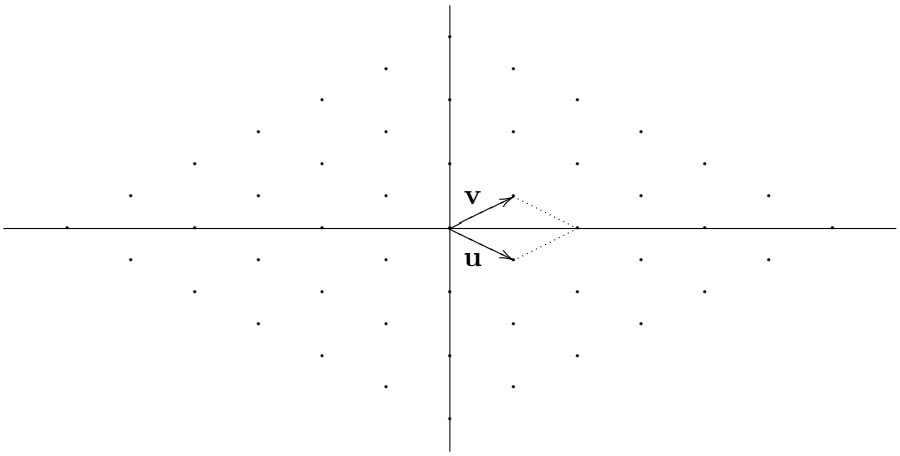


FIGURE 2.4. Centred rectangular lattice

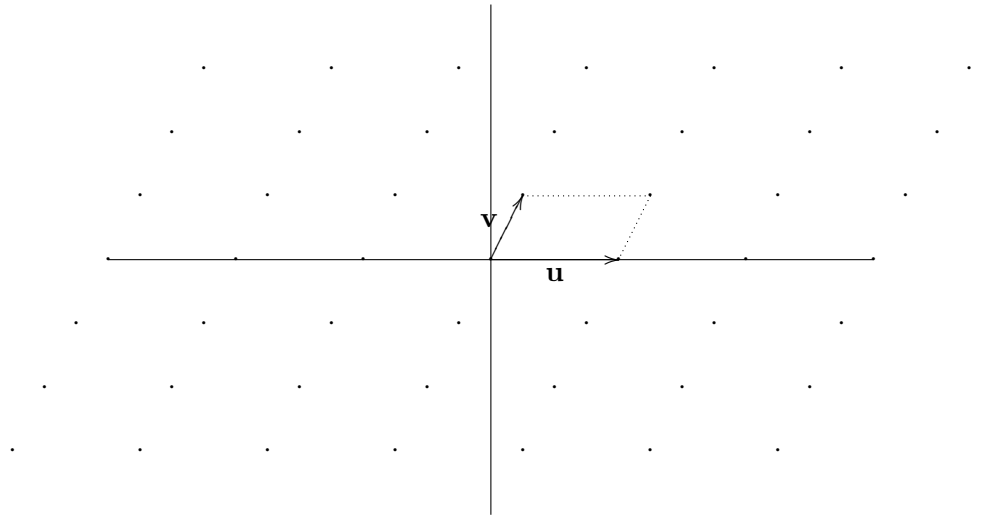
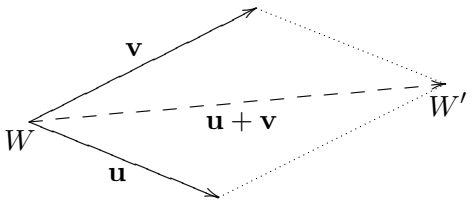


FIGURE 2.5. Parallelogram lattice

where $H = \mathrm{Trans}_{m\mathbf{u}+n\mathbf{v}}$. So to understand the symmetries of $\mathcal{W}$ we can confine attention to a *fundamental region* of form



with W at one of the vertices where W' has position vector $\mathbf{w}+\mathbf{u}+\mathbf{v}$; this is the subset

$$\{\mathbf{w} + s\mathbf{u} + t\mathbf{v} : 0 \leqslant s < 1,\ 0 \leqslant t < 1\} \subseteq \mathbb{R}^2.$$

An alternative is the fundamental region centred at W,

$$\{\mathbf{w} + s\mathbf{u} + t\mathbf{v} : -1/2 \leqslant s < 1/2,\ -1/2 \leqslant t < 1/2\} \subseteq \mathbb{R}^2.$$

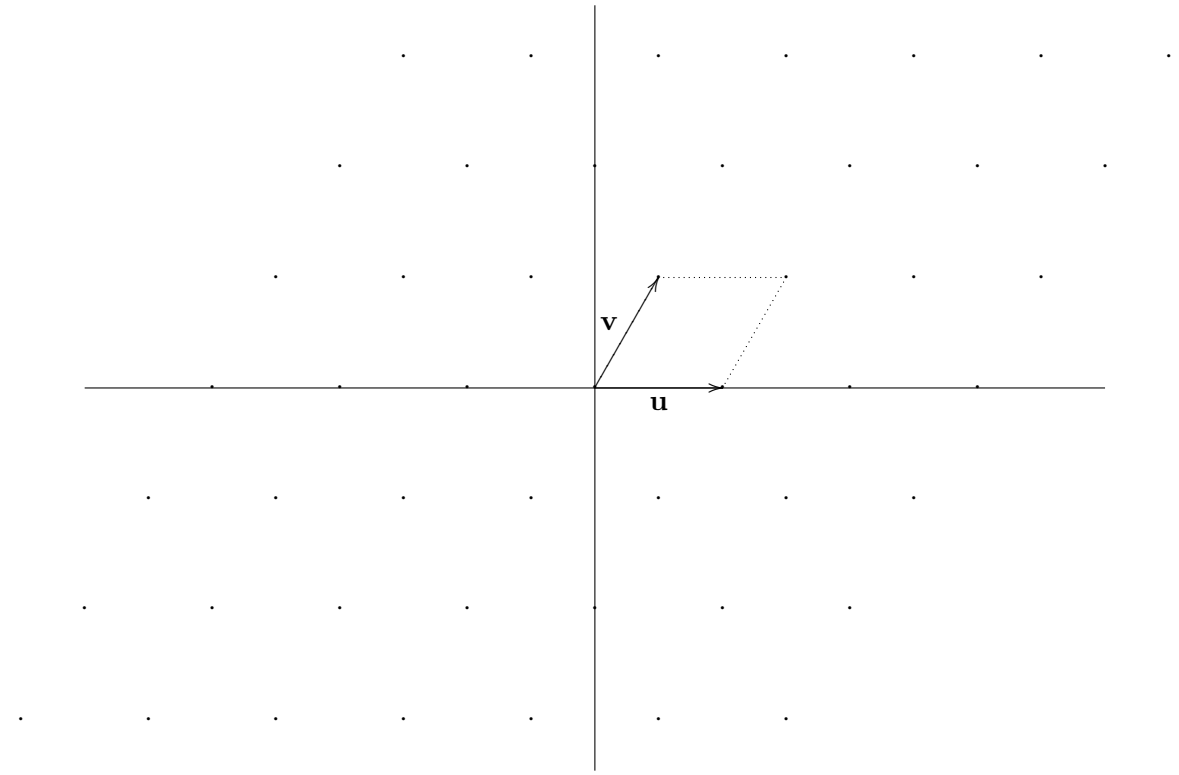
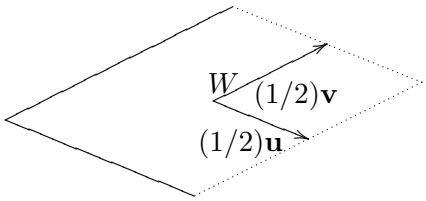


FIGURE 2.6. Hexagonal lattice



Such a fundamental region $\mathcal{F}$ is useful because the position vector $\mathbf{p}$ every point in the plane can be uniquely expressed in the form

$$\mathbf{p} = \mathbf{p}' + m\mathbf{u} + n\mathbf{v}$$

for some point $\mathbf{p}' \in \mathcal{F}$ and $m, n \in \mathbb{Z}$. This allows us to *tile* the plane with copies of a figure located within a fundamental region. The square lattice design of Figure 2.7 is obtained by translating (using vectors $m\mathbf{u} + n\mathbf{v}$ for $(m, n \in \mathbb{Z})$) the pattern ♣ ♢ / ♡ ♠ contained in the fundamental region

$$\{s\mathbf{u} + t\mathbf{v} : -1/2 \leqslant s < 1/2,\ -1/2 \leqslant t < 1/2\}$$

to tile the whole plane.

Given a wallpaper pattern $\mathcal{W}$ with translation lattice $\mathrm{span}_{\mathbb{Z}}(\mathbf{u}, \mathbf{v})$, associated with a point P in the is the subgroup $\mathrm{Euc}(2)_{\mathcal{W},P} \leqslant \mathrm{Euc}(2)_{\mathcal{W}}$ of symmetries that fix P. In particular, when $P = O$, we can consider

$$\mathrm{O}(2)_{\mathcal{W}} = \mathrm{Euc}(2)_{\mathcal{W},O} \cap \mathrm{O}(2) \leqslant \mathrm{O}(2),$$

called the *holohedry subgroup* of $\mathcal{W}$. If $P \in \mathcal{W}$ the *point group of P in $\mathcal{W}$* is the subgroup

$$\mathrm{O}(2)_{\mathcal{W},P} = \{(A \mid \mathbf{0}) \in \mathrm{O}(2) : (A \mid \mathbf{t}) \in \mathrm{Euc}(2)_{\mathcal{W},P} \text{ for some } \mathbf{t} \in \mathbb{R}^2\} \leqslant \mathrm{O}(2).$$

Note that if $(A \mid \mathbf{t}) \in \mathrm{Euc}(2)_{\mathcal{W}}$ it is not necessarily true that $(A \mid \mathbf{0}) \in \mathrm{Euc}(2)_{\mathcal{W}}$. The *point group of $\mathcal{W}$* is

$$\overline{\mathrm{O}(2)}_{\mathcal{W}} = \{(A \mid \mathbf{0}) : (A \mid \mathbf{t}) \in \mathrm{Euc}(2)_{\mathcal{W}} \text{ for some } \mathbf{t} \in \mathbb{R}^2\} \leqslant \mathrm{O}(2).$$

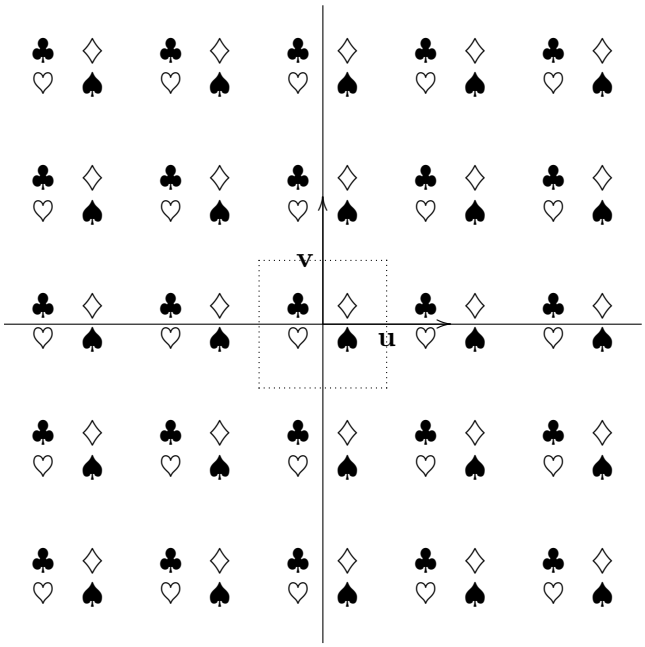


FIGURE 2.7. Square lattice pattern obtained from a fundamental region

If there is a point P for which $\mathrm{O}(2)_{\mathcal{W},P} \leqslant \mathrm{Euc}(2)_{\mathcal{W}}$ then $\mathcal{W}$ is called *symmomorphic*.

We now describe the seventeen distinct wallpaper patterns up to similarity of their symmetry groups; examples of each type are shown in Figures 2.8–2.24. There is a standard notation used to describe these and names such as 'pm' will be used to allow interested readers to pursue the details in books and other sources.

