K-Theory and Noncommutative Geometry

Lecture 5 Cyclic Cohomology for Hopf Algebras

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References

The main paper is one which was already cited in Lecture1:

A. Connes and H. Moscovici, *Hopf algebras, cyclic cohomology and the transverse index theorem*, Comm. Math. Phys. **198** (1998), 199–246.

Various papers (available on the ArXiv server) provide improvements and surveys:

A. Connes and Moscovici, math.QA/9904154, math.QA/05013, math.OA/0002125, . . .

The following elegant paper develops cyclic theory for Hopf algebras from the point of view of Cuntz-Quillen theory:

M. Crainic, *Cyclic cohomology for Hopf algebras*, J. Pure Appl. Alg. **166** (2002), 29–66. math.QA/9812113.

Overview

Theme of the Lecture: There is a general construction

$$\underbrace{HC^*(\mathcal{H})}_{\text{HC}^* \text{ for Hopf algebras}} \xrightarrow[]{\text{Character map} \atop \tau^{\natural}} \underbrace{HC^*(A)}_{\text{HC}^* \text{ for algebras}}$$

which accounts for many geometric constructions of cyclic cocycles.

The index cocycles of the last lecture (analytic constructions) tend to be exceptions.

Next Lecture: We shall show that the residue cocycle is a Hopf cocycle.

Roughly speaking, to prove an index theorem is to identify an index cocycle with an explicit Hopf cocycle (say at the level of cohomology).

Review: Cyclic Theory

Recall that $\varphi\colon A^{\otimes^{n+1}} \to \mathbb{C}$ is a *cyclic n-cocycle* if

- $\varphi(a^0, a^1, \ldots, a^n) = (-1)^n \varphi(a^n, a^0, \ldots, a^{n-1})$
- $b\varphi(a^0, \ldots, a^{n+1}) = 0$, where

$$\begin{aligned} b\varphi(a^0, \ldots, a^{n+1}) = {} & \varphi(a^0 a^1, \ldots, a^{n+1}) \\ & - \varphi(a^0, a^1 a^2, \ldots, a^{n+1}) \\ & + \ldots \\ & + (-1)^{n+1} \varphi(a^{n+1} a^0, \ldots, a^n). \end{aligned}$$

The formula

$$\langle \varphi, p \rangle = \varphi(p, p, \ldots, p)$$

determines a pairing

$$HC^{2n}(A) \otimes K_0(A) \to \mathbb{C},$$

between cyclic cohomology (cyclic cocycles modulo coboundaries) and K-theory.

Examples of Cyclic Cocycles

From Lecture 3 . . .

$\mathfrak{g}$	Lie algebra
$\mathfrak{g} \otimes A \to A$	Action of $\mathfrak{g}$ by derivations
$\tau \colon A \to \mathbb{C}$	Invariant trace : $\tau(X(a)) = 0$

The homology of $\mathfrak{g}$ with coefficients $\mathbb{C}$ is computed from the 'Chevalley-Eilenberg' complex

$$\mathfrak{g} \xleftarrow{\delta} \mathfrak{g} \wedge \mathfrak{g} \xleftarrow{\delta} \mathfrak{g} \wedge \mathfrak{g} \wedge \mathfrak{g} \xleftarrow{\delta} \cdots$$

where

$$\delta(X_1 \wedge \cdots \wedge X_n) =$$
$$\sum_{i<j} (-1)^{i+j} [X_i, X_j] \wedge X_1 \wedge \cdots \wedge \widehat{X_i} \wedge \cdots \wedge \widehat{X_j} \wedge \cdots \wedge X_n.$$

Now, embed $\wedge^n \mathfrak{g}$ into $\otimes^n \mathfrak{g}$ by total antisymmetrization.

$$X_1 \wedge \cdots \wedge X_n \mapsto \sum_{\sigma} (-1)^{\sigma} X_{\sigma(1)} \otimes \cdots \otimes X_{\sigma(n)}.$$

Proposition. *The map from* $\otimes^n \mathfrak{g}$ *into* $\mathrm{Hom}(A^{\otimes^n}, \mathbb{C})$ *defined by the formula*

$$\phi_{X_1 \otimes \cdots \otimes X_n}(a^0, \ldots, a^n) = \tau\big(a^0 X_1(a^1) \ldots X_n(a^n)\big).$$

takes Lie algebra cycles to cyclic cocycles.

Proof of Cyclicity. Tricky. From

$$\begin{aligned} 0 &= \tau\left(X_n\left(a^0 X_1(a^1) \ldots X_{n-1}(a^{n-1}) a^n \right)\right) \\ &= \tau\left(X_n(a^0) X_1(a^1) \ldots X_{n-1}(a^{n-1}) a^n \right) \\ &\quad + \sum \tau\left(a^0 X_1(a^1) \ldots X_n(X_i(a^i)) \ldots X_{n-1}(a^{n-1}) a^n \right) \\ &\quad + \tau\left(a^0 X_1(a^1) \ldots X_{n-1}(a^{n-1}) X_n(a^n) \right) \end{aligned}$$

we get, for $c \in \wedge^n \mathfrak{g}$, $\phi_c - \lambda \phi_c = (-1)^{n-1} \psi_{\delta c}$, where

$$\psi_{Y_1 \otimes \cdots \otimes Y_{n-1}}(a^0, \ldots, a^n) = \psi(a^0 Y_1(a^1) \ldots Y^{n-1}(a^{n-1}) a^n).$$

$\square$

Proof that Coboundary is Zero. Easy. One has $b\varphi_c = 0$ for any Lie algebra chain c (not necessarily a cycle). □

Example. Let X_1 and X_2 be derivations on A, let τ be an invariant trace, and let

$$\phi(a^0, a^1, a^2) = \tau\big(a^0 X_1(a^1) X_2(a^2)\big) - \tau\big(a^0 X_2(a^1) X_1(a^2)\big).$$

One has

$$\phi(a^0, a^1, a^2) - \phi(a^2, a^1, a^0) = \tau\big(a^0 Y(a^1) a^2\big),$$

where

$$Y = [X_1, X_2] = -\delta\big(X_1 \wedge X_2\big).$$

So if X_1 and X_2 are commuting derivations then ϕ is a cyclic 2-cocycle.

(The irrational rotation algebra carries such a cyclic 2-cocycle.)

Index Theory

$\Psi(\mathcal{