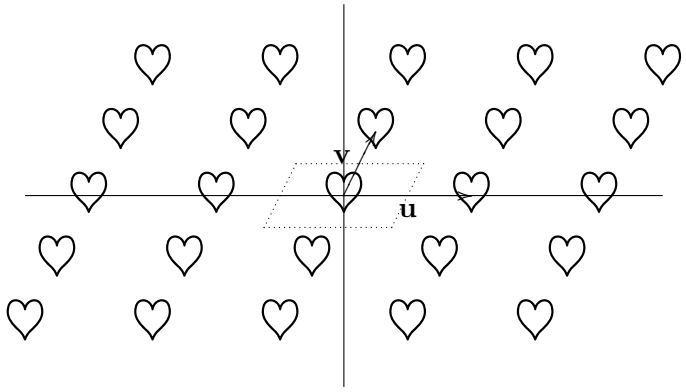


FIGURE 2.8. p1

Pattern p1. The symmetry group of the fundamental region here is trivial, so the holohedry group is $\{I\}$. The full symmetry group is given by the translation lattice,

$$\{I \mid m\mathbf{u} + n\mathbf{v} : m, n \in \mathbb{Z}\}.$$

Pattern p2. The fundamental region has a half rotation symmetry, so the holohedry group is

$$\{(I \mid \mathbf{0}), (-I \mid \mathbf{0})\}.$$

The full symmetry group is

$$\{(I \mid m\mathbf{u} + n\mathbf{v}) : m, n \in \mathbb{Z}\} \cup \{(-I \mid m\mathbf{u} + n\mathbf{v}) : m, n \in \mathbb{Z}\}$$

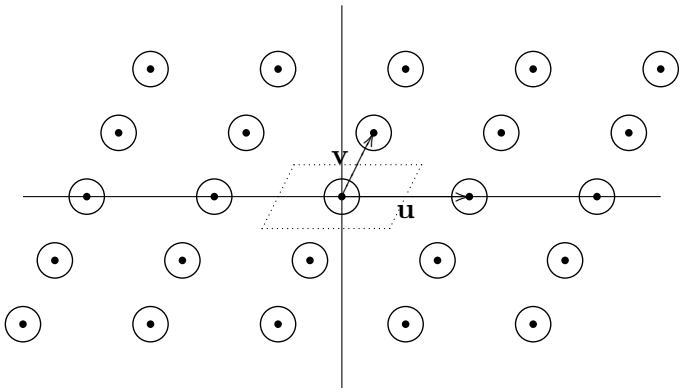


FIGURE 2.9. p2

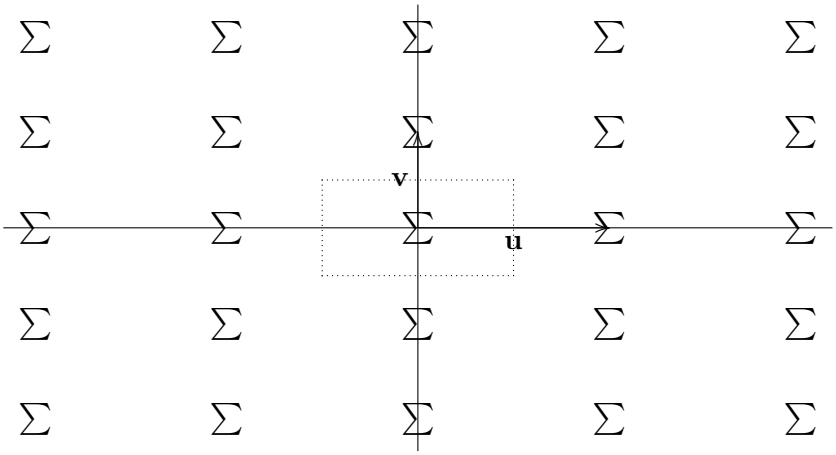


FIGURE 2.10. pm

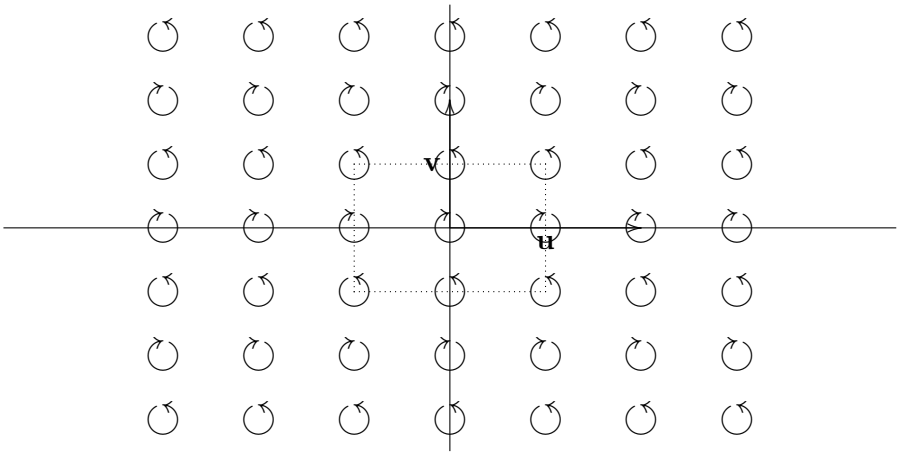


FIGURE 2.11. pg

which consists of the translations together with half rotations about the points

$$m\mathbf{u}+n\mathbf{v},\ (m+1/2)\mathbf{u}+n\mathbf{v},\ m\mathbf{u}+(n+1/2)\mathbf{v},\ (m+1/2)\mathbf{u}+(n+1/2)\mathbf{v}\quad (m,n\in\mathbb{Z}).$$

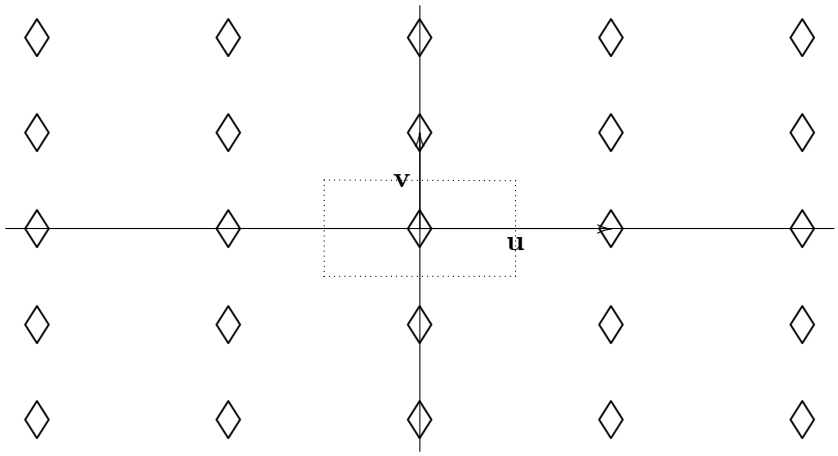


FIGURE 2.12. pmm

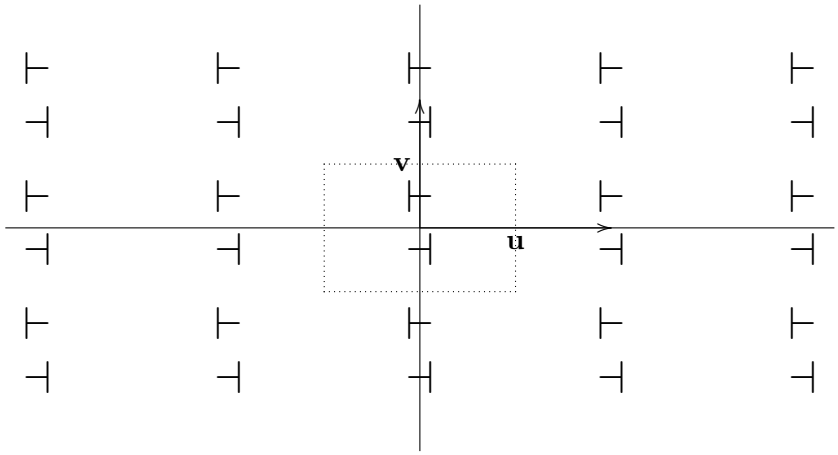


FIGURE 2.13. pgm

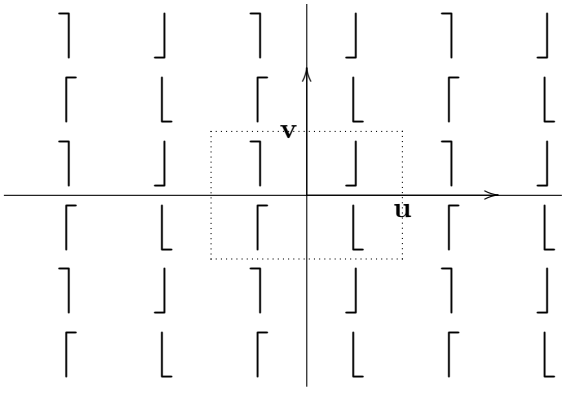


FIGURE 2.14. p2g

Pattern pm. The fundamental region has a reflection in the x-axis, so the holohedry group is

$$\{(I \mid \mathbf{0}), (S_x \mid \mathbf{0})\},$$

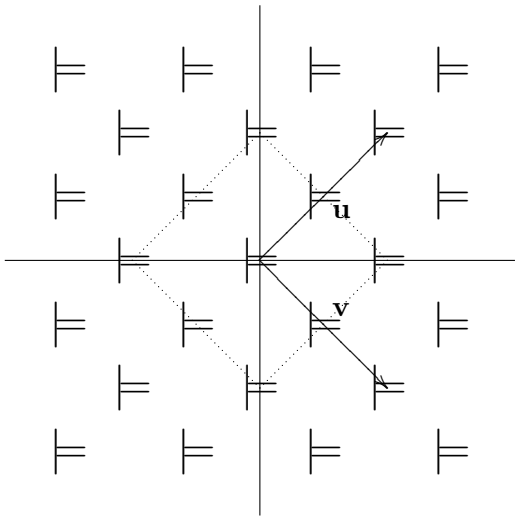


FIGURE 2.15. cm

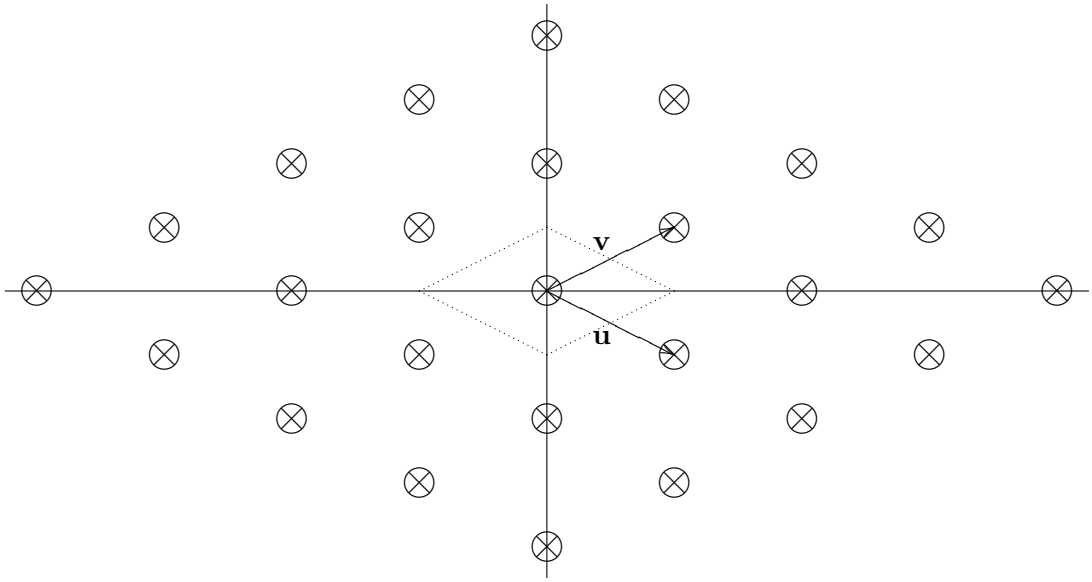


FIGURE 2.16. cmm

where $S_x = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. The full symmetry group

$$\{(I \mid m\mathbf{u} + n\mathbf{v}) : m, n \in \mathbb{Z}\} \cup \{(S_x \mid m\mathbf{u} + n\mathbf{v}) : m, n \in \mathbb{Z}\}$$

consists of translations and glide reflections in each of the horizontal lines

$$\{t\mathbf{u} + n\mathbf{v} : t \in \mathbb{R}\},\ \{t\mathbf{u} + (n + 1/2)\mathbf{v} : t \in \mathbb{R}\} \quad (n \in \mathbb{Z}).$$

Pattern pg. The fundamental region has no reflections or non-trivial rotations, but there are glide reflections in the y-axis with Seitz symbols of form $(S_y \mid (n + (1/2))\mathbf{v})$ where $S_y = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ and $n \in \mathbb{Z}$. The holohedry group is trivial and the full symmetry group is

$$\{(I \mid m\mathbf{u} + n\mathbf{v}) : m, n \in \mathbb{Z}\} \cup \{(S_y \mid m\mathbf{u} + (n + (1/2))\mathbf{v}) : m, n \in \mathbb{Z}\}$$

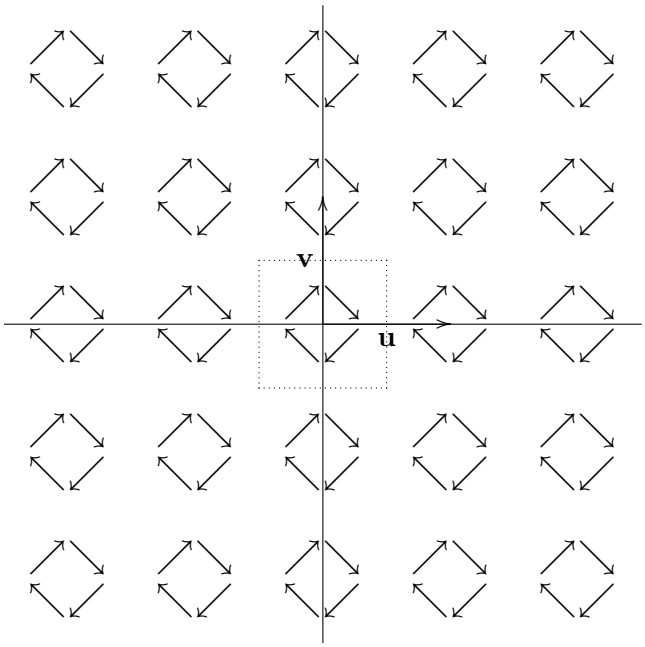


FIGURE 2.17. p4

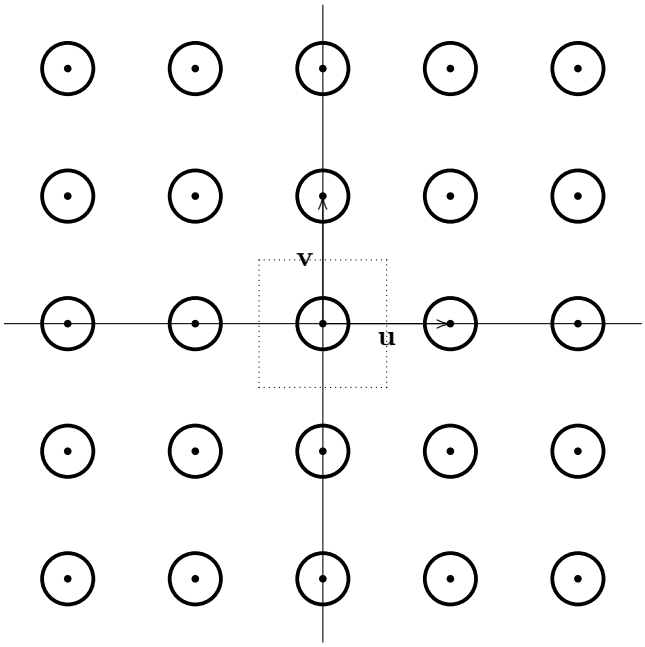


FIGURE 2.18. p4m

which consists of translations together with glide reflections in the vertical lines

$$\{m\mathbf{u} + t\mathbf{v} : t \in \mathbb{R}\},\ \{(m + 1/2)\mathbf{u} + t\mathbf{v} : t \in \mathbb{R}\} \quad (m \in \mathbb{Z}).$$

Pattern pmm. The fundamental region has reflection symmetries in the x and y-axes with Seitz symbols $(S_x \mid \mathbf{0})$ and $(S_y \mid \mathbf{0})$. These compose to give the half rotation $(-I \mid \mathbf{0})$. So the

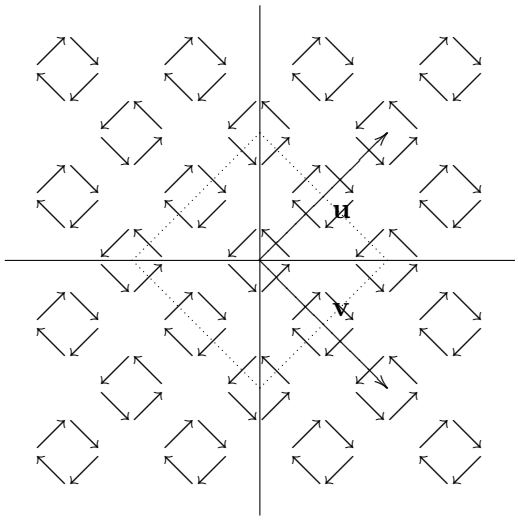


FIGURE 2.19. p4g

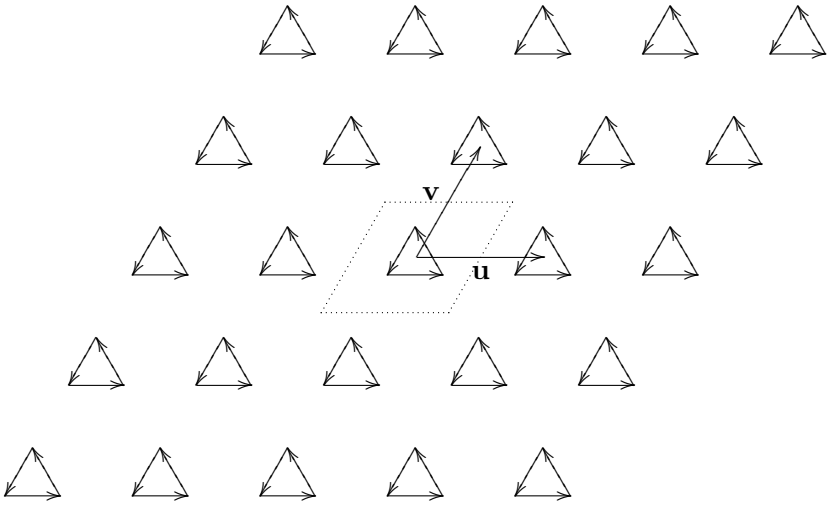


FIGURE 2.20. p3

holohedry group is the dihedral group D_4 and the full symmetry group is

$$\{(I \mid m\mathbf{u}+n\mathbf{v}) : m,n \in \mathbb{Z}\} \cup \{(-I \mid m\mathbf{u}+n\mathbf{v}) : m,n \in \mathbb{Z}\} \\ \cup \{(S_x \mid m\mathbf{u}+n\mathbf{v}) : m,n \in \mathbb{Z}\} \cup \{(S_y \mid m\mathbf{u}+n\mathbf{v}) : m,n \in \mathbb{Z}\}$$

which consists of translations $(I \mid m\mathbf{u}+n\mathbf{v})$, half rotations $(-I \mid m\mathbf{u}+n\mathbf{v})$ and glide reflections $(S_x \mid m\mathbf{u}+n\mathbf{v})$, $(S_y \mid m\mathbf{u}+n\mathbf{v})$.

Pattern pgm. The fundamental region has a half rotation symmetry so the holohedry group is

$$\{(I \mid \mathbf{0}), (-I \mid \mathbf{0})\}.$$

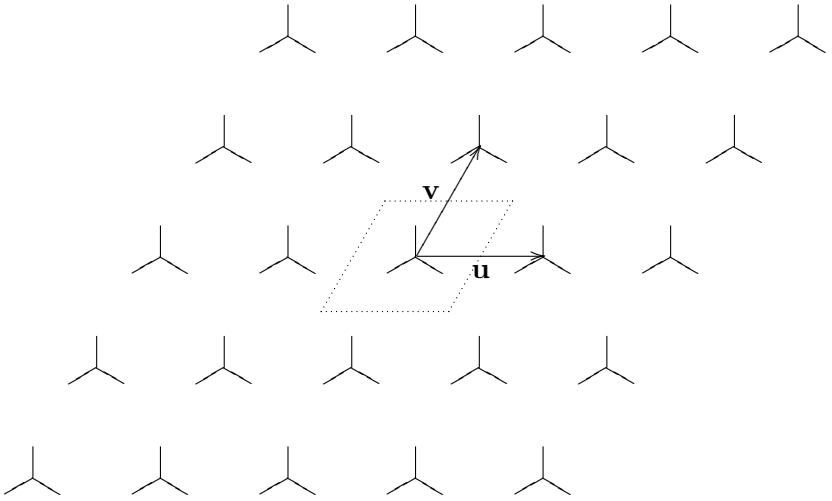


FIGURE 2.21. p3m1

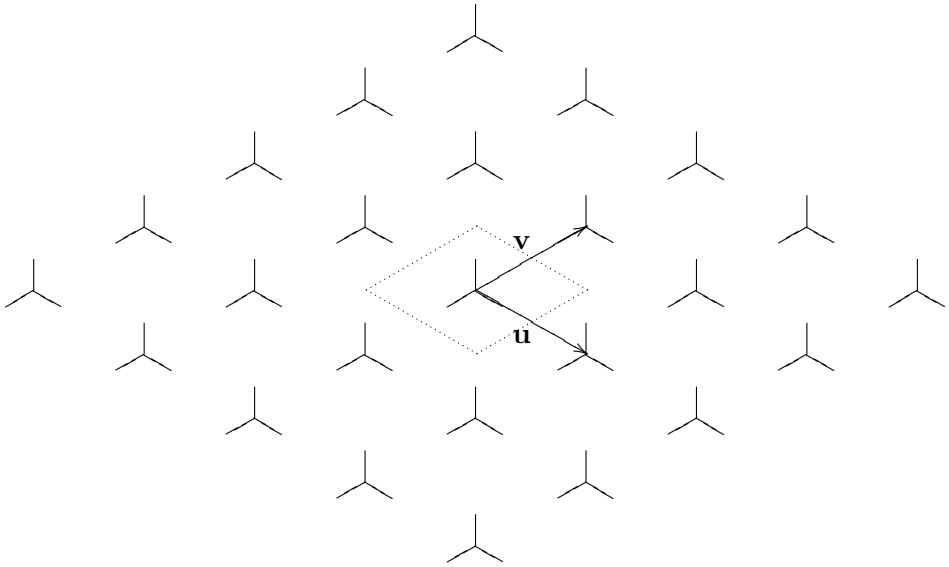


FIGURE 2.22. p31m

There is a glide reflection in the y-axis $(S_y \mid (1/2)\mathbf{v})$ which squares to $(I \mid \mathbf{v})$, and the reflection $(S_x \mid (1/2)\mathbf{v})$ in a horizontal line. The full symmetry group is

$$\{(I \mid m\mathbf{u} + n\mathbf{v}) : m, n \in \mathbb{Z}\} \cup \{(-I \mid m\mathbf{u} + n\mathbf{v}) : m, n \in \mathbb{Z}\}$$
$$\cup \{(S_y \mid m\mathbf{u} + (n + (1/2))\mathbf{v}) : m, n \in \mathbb{Z}\} \cup \{(S_x \mid m\mathbf{u} + (n + (1/2))\mathbf{v}) : m, n \in \mathbb{Z}\}.$$

Pattern p2g. The fundamental region has no reflections but it has a half rotation $(-I \mid \mathbf{0})$. There are glide reflections $(S_y \mid (m + (1/2))\mathbf{u} + (n + (1/2))\mathbf{v})$, $(S_x \mid (m + (1/2))\mathbf{u} + (n + (1/2)\mathbf{v})$ for $m, n \in \mathbb{Z}$. The holohedry group is

$$\{(I \mid \mathbf{0}), (-I \mid \mathbf{0})\}.$$

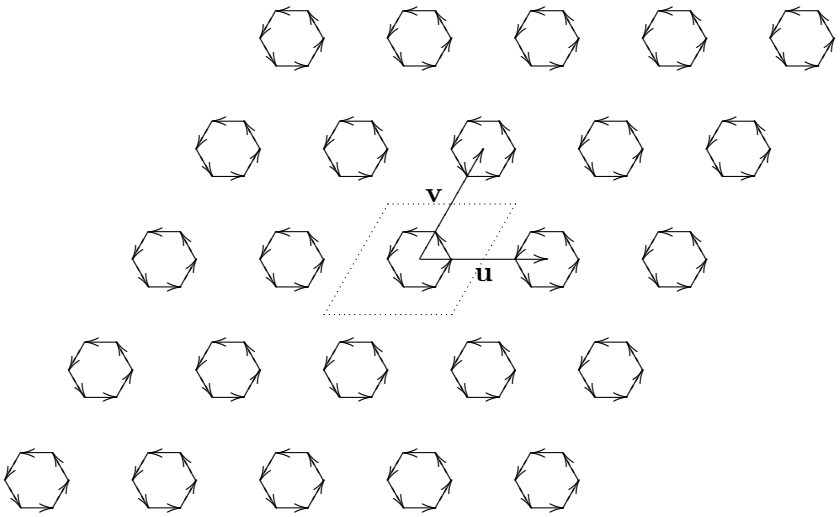


FIGURE 2.23. p6

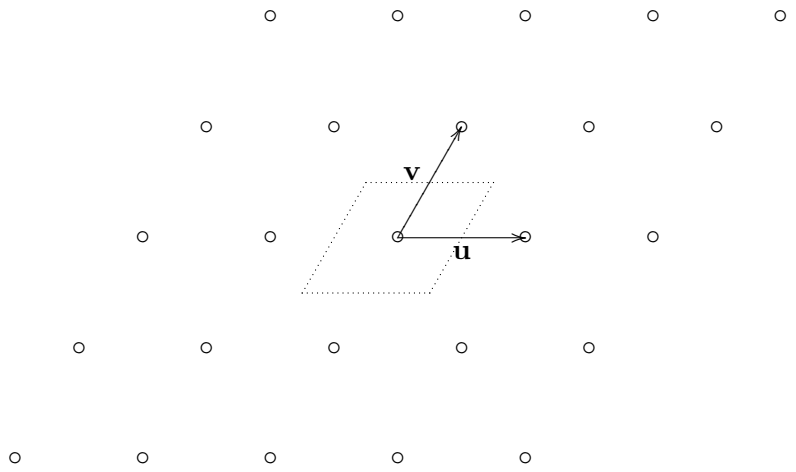


FIGURE 2.24. p6m

The whole symmetry group is

$$\{(I \mid m\mathbf{u}+n\mathbf{v}) : m,n \in \mathbb{Z}\} \cup \{(-I \mid m\mathbf{u}+n\mathbf{v}) : m,n \in \mathbb{Z}\}$$
$$\cup \{(S_y \mid (m+(1/2))\mathbf{u}+(n+(1/2))\mathbf{v}) : m,n \in \mathbb{Z}\}$$
$$\cup \{(S_x \mid (m+(1/2))\mathbf{u}+(n+(1/2))\mathbf{v}) : m,n \in \mathbb{Z}\}.$$

Pattern cm. The fundamental region has a reflection $(S_x \mid \mathbf{0})$ in the x-axis and the holohedry group is

$$\{(I \mid \mathbf{0}), (S_x \mid \mathbf{0})\}.$$

There are glide reflections $(S_x \mid \mathbf{u})$ and $(S_x \mid \mathbf{v})$ in lines parallel to the x-axis and which compose to give the translation $(I \mid \mathbf{u}+\mathbf{v})$. The whole symmetry group is

$$\{(I \mid m\mathbf{u}+n\mathbf{u}) : m,n \in \mathbb{Z}\} \cup \{(S_x \mid m\mathbf{u}+n\mathbf{v}) : m,n \in \mathbb{Z}\}.$$

Pattern cmm. The fundamental region has reflections in the x and y-axes as well as the half rotation about the origin. So the holohedry group is

$$\{(I \mid \mathbf{0}), (-I \mid \mathbf{0}), (S_x \mid \mathbf{0}), (S_y \mid \mathbf{0})\}.$$

The whole symmetry group is

$$\begin{aligned}\{(I \mid m\mathbf{u}+n\mathbf{v}) : m,n \in \mathbb{Z}\} \cup \{(-I \mid m\mathbf{u}+n\mathbf{v}) : m,n \in \mathbb{Z}\} \\ \cup \{(S_y \mid m\mathbf{u}+n\mathbf{v}) : m,n \in \mathbb{Z}\} \cup \{(S_x \mid m\mathbf{u}+n\mathbf{v}) : m,n \in \mathbb{Z}\}.\end{aligned}$$

Pattern p4. The fundamental region has rotations through $\pm\pi/2$ and a half rotation but no reflections. The holohedry group is

$$\{(I \mid \mathbf{0}), (-I \mid \mathbf{0}), (R_{1/4} \mid \mathbf{0}), (-R_{1/4} \mid \mathbf{0})\},$$

where $R_{1/4} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. The full symmetry group is

$$\begin{aligned}\{(I \mid m\mathbf{u}+n\mathbf{u}) : m,n \in \mathbb{Z}\} \cup \{(-I \mid m\mathbf{u}+n\mathbf{u}) : m,n \in \mathbb{Z}\} \\ \cup \{(R_{1/4} \mid m\mathbf{u}+n\mathbf{u}) : m,n \in \mathbb{Z}\} \cup \{(-R_{1/4} \mid m\mathbf{u}+n\mathbf{u}) : m,n \in \mathbb{Z}\}.\end{aligned}$$

Pattern p4m. The fundamental region has rotations through $\pm\pi/2$ and a half rotation as well as reflections in the x and y-axes and reflections in the diagonals. The holohedry group is a dihedral group D_8,

$$\{(I \mid \mathbf{0}), (-I \mid \mathbf{0}), (R_{1/4} \mid \mathbf{0}), (-R_{1/4} \mid \mathbf{0})\} \cup \{(S_x \mid \mathbf{0}), (S_y \mid \mathbf{0}), (R_{1/4}S_x \mid \mathbf{0}), (R_{1/4}S_y \mid \mathbf{0})\},$$

where $R_{1/4} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. The full symmetry group is

$$\begin{aligned}\{(I \mid m\mathbf{u}+n\mathbf{u}) : m,n \in \mathbb{Z}\} \cup \{(-I \mid m\mathbf{u}+n\mathbf{u}) : m,n \in \mathbb{Z}\} \\ \cup \{(R_{1/4} \mid m\mathbf{u}+n\mathbf{u}) : m,n \in \mathbb{Z}\} \cup \{(-R_{1/4} \mid m\mathbf{u}+n\mathbf{u}) : m,n \in \mathbb{Z}\} \\ \cup \{(S_x \mid m\mathbf{u}+n\mathbf{u}) : m,n \in \mathbb{Z}\} \cup \{(-S_x \mid m\mathbf{u}+n\mathbf{u}) : m,n \in \mathbb{Z}\} \\ \cup \{(R_{1/4}S_x \mid m\mathbf{u}+n\mathbf{u}) : m,n \in \mathbb{Z}\} \cup \{(-R_{1/4}S_x \mid m\mathbf{u}+n\mathbf{u}) : m,n \in \mathbb{Z}\} \\ \cup \{(R_{1/4}S_x \mid m\mathbf{u}+n\mathbf{u}) : m,n \in \mathbb{Z}\} \cup \{(-R_{1/4}S_x \mid m\mathbf{u}+n\mathbf{u}) : m,n \in \mathbb{Z}\} \\ \cup \{(S_y \mid m\mathbf{u}+n\mathbf{u}) : m,n \in \mathbb{Z}\} \cup \{(-S_y \mid m\mathbf{u}+n\mathbf{u}) : m,n \in \mathbb{Z}\} \\ \cup \{(R_{1/4}S_y \mid m\mathbf{u}+n\mathbf{u}) : m,n \in \mathbb{Z}\} \cup \{(-R_{1/4}S_y \mid m\mathbf{u}+n\mathbf{u}) : m,n \in \mathbb{Z}\} \\ \cup \{(R_{1/4}S_y \mid m\mathbf{u}+n\mathbf{u}) : m,n \in \mathbb{Z}\} \cup \{(-R_{1/4}S_y \mid m\mathbf{u}+n\mathbf{u}) : m,n \in \mathbb{Z}\}.\end{aligned}$$

Pattern p4g. The fundamental region has two quarter and a half rotation about the origin, but no reflections. So the holohedry group is

$$\{(I \mid \mathbf{0}), (-I \mid \mathbf{0}), (R_{1/4} \mid \mathbf{0}), (R_{-1/4} \mid \mathbf{0})\}.$$

There is also the glide reflection $(S_x \mid (1/2)\mathbf{u}+\mathbf{v})$ whose square is $(S_x \mid (1/2)\mathbf{u}+\mathbf{v})^2 = (I \mid \mathbf{u})$. The whole symmetry group is

$$\begin{aligned}\{(I \mid m\mathbf{u}+n\mathbf{v}) : m,n \in \mathbb{Z}\} \cup \{(-I \mid m\mathbf{u}+n\mathbf{v}) : m,n \in \mathbb{Z}\} \\ \cup \{(R_{1/4} \mid m\mathbf{u}+n\mathbf{v}) : m,n \in \mathbb{Z}\} \cup \{(-R_{1/4} \mid m\mathbf{u}+n\mathbf{v}) : m,n \in \mathbb{Z}\} \\ \cup \{(S_x \mid (m+(1/2))\mathbf{u}+n\mathbf{v}) : m,n \in \mathbb{Z}\} \cup \{(S_y \mid (m+(1/2))\mathbf{u}+n\mathbf{v}) : m,n \in \mathbb{Z}\} \\ \cup \{(R_{1/4}S_x \mid m\mathbf{u}+(n+(1/2))\mathbf{v}) : m,n \in \mathbb{Z}\} \cup \{(R_{1/4}S_y \mid m\mathbf{u}+(n+(1/2)\mathbf{v}) : m,n \in \mathbb{Z}\}.\end{aligned}$$

Here the first 4 subsets consist of rotations while the last 4 consists of glide reflections.

Pattern p3. The fundamental region has no non-trivial symmetries, but there are rotational symmetries $(R_{1/3} \mid \mathbf{0})$ and $(R_{-1/3} \mid \mathbf{0})$ about the origin which is at the centre of a triangle, where $R_{1/3} = \begin{bmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{bmatrix}$. The holohedry group is

$$\{(I \mid \mathbf{0}), (R_{1/3} \mid \mathbf{0}), (R_{-1/3} \mid \mathbf{0})\},$$

while the full symmetry group consists of translations and rotations,

$$\{(I \mid m\mathbf{u} + n\mathbf{u}) : m, n \in \mathbb{Z}\} \cup \{(R_{1/3} \mid m\mathbf{u} + n\mathbf{u}) : m, n \in \mathbb{Z}\} \cup \{(R_{-1/3} \mid m\mathbf{u} + n\mathbf{u}) : m, n \in \mathbb{Z}\}.$$

Pattern p3m1. The fundamental region has no non-trivial symmetries, but there are rotational symmetries $(R_{1/3} \mid \mathbf{0})$ and $(R_{-1/3} \mid \mathbf{0})$ about the origin which is at the centre of a triangle, where $R_{1/3} = \begin{bmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{bmatrix}$. There is also a reflection symmetry $(S_y \mid \mathbf{0})$ in the y-axis which has the effect

$$(S_y \mid \mathbf{0})\mathbf{u} = -\mathbf{u}, \quad (S_y \mid \mathbf{0})\mathbf{v} = -\mathbf{u} + \mathbf{v}.$$

The holohedry group is the dihedral group

$$\{(I \mid \mathbf{0}), (R_{1/3} \mid \mathbf{0}), (R_{-1/3} \mid \mathbf{0})\} \cup \{(S_y \mid \mathbf{0}), (R_{1/3}S_y \mid \mathbf{0}), (R_{-1/3}S_y \mid \mathbf{0})\},$$

where the last 3 symmetries are reflections in the 3 legs of a  symbol. The full symmetry group consists of translations, rotations and glide reflections,

$$\begin{aligned}\{(I \mid m\mathbf{u}+n\mathbf{u}) : m, n \in \mathbb{Z}\} \cup \{(R_{1/3} \mid m\mathbf{u}+n\mathbf{u}) : m, n \in \mathbb{Z}\} \cup \{(R_{-1/3} \mid m\mathbf{u}+n\mathbf{u}) : m, n \in \mathbb{Z}\} \\ \cup \{(S_y \mid m\mathbf{u}+n\mathbf{u}) : m, n \in \mathbb{Z}\} \cup \{(S_yR_{1/3} \mid m\mathbf{u}+n\mathbf{u}) : m, n \in \mathbb{Z}\} \\ \cup \{(S_yR_{-1/3} \mid m\mathbf{u}+n\mathbf{u}) : m, n \in \mathbb{Z}\}.\end{aligned}$$

The centres of rotation are not on the glide reflection axes.

Pattern p31m. The fundamental region has no non-trivial symmetries, but there are rotational symmetries $(R_{1/3} \mid \mathbf{0})$ and $(R_{-1/3} \mid \mathbf{0})$ about the origin which is at the centre of a triangle, where $R_{1/3} = \begin{bmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{bmatrix}$. There is also a reflection symmetry $(S_y \mid \mathbf{0})$ in the y-axis which has the effect

$$(S_y \mid \mathbf{0})\mathbf{u} = -\mathbf{v}, \quad (S_y \mid \mathbf{0})\mathbf{v} = -\mathbf{u}.$$

The holohedry group is again the dihedral group

$$\{(I \mid \mathbf{0}), (R_{1/3} \mid \mathbf{0}), (R_{-1/3} \mid \mathbf{0})\} \cup \{(S_y \mid \mathbf{0}), (R_{1/3}S_y \mid \mathbf{0}), (R_{-1/3}S_y \mid \mathbf{0})\},$$

where the last 3 symmetries are reflections in the 3 legs of a  symbol. The full symmetry group consists of translations, rotations and glide reflections,

$$\begin{aligned}\{(I \mid m\mathbf{u}+n\mathbf{u}) : m, n \in \mathbb{Z}\} \cup \{(R_{1/3} \mid m\mathbf{u}+n\mathbf{u}) : m, n \in \mathbb{Z}\} \cup \{(R_{-1/3} \mid m\mathbf{u}+n\mathbf{u}) : m, n \in \mathbb{Z}\} \\ \cup \{(S_y \mid m\mathbf{u}+n\mathbf{u}) : m, n \in \mathbb{Z}\} \cup \{(S_yR_{1/3} \mid m\mathbf{u}+n\mathbf{u}) : m, n \in \mathbb{Z}\} \\ \cup \{(S_yR_{-1/3} \mid m\mathbf{u}+n\mathbf{u}) : m, n \in \mathbb{Z}\}.\end{aligned}$$

Some centres of rotation are on glide reflection axes.

Pattern p6. There are rotational symmetries $(R_{\pm 1/6} \mid \mathbf{0})$, $(R_{\pm 1/3} \mid \mathbf{0})$ and $(-I \mid \mathbf{0})$ about the origin which is at the centre of a hexagon, where $R_{1/6} = \begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix}$. There are no reflection symmetries. The holohedry group is the cyclic group

$$\{(I \mid \mathbf{0}), (-I \mid \mathbf{0}), (R_{1/3} \mid \mathbf{0}), (R_{-1/3} \mid \mathbf{0}), (R_{1/6} \mid \mathbf{0}), (R_{-1/6} \mid \mathbf{0})\}.$$

The full symmetry group consists of translations and rotations,

$$\{(I \mid m\mathbf{u}+n\mathbf{u}) : m,n \in \mathbb{Z}\} \cup \{(R_{1/3} \mid m\mathbf{u}+n\mathbf{u}) : m,n \in \mathbb{Z}\} \cup \{(R_{-1/3} \mid m\mathbf{u}+n\mathbf{u}) : m,n \in \mathbb{Z}\}$$
$$\cup \{(R_{1/6} \mid m\mathbf{u}+n\mathbf{u}) : m,n \in \mathbb{Z}\} \cup \{(R_{-1/6} \mid m\mathbf{u}+n\mathbf{u}) : m,n \in \mathbb{Z}\}.$$

Pattern p6m. There are rotational symmetries $(R_{\pm 1/6} \mid \mathbf{0})$, $(R_{\pm 1/3} \mid \mathbf{0})$ and $(-I \mid \mathbf{0})$ about the origin which is at the centre of a hexagon, where $R_{1/6} = \begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix}$. There is also a reflection symmetry $(S_y \mid \mathbf{0})$ in the y-axis which has the effect

$$(S_y \mid \mathbf{0})\mathbf{u} = -\mathbf{u}, \quad (S_y \mid \mathbf{0})\mathbf{v} = -\mathbf{u}+\mathbf{v}.$$

The holohedry group is the dihedral group

$$\{(I \mid \mathbf{0}), (-I \mid \mathbf{0}), (R_{1/3} \mid \mathbf{0}), (R_{-1/3} \mid \mathbf{0}), (R_{1/6} \mid \mathbf{0}), (R_{-1/6} \mid \mathbf{0})\}$$
$$\cup \{(S_y \mid \mathbf{0}), (-S_y \mid \mathbf{0}), (S_yR_{1/3} \mid \mathbf{0}), (S_yR_{-1/3} \mid \mathbf{0}), (S_yR_{1/6} \mid \mathbf{0}), (S_yR_{-1/6} \mid \mathbf{0})\}.$$

The full symmetry group consists of translations, rotations and glide reflections,

$$\{(I \mid m\mathbf{u}+n\mathbf{u}) : m,n \in \mathbb{Z}\} \cup \{(R_{1/3} \mid m\mathbf{u}+n\mathbf{u}) : m,n \in \mathbb{Z}\} \cup \{(R_{-1/3} \mid m\mathbf{u}+n\mathbf{u}) : m,n \in \mathbb{Z}\}$$
$$\cup \{(R_{1/6} \mid m\mathbf{u}+n\mathbf{u}) : m,n \in \mathbb{Z}\} \cup \{(R_{-1/6} \mid m\mathbf{u}+n\mathbf{u}) : m,n \in \mathbb{Z}\}$$
$$\cup \{(S_y \mid m\mathbf{u}+n\mathbf{u}) : m,n \in \mathbb{Z}\} \cup \{(S_yR_{1/3} \mid m\mathbf{u}+n\mathbf{u}) : m,n \in \mathbb{Z}\}$$
$$\cup \{(S_yR_{-1/3} \mid m\mathbf{u}+n\mathbf{u}) : m,n \in \mathbb{Z}\} \cup \{(S_yR_{1/6} \mid m\mathbf{u}+n\mathbf{u}) : m,n \in \mathbb{Z}\}$$
$$\cup \{(S_yR_{-1/6} \mid m\mathbf{u}+n\mathbf{u}) : m,n \in \mathbb{Z}\}.$$

Exercises on Chapter 2

2.1. In this question, all permutations are elements of the symmetric group S_6.
(a) Determine the following, in each case expressing the answer in a similar form:

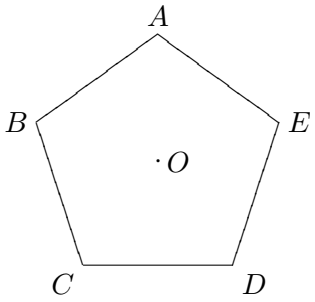
$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 1 & 5 & 6 & 4 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 4 & 2 & 3 & 1 & 5 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 1 & 5 & 6 & 4 \end{pmatrix}^{-1}.$$

(b) Express the permutation $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 1 & 5 & 6 & 4 \end{pmatrix}$ as a product of disjoint cycles.
(c) Determine $\operatorname{sgn} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 1 & 5 & 6 & 4 \end{pmatrix}$.

2.2. Calculate the following products in the symmetric group S_6, giving the answers as products of disjoint cycles:

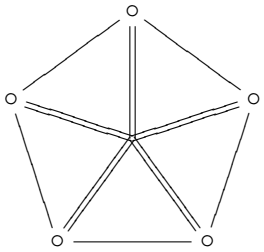
$$(2\ 3\ 5\ 6)(1\ 6\ 2\ 3), \quad (2\ 3)(1\ 6\ 2)(5\ 6\ 2\ 4), \quad (5\ 6\ 2\ 4)^{-1}.$$

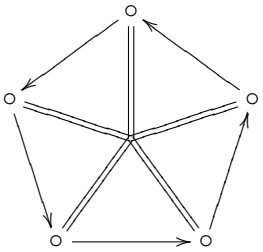
2.3. (a) Consider a regular pentagon $\mathcal{P}$ with vertices A, B, C, D, E appearing in anti-clockwise order around its centre which is at the origin O.



Find all ten symmetries of $\mathcal{P}$, describing them geometrically and in permutation notation.
(b) Work out the effect of the two possible compositions of reflection in the line OA with reflection in the line OC.
(c) Work out the effect of the two possible compositions of reflection in the line OA with rotation through $3/5$ of a turn anti-clockwise.

2.4. Determine the symmetry groups of each of the following plane figures.

(i) 

(ii) 

2.5. (a) Let $S \subseteq \mathbb{R}^2$ be a non-empty subset and $P \in S$ any point in S. Show that

$$\Gamma_{S,P} = \{\alpha \in \mathrm{Euc}(2)_S : \alpha(P) = P\} \subseteq \mathrm{Euc}(2)_S$$

is a subgroup of $\mathrm{Euc}(2)_S$.
(b) When S is a square with vertices A, B, C, D, determine $\Gamma_{S,D}$. [*See Example 2.24.*]
(c) When S is a rectangle with vertices A, B, C, D and sides $|AB| = |CD| = 2|AD| = 2|BC|$, determine $\Gamma_{S,D}$. [*See Example 2.25 in the Notes.*]

2.6. Let

$$\mathbb{T} = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\} \subseteq \mathbb{R}^2$$

be the unit circle. Determine the symmetry subgroup $\mathrm{Euc}(2)_{\mathbb{T}}$. Does $\mathrm{Euc}(2)_{\mathbb{T}}$ have any finite subgroups?

2.7. If $H\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is a similarity transformation, show that H preserves angles between lines.

2.8. Show that the set of all similarity transformations $\mathbb{R}^2 \longrightarrow \mathbb{R}^2$ forms a group $(\Sigma(2), \circ)$ under composition and that $\mathrm{Euc}(2) \leqslant \Sigma(2)$. Find another interesting subgroup of $\Sigma(2)$.

2.9. Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be two lines in the plane. Show that the two reflections in these lines, $\mathrm{Refl}_{\mathcal{L}_1}$ and $\mathrm{Refl}_{\mathcal{L}_2}$, are similar.

2.10. Let $\mathrm{Rot}_{C,\theta}$ be a non-trivial rotation through angle θ about the point C with position vector $\mathbf{c}$. If $\mathbf{t}$ is a non-zero vector, show that $\mathrm{Trans}_{\mathbf{t}} \circ \mathrm{Rot}_{C,\theta}$ is rotation through θ about the point C' with position vector

$$\mathbf{c}' = \mathbf{c} + \frac{1}{2}\begin{bmatrix} 1 & -\cot(\theta/2) \\ \cot(\theta/2) & 1 \end{bmatrix}\mathbf{t}.$$

2.11. Consider the cyclic subgroup $\Gamma \leqslant \mathrm{Euc}(2)$ generated by the isometry $\gamma = (C \mid \mathbf{w})$, where

$$C = \begin{bmatrix} -1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} -\sqrt{3}/2 \\ 3/2 \end{bmatrix}.$$

(a) Show that Γ has order 3 and list its elements.
(b) Use the idea in the proof of Theorem 2.32 to find a fixed point of Γ. Is it the only one?
(c) Find a suitable similarity transformation ψ for which $\psi_*\Gamma \leqslant \mathrm{O}(2)$.

2.12. Consider the finite subgroup $\Gamma \leqslant \mathrm{Euc}(2)$ of order 8 generated by the isometries $\alpha = (A \mid \mathbf{u})$ and $\beta = (B \mid \mathbf{v})$, where

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

(a) By considering enough isometries of the form $\alpha^r\beta^s$, find the Seitz symbols of all 8 elements of Γ.
(b) Use the idea in the proof of Theorem 2.32 to find a fixed point of Γ. Is it the only one?
(c) Find a suitable similarity transformation φ for which $\varphi_*\Gamma \leqslant \mathrm{O}(2)$.

2.13. Let $\Gamma \leqslant \mathrm{Euc}(2)$ be a subgroup containing the isometries $F, G\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2$.
(a) If F and G are reflections in two distinct parallel lines, show that there is no point fixed by all the elements of Γ. Deduce that Γ is infinite.
(b) If F is the reflection in a line $\mathcal{L}$ and G is a non-trivial rotation about a point $\mathbf{p}$ *not on* $\mathcal{L}$, show that Γ is infinite.

2.14. Let $\Gamma \leqslant \mathrm{Euc}(2)$ be a subgroup containing the isometries $F, G\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ and suppose that these generate Γ in the sense that every element of Γ is obtained by repeatedly composing powers of F and G. If a point $\mathbf{p}$ is fixed by both F and G, show that it is fixed by every element of Γ.

2.15. Classify each of the frieze patterns in Figure 2.25 as one of the 7 types.

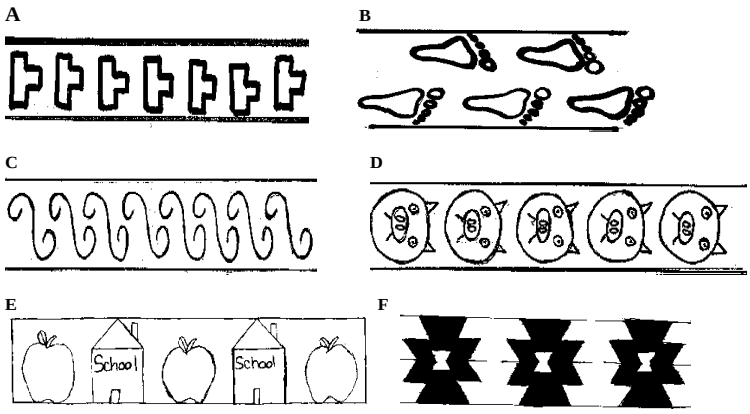
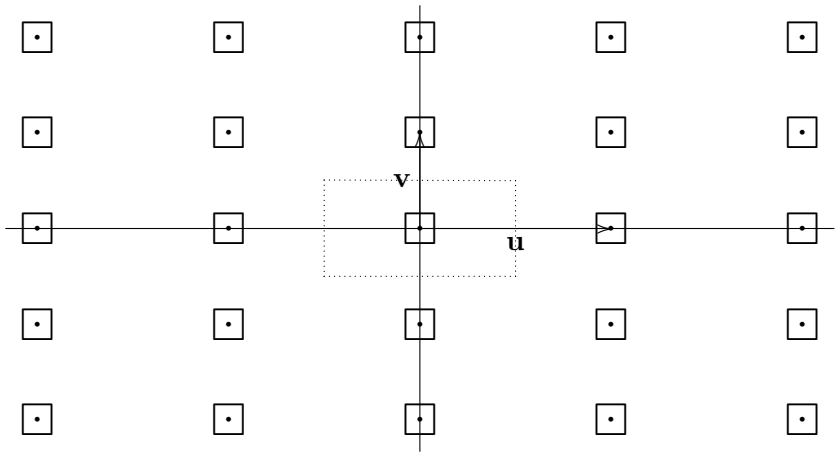


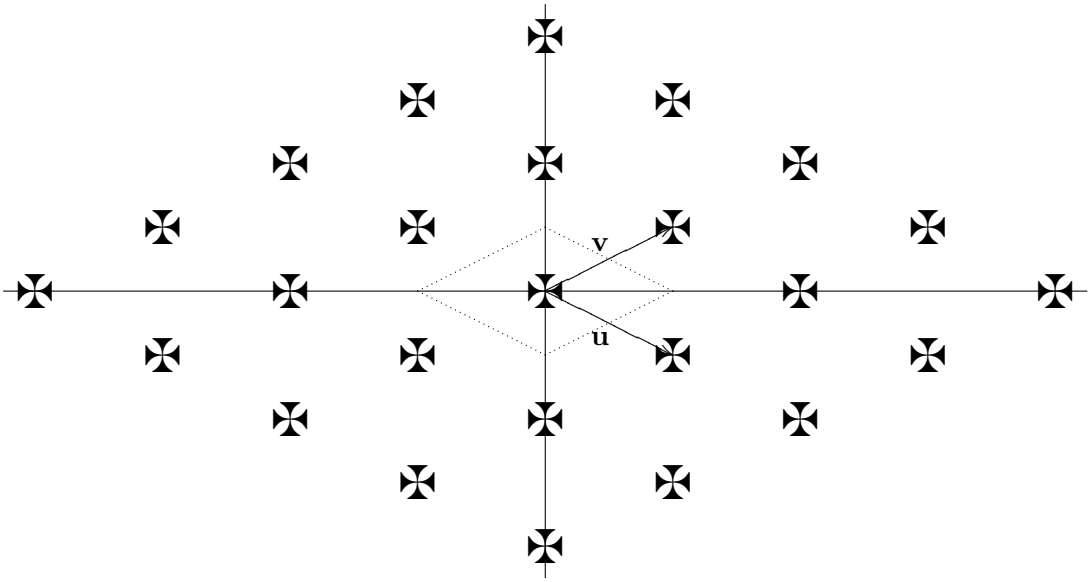
FIGURE 2.25. Examples of freize patterns

2.16. Discuss the symmetry groups of the following wallpaper patterns using the indicated vectors **u** and **v** as generators for the translation subgroup and where the dashed lines indicate a fundamental region. Classify each pattern as one of the 17 basic types.

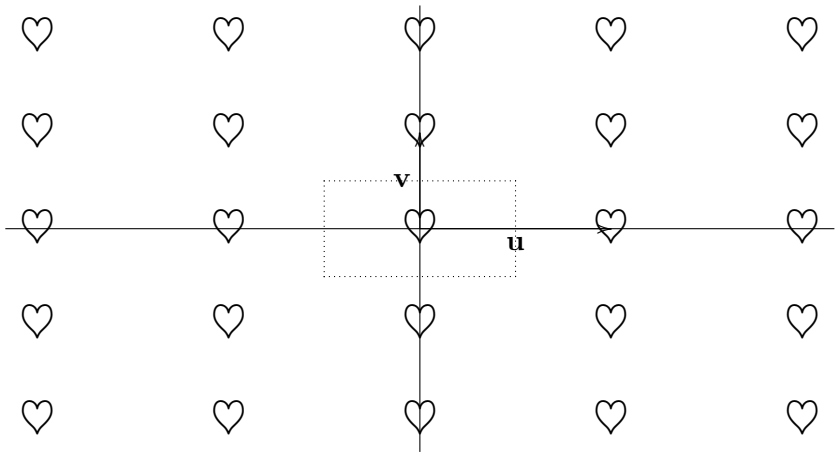
(a)



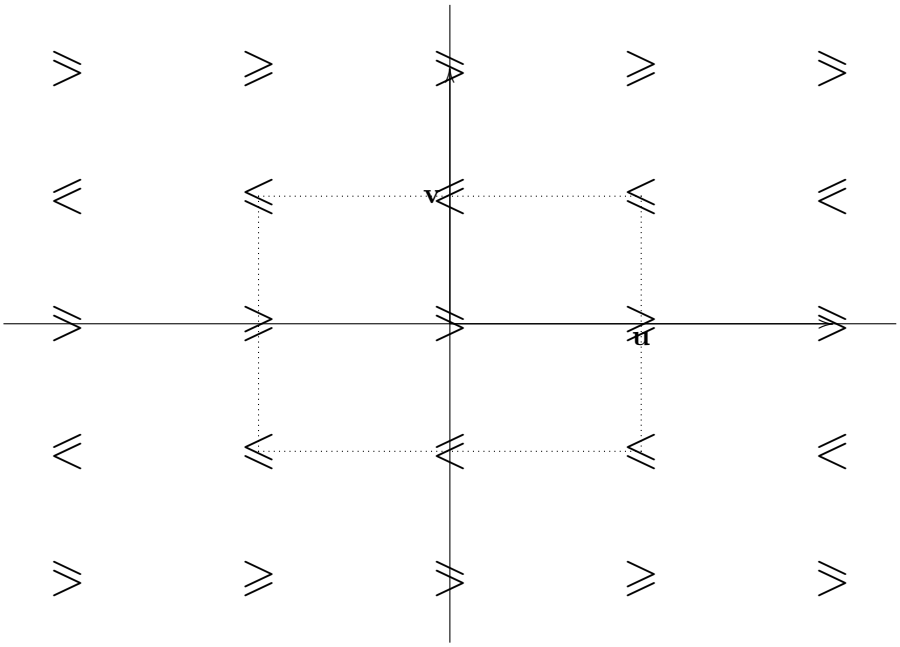
(b)



(c)



(d)



CHAPTER 3

Isometries in 3 dimensions

1. Some 3-dimensional vector geometry

We will use similar notation to that of Chapter 1, denoting the position vector of a point P in 3-space $\mathbf{p}$, etc. We will also identify P with its position vector $\mathbf{p} \in \mathbb{R}^3$ and the position vector (x, y, z) with the column vector $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$. We will often use the standard unit vectors $\mathbf{i} = (1, 0, 0)$, $\mathbf{j} = (0, 1, 0)$, $\mathbf{k} = (0, 0, 1)$.

The *scalar product* of the vectors $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$ is the real number

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + v_2 u_2 + v_3 u_3 \in \mathbb{R}.$$

The *length* of $\mathbf{v}$ is the non-negative real number

$$|\mathbf{v}| = \sqrt{\mathbf{v} \cdot \mathbf{v}} = \sqrt{v_1^2 + v_2^2 + v_3^2}.$$

The *angle between* $\mathbf{u}$ *and* $\mathbf{v}$ is

$$\cos^{-1} \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}|\,|\mathbf{v}|} \in [0, \pi].$$

In place of lines in $\mathbb{R}^2$ we now have *planes* in $\mathbb{R}^3$. Such a plane $\mathcal{P}$ is specified by an implicit equation of the form

$$ax + by + cz = d,$$

where $(a, b, c) \neq \mathbf{0}$. Thus we have

$$\mathcal{P} = \{(x, y, z) \in \mathbb{R}^3 : ax + by + cz = d\}. \tag{3.1a}$$

It is worth remarking that the vector (a, b, c) is perpendicular to $\mathcal{P}$. An alternative way to write the implicit equation is

$$(a, b, c) \cdot (x, y, z) = d,$$

so we also have

$$\mathcal{P} = \{\mathbf{x} \in \mathbb{R}^3 : (a, b, c) \cdot \mathbf{x} = d\}. \tag{3.1b}$$

To determine d it suffices to know any point $\mathbf{x}_0$ on $\mathcal{P}$, then $d = (a, b, c) \cdot \mathbf{x}_0$.

Suppose that the vectors $\mathbf{u}$ and $\mathbf{v}$ are parallel to $\mathcal{P}$ and so perpendicular to (a, b, c) and also that neither is a scalar multiple of the other (hence they are *linearly independent*) then we can use the parametric equation

$$\mathbf{x} = s\mathbf{u} + t\mathbf{v} + \mathbf{x}_0,$$

where $s, t \in \mathbb{R}$ and $\mathbf{x}_0$ is some point on $\mathcal{P}$. It is often useful to take $\mathbf{u}$ and $\mathbf{v}$ to be unit vectors. Then

$$\mathcal{P} = \{s\mathbf{u} + t\mathbf{v} + \mathbf{x}_0 \in \mathbb{R}^3 : s, t \in \mathbb{R}\}. \tag{3.1c}$$

Given this parametric form for the plane $\mathcal{P}$, it is possible to find a vector normal to it using the *vector* or *cross product* of $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$, which is the vector

$$\mathbf{u} \times \mathbf{v} = (D_1, D_2, D_3),$$

for which

$$D_1 = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} = u_2 v_3 - u_3 v_2,\ D_2 = -\begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} = u_3 v_1 - u_1 v_3,\ D_3 = \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} = u_1 v_2 - u_2 v_1.$$

A useful way of writing the cross product involves 3×3 *determinants* which are defined by

(3.2a)
$$\det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{21} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}.$$

In fact, we can expand along any row, using the alternative but equivalent formulæ

(3.2b)
$$\det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = -a_{21} \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{22} \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} - a_{23} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix},$$

(3.2c)
$$\det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{31} \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} - a_{32} \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} + a_{33} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}.$$

Note the pattern of signs appearing in front of the terms $a_{ij} \begin{vmatrix} * & * \\ & * \end{vmatrix}$ which can be remembered from

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}.$$

Then

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ \mathbf{i} & \mathbf{j} & \mathbf{k} \end{vmatrix} = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \mathbf{k} = D_1\mathbf{i} + D_2\mathbf{j} + D_3\mathbf{k}. \tag{3.3}$$

3.1. Proposition. *The vector product has the following properties. For* $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$ *and* $t \in \mathbb{R}$,

(a) $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}$,

(b) $(t\mathbf{u}) \times \mathbf{v} = t(\mathbf{u} \times \mathbf{v})$,

(c) $\mathbf{v} \times \mathbf{u} = -\mathbf{u} \times \mathbf{v}$,

(d) $\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) = 0$,

(e) $\mathbf{u} \times \mathbf{v} \neq \mathbf{0}$ if $\mathbf{u}$ and $\mathbf{v}$ are linearly independent.

3.2. Corollary. $\mathbf{u} \times \mathbf{v}$ *is normal to* $\mathbf{u}$ *and* $\mathbf{v}$.

Proof. This follows from (c) and (d). □

Thus for the above plane $\mathcal{P}$, $\mathbf{u} \times \mathbf{v}$ is a vector normal to $\mathcal{P}$ and so we obtain the implicit equation

$$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{x} = (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{x}_0,$$

where $\mathbf{x}_0$ is any known point on $\mathcal{P}$.

Given an implicit equation for a plane of the form

$$\mathbf{w} \cdot \mathbf{x} = d$$

together with a non-zero vector $\mathbf{u}$ parallel to $\mathcal{P}$, we can use the vector product to find another vector parallel to $\mathcal{P}$, namely

$$\mathbf{v} = \mathbf{w} \times \mathbf{u}.$$

The vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ (taken in that order) form a *right handed system* in that they have the same orientation as the first finger, second finger and thumb of a right hand, or equivalently

are oriented like the standard basis vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$ or the positive x, y and z-axes. We also have the formulæ

$$\mathbf{w} = \mathbf{u} \times \mathbf{v}, \quad \mathbf{u} = \mathbf{v} \times \mathbf{w}.$$

Each of the sequences $\mathbf{v}, \mathbf{w}, \mathbf{u}$ and $\mathbf{w}, \mathbf{u}, \mathbf{v}$ is a right handed system, while the sequences $\mathbf{u}, \mathbf{w}, \mathbf{v}$, $\mathbf{v}, \mathbf{u}, \mathbf{w}$ and $\mathbf{w}, \mathbf{v}, \mathbf{u}$ are all *left handed*. If the vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are mutually normal unit vectors they are said to form a right or left handed *orthonormal system*. Here is a useful result for checking whether a system of mutually normal unit vectors is right or left handed.

3.3. Proposition. *A sequence of mutually normal unit vectors* $\mathbf{u}, \mathbf{v}, \mathbf{w}$ *is a right handed orthonormal system if and only if*

$$\begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = 1.$$

Proof. This follows from the fact that

$$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}. \tag{3.4}$$

This quantity is often called the *vector triple product* of $\mathbf{u}, \mathbf{v}, \mathbf{w}$ and written $[\mathbf{u}, \mathbf{v}, \mathbf{w}]$. □

3.4. Example. Find implicit and parametric equations for the plane $\mathcal{P}$ containing the points with position vectors $\mathbf{p} = (1, 0, 1)$, $\mathbf{q} = (1, 1, 1)$ and $\mathbf{r} = (0, 1, 0)$.

Solution. Let us begin with a parametric equation. Notice that the vectors

$$\mathbf{u} = \mathbf{q} - \mathbf{p} = (0, 1, 0), \quad \mathbf{v} = \mathbf{r} - \mathbf{p} = (-1, 1, -1)$$

are parallel to $\mathcal{P}$ and linearly independent since neither is a scalar multiple of the other. Thus a parametric equation is

$$\mathbf{x} = s(0, 1, 0) + t(-1, 1, -1) + (1, 0, 1) = (1 - t, s + t, 1 - t) \quad (s, t \in \mathbb{R}).$$

To obtain an implicit equation we need a vector normal to $\mathcal{P}$. For this we can use

$$\mathbf{w} = \mathbf{u} \times \mathbf{v} = (-1, 0, 1).$$

This gives the equation

$$(-1, 0, 1) \cdot \mathbf{x} = (-1, 0, 1) \cdot (1, 0, 1) = 0$$

since $\mathbf{p} = (1, 0, 1)$ is in $\mathcal{P}$. On writing $\mathbf{x} = (x, y, z)$ this becomes

$$-x + z = 0.$$

In this example we could start with $\mathbf{w} = (-1, 0, 1)$ and $\mathbf{u} = (0, 1, 0)$ then produce a second vector parallel to $\mathcal{P}$, namely

$$\mathbf{w} \times \mathbf{u} = (-1, 0, 1) \times (0, 1, 0) = (1, 0, -1).$$

These three vectors are mutually perpendicular. □

2. Isometries of 3-dimensional space

3.5. Definition. An *isometry* of $\mathbb{R}^3$ is a distance preserving function $F\colon \mathbb{R}^3 \longrightarrow \mathbb{R}^3$.

Here, the phrase *distance preserving* means that for points P and Q with position vectors $\mathbf{p}$ and $\mathbf{q}$,

$$|F(P)F(Q)| = |PQ|, \qquad \textit{i.e.}, \quad |F(\mathbf{p}) - F(\mathbf{q})| = |\mathbf{p} - \mathbf{q}|.$$

3.6. Proposition. *An isometry* $F\colon \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ *preserves angles between lines and vectors.*

We have a similar result to a familiar one for $\mathbb{R}^2$.

3.7. THEOREM. *Every isometry* $F\colon \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ *has the form*

$$F(\mathbf{x}) = A\mathbf{x} + \mathbf{t},$$

where A *is a* 3×3 *real orthogonal matrix and* $\mathbf{t} \in \mathbb{R}^3$. *Furthemore,* A *and* $\mathbf{t}$ *are uniquely detrmined by* F.

We will use the Seitz symbol $(A \mid \mathbf{t})$ to denote the isometry $\mathbb{R}^3 \longrightarrow \mathbb{R}^3$ defined by

$$\mathbf{x} \longmapsto A\mathbf{x} + \mathbf{t}.$$

As in the 2-dimensional situation, a 3×3 real orthogonal matrix A has determinant $\det A = \pm 1$. There are several types of isometries in $\mathbb{R}^3$, some very different from those occurring in $\mathbb{R}^2$.

Translations. These have Seitz symbols of form $(I_3 \mid \mathbf{t})$ and behave in similar ways to translations of $\mathbb{R}^2$.

Reflections. In $\mathbb{R}^3$, the *reflection in a plane* $\mathcal{P}$ has Seitz symbol of the form $(S \mid 2\mathbf{w})$ where $\mathbf{w}$ is the position vector of a point on $\mathcal{P}$ and is perpendicular to this plane, while the orthogonal matrix S has the effect

$$S\mathbf{w} = -\mathbf{w}, \qquad S\mathbf{u} = \mathbf{u} \quad \text{for } \mathbf{u} \text{ a vector parallel to } \mathcal{P}.$$

Such an orthogonal matrix has determinant $\det S = -1$.

Glide reflections. The Seitz symbol of a *glide reflection in a plane* $\mathcal{P}$ has the form

$$(S \mid 2\mathbf{w} + \mathbf{u}) = (I_3 \mid \mathbf{u})(S \mid 2\mathbf{w}),$$

where $(S \mid 2\mathbf{w})$ is reflection in the plane $\mathcal{P}$ containing the point with position vector $\mathbf{w}$ which is also perpendicular to $\mathcal{P}$, while $\mathbf{u}$ is parallel to $\mathcal{P}$. This is very similar to the situation with a glide reflection in a line in $\mathbb{R}^2$. We will usually think of reflections as glide reflections.

Rotations. The Seitz symbol of a *rotation about a line* has Seitz symbol of the form $(R \mid \mathbf{t})$ where R is orthogonal and $\det R = 1$. Using ideas about eigenvalues and eigenvectors it can be shown that for such a matrix, either $R = I_3$ or $(R \mid \mathbf{0})$ represents a rotation about the line through the origin

$$\mathcal{L}_1 = \{\mathbf{x} \in \mathbb{R}^3 : R\mathbf{x} = \mathbf{x}\}.$$

We usually refer to $\mathcal{L}_1$ as the *axis of rotation* of $(R \mid \mathbf{0})$ or even of R.

If we choose a unit vector $\mathbf{v}_1 \in \mathcal{L}_1$ and any non-zero vector $\mathbf{v}_2$ perpendicular to $\mathcal{L}_1$, then $\mathbf{v}_3 = \mathbf{v}_1 \times \mathbf{v}_2$ is perpendicular to $\mathcal{L}_1$ and in fact

$$\mathbf{v}_3 \cdot \mathbf{v}_2 = 0 = \mathbf{v}_3 \cdot \mathbf{v}_1, \quad |\mathbf{v}_3| = |\mathbf{v}_2|.$$

In practise we usually take $|\mathbf{v}_3| = |\mathbf{v}_2| = 1$. These three vectors form a right handed *orthonormal basis* of $\mathbb{R}^3$ and every vector $\mathbf{x} \in \mathbb{R}^3$ can be uniquely expressed as

$$\mathbf{x} = x_1'\mathbf{v}_1 + x_2'\mathbf{v}_2 + x_3'\mathbf{v}_3$$

where

$$x_1' = \mathbf{v}_1 \cdot \mathbf{x}, \quad x_2' = \mathbf{v}_2 \cdot \mathbf{x}, \quad x_3' = \mathbf{v}_3 \cdot \mathbf{x}.$$

The effect of the rotation $(R \mid \mathbf{0})$ on an arbitrary vector is given by

$$\begin{aligned}(R \mid \mathbf{0})(x_1'\mathbf{v}_1 + x_2'\mathbf{v}_2 + x_3'\mathbf{v}_3) &= x_1'\mathbf{v}_1 + x_2'(\cos\theta\mathbf{v}_2 + \sin\theta\mathbf{v}_3) + x_3'(-\sin\theta\mathbf{v}_2 + \cos\theta\mathbf{v}_3) \\ &= x_1'\mathbf{v}_1 + (x_2'\cos\theta - x_3'\sin\theta)\mathbf{v}_2 + (x_2'\sin\theta + x_3'\cos\theta)\mathbf{v}_3\end{aligned} \tag{3.5}$$

for some angle $\theta \in \mathbb{R}$. A convenient way to denote this expressions is by

$$(R \mid \mathbf{0})(x_1'\mathbf{v}_1 + x_2'\mathbf{v}_2 + x_3'\mathbf{v}_3) = \begin{bmatrix}\mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3\end{bmatrix} \begin{bmatrix}1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta\end{bmatrix} \begin{bmatrix}x_1' \\ x_2' \\ x_3'\end{bmatrix}, \tag{3.6}$$

where $\begin{bmatrix}\mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3\end{bmatrix}$ is really an example of a *block form matrix* that some students may have met in a course on Linear Algebra. The matrix in the middle on the right hand side contains the familiar 2×2 block

$$\begin{bmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{bmatrix}$$

which represents a rotation through θ in the plane spanned by $\mathbf{v}_2$ and $\mathbf{v}_3$, so θ is the angle of rotation about $\mathcal{L}_1$.

The more general type of rotation is about a line $\mathcal{L}'$ parallel to a line $\mathcal{L}_1$ through the origin. In this situation, the Seitz symbol $(R \mid \mathbf{t})$ has translation vector $\mathbf{t}$ perpendicular to $\mathcal{L}_1$ and $\mathcal{L}'$. This means that $\mathbf{t}$ is in the plane $\mathcal{P}$ and so it can be expressed as

$$\mathbf{t} = t_2\mathbf{v}_2 + t_3\mathbf{v}_3.$$

Then the point in $\mathcal{P}$ with position vector $\mathbf{c} = c_2\mathbf{v}_2 + c_3\mathbf{v}_3$ satisfying

$$(R \mid \mathbf{t})\mathbf{c} = \mathbf{c}$$

is found by solving the equation

$$R\mathbf{c} + \mathbf{t} = \mathbf{c},$$

or equivalently

$$(I_3 - R)\mathbf{c} = \mathbf{t}.$$

In matrix form this becomes

$$\begin{bmatrix}\mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3\end{bmatrix}\begin{bmatrix}0 & 0 & 0 \\ 0 & 1-\cos\theta & \sin\theta \\ 0 & -\sin\theta & 1-\cos\theta\end{bmatrix}\begin{bmatrix}0 \\ c_2 \\ c_3\end{bmatrix} = \begin{bmatrix}\mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3\end{bmatrix}\begin{bmatrix}0 \\ t_2 \\ t_3\end{bmatrix},$$

or equivalently,

$$\begin{bmatrix}1-\cos\theta & \sin\theta \\ -\sin\theta & 1-\cos\theta\end{bmatrix}\begin{bmatrix}c_2 \\ c_3\end{bmatrix} = \begin{bmatrix}t_2 \\ t_3\end{bmatrix}.$$

Since

$$\det\begin{bmatrix}1-\cos\theta & \sin\theta \\ -\sin\theta & 1-\cos\theta\end{bmatrix} = (1-\cos\theta)^2 + \sin^2\theta = 2(1-\cos\theta),$$

this determinant is non-zero if $R \neq I_3$, and the equation then has the unique solution

$$\begin{bmatrix}c_2 \\ c_3\end{bmatrix} = \begin{bmatrix}1-\cos\theta & \sin\theta \\ -\sin\theta & 1-\cos\theta\end{bmatrix}^{-1}\begin{bmatrix}t_2 \\ t_3\end{bmatrix},$$

using essentially the same algebra in the 2-dimensional situation of rotation about a point. The vector $\mathbf{c}$ is the position vector of a point on the axis of rotation $\mathcal{L}'$.

Screw rotations. There are some new types of isometry in $\mathbb{R}^3$ which have no analogues in $\mathbb{R}^2$. The first of these is *screw rotation*. Such an isometry has Seitz symbol $(R \mid \mathbf{t})$ where $\det R = 1$ and we will write

$$\mathbf{t} = \mathbf{t}' + \mathbf{w},$$

with $\mathbf{t}'$ normal to the axis of rotation $\mathcal{L}$ of R and $\mathbf{w}$ parallel to it. Then

$$(R \mid \mathbf{t}) = (R \mid \mathbf{t}' + \mathbf{w}) = (I_3 \mid \mathbf{w})(R \mid \mathbf{t}'),$$

which represents a rotation about a line parallel to $\mathcal{L}$ followed by a translation parallel to $\mathcal{L}$.

If $\mathbf{w} \neq \mathbf{0}$, we usually think of the direction of $\mathbf{w}$ as that of a forward pointing screw and measure the angle of rotation as positive if the plane turns like a right handed screw driver. This means that if we assume that the unit vector $\hat{\mathbf{w}}$ together with a pair of normal unit vectors $\hat{\mathbf{u}}, \hat{\mathbf{v}}$ perpendicular to $\mathcal{L}$ form a right handed system $\hat{\mathbf{u}}, \hat{\mathbf{v}}, \hat{\mathbf{w}}$, then the angle of rotation θ is given by

$$R\hat{\mathbf{u}} = \cos\theta\hat{\mathbf{u}} + \sin\theta\hat{\mathbf{v}}, \quad R\hat{\mathbf{v}} = -\sin\theta\hat{\mathbf{u}} + \cos\theta\hat{\mathbf{v}}.$$

3.8. Example. Describe the screw rotation whose Seitz symbol is $(R \mid \mathbf{t})$ where

$$R = \begin{bmatrix} \sqrt{3}/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ -1/2 & 0 & \sqrt{3}/2 \end{bmatrix}, \quad \mathbf{t} = (0, 7, 0),$$

giving its angle and axis of rotation and the translation parallel to the latter.

Solution. Notice that

$$R = \begin{bmatrix} \cos \pi/6 & 0 & \sin \pi/6 \\ 0 & 1 & 0 \\ -\sin \pi/6 & 0 & \cos \pi/6 \end{bmatrix}, \quad \det R = 1,$$

so this matrix represents a rotation. First we find the axis of rotation of the matrix R. Clearly this is the y-axis, so let us take the unit vector $\mathbf{j} = (0, 1, 0)$ parallel to it. Now the unit vectors $\mathbf{i} = (1, 0, 0)$ and $\mathbf{k} = (0, 0, 1)$ are normal and also normal to $\mathbf{j}$. We have to take a right handed system made up from $\pm\mathbf{i}, \pm\mathbf{j}, \pm\mathbf{k}$. The system $\mathbf{i}, \mathbf{k}, \mathbf{j}$ is left handed, but since $\mathbf{i} \times (-\mathbf{k}) = \mathbf{j}$, we can replace $\mathbf{k}$ by $-\mathbf{k}$ to get the right handed orthonormal system $\mathbf{i}, -\mathbf{k}, \mathbf{j}$. For this we have

$$R\mathbf{i} = \cos \pi/6\mathbf{i} + \sin \pi/6(-\mathbf{k}), \quad R(-\mathbf{k}) = -\sin \pi/6\mathbf{i} + \cos \pi/6(-\mathbf{k}).$$

Hence R represents a rotation through $\pi/6$ about the y-axis where we measure positive angles so that $\mathbf{i}$ turns towards $-\mathbf{k}$. The translation vector $\mathbf{t} = (0, 7, 0) = 7\mathbf{j}$ points in the same direction as $\mathbf{j}$. □

Screw reflections. A *screw reflection* has Seitz symbol of the form $(A \mid \mathbf{t})$ where A is orthogonal and $\det A = -1$. Then there is a unit vector $\mathbf{v}_1$ for which

$$A\mathbf{v}_1 = -\mathbf{v}_1,$$

and a unit vector $\mathbf{v}_2$ normal to $\mathbf{v}_1$ together with the unit vector $\mathbf{v}_3 = \mathbf{v}_1 \times \mathbf{v}_2$ for which

$$A\mathbf{v}_2 = \cos\theta\mathbf{v}_2 + \sin\theta\mathbf{v}_3, \quad A\mathbf{v}_3 = -\sin\theta\mathbf{v}_2 + \cos\theta\mathbf{v}_3.$$

information can be rewritten in the form

$$\begin{bmatrix} A\mathbf{v}_1 & A\mathbf{v}_2 & A\mathbf{v}_3 \end{bmatrix} = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

Geometrically this represents the composition of reflection in the plane perpendicular to $\mathbf{v}_1$ composed with rotation about the line through the origin parallel to $\mathbf{v}_1$.

If we now write

$$\mathbf{t} = \mathbf{u} + 2\mathbf{w}$$

where $\mathbf{w}$ is parallel to $\mathbf{v}_1$ and $\mathbf{u}$ is perpendicular to $\mathbf{v}_1$, then $(A \mid \mathbf{t})$ represents the composition of reflection in the plane

$$\mathbf{v}_1 \cdot \mathbf{x} = \mathbf{v}_1 \cdot \mathbf{w}$$

and rotation about the axis parallel to $\mathbf{v}_1$ and passing through the point with position vector $\mathbf{c}$ and satisfying the two conditions

$$\mathbf{v}_1 \cdot \mathbf{c} = 0, \quad (I - A)\mathbf{c} = \mathbf{u}.$$

3.9. Definition. Suppose we are considering a Seitz symbol $(A \mid \mathbf{0})$ or equivalently the orthogonal matrix $A \neq I_3$. If $\det A = 1$, then the line $\mathcal{L}$ through the origin whose vectors are fixed by A is called the *line of rotation of* A. On the other hand, if $\det A = -1$, then the plane $\mathcal{P}$ through the origin for which points not in it are reflected through it by A is called the *plane of reflection of* A. We can find this by first determining a vector perpendicular to $\mathcal{P}$ by solving the equation $A\mathbf{v} = -\mathbf{v}$, then taking the equation $\mathbf{v} \cdot x = 0$ to define $\mathcal{P}$.

3. The Euclidean group of $\mathbb{R}^3$

The set of all isometries $\mathbb{R}^3 \longrightarrow \mathbb{R}^3$ is denoted Euc(3) and forms the 3-dimensional Euclidean group $(\mathrm{Euc}(3), \circ)$ under composition of functions. Much of the general theory for Euc(2) discussed in Chapter 1 carries over to Euc(3) with obvious minor modifications. In particular, the following generalization of Theorem 2.32 is true.

3.10. Theorem. *Let $\Gamma \leqslant \mathrm{Euc}(3)$ be a finite subgroup. Then there is a point of $\mathbb{R}^2$ fixed by every element of Γ.*

There are obvious notions of similarity generalizing those for Euc(2) to Euc(3).

We define the 3×3 *orthogonal group* to be

$$\mathrm{O}(3) = \{(A \mid \mathbf{0}) : A^T A = I_3\} \leqslant \mathrm{Euc}(3)$$

and the 3×3 *special orthogonal group* to be

$$\mathrm{SO}(3) = \{(A \mid \mathbf{0}) : A^T A = I_3,\ \det A = 1\} \leqslant \mathrm{O}(3) \leqslant \mathrm{Euc}(3).$$

Notice that we can view Euc(2) as a subgroup of Euc(3), *i.e.*, $\mathrm{Euc}(2) \leqslant \mathrm{Euc}(3)$, by thinking of Euc(2) as consisting of isometries of $\mathbb{R}^3$ that fix all the points on the z-axis. Thus

$$\mathrm{Euc}(2) = \left\{ (A \mid (t_1, t_2, 0)) : A = \begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\} \leqslant \mathrm{Euc}(3).$$

In particular we have $\mathrm{O}(2) \leqslant \mathrm{O}(3)$ and $\mathrm{SO}(2) \leqslant \mathrm{SO}(3)$.

As far as the finite subgroups of Euc(3) are concerned, the following result classifies them up to similarity.

3.11. Proposition. *Let $\Gamma \leqslant \mathrm{Euc}(3)$ be a finite subgroup. Then Γ is similar to either a subgroup of* O(2) *or to the symmetry group of one of the* Platonic solids.

The *Platonic solids* are the five regular solids, *i.e.*, the *tetrahedron*, the *cube*, the *octahedron*, the *icosahedron* and the *dodecahedron*. These have the following numbers of vertices, edges and faces.

	vertices	edges	faces	edges on each face
Tetrahedron	4	6	4	3
Cube	8	12	6	4
Octahedron	6	12	8	3
Icosahedron	12	30	20	3
Dodecahedron	20	30	12	5

For an interactive introduction to these see the **Mathworld** web page

`http://mathworld.wolfram.com/topics/PlatonicSolids.html`

and also

`http://www.venuemedia.com/mediaband/collins/cube.html`

Exercises on Chapter 3

3.1. In $\mathbb{R}^3$, find implicit and parametric forms for the plane $\mathcal{P}$ which contains the three points $A(1,1,-2)$, $B(0,2,1)$, $C(1,-1,2)$.

3.2. For each of the following matrices, show that it represents an isometry of $\mathbb{R}^3$ which fixes the origin and decide whether it represents a rotation about a line or a reflection in a plane. In each case, find the line of rotation or plane of reflection.

$$\text{(a)} \quad \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{6} & 1/\sqrt{6} & -2/\sqrt{6} \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \end{bmatrix}, \quad \text{(b)} \quad \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & -1/\sqrt{3} \\ 1/\sqrt{6} & 1/\sqrt{6} & 2/\sqrt{6} \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \end{bmatrix}.$$

3.3. If $S = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix}$ and $\mathbf{t} = (1,0,1)$, show that the isometry of $\mathbb{R}^3$ whose Seitz symbol is $(S \mid \mathbf{t})$ represents a glide reflection. Find the reflecting plane and the translation parallel to it.

3.4. Show that the isometry of $\mathbb{R}^3$ whose Seitz symbol is $(R \mid \mathbf{t})$ represents a screw rotation, where

$$R = \begin{bmatrix} -1/2 & 0 & -\sqrt{3}/2 \\ 0 & 1 & 0 \\ \sqrt{3}/2 & 0 & -1/2 \end{bmatrix}, \quad \mathbf{t} = (1,1,0).$$

Find the angle and axis of rotation and the translation parallel to it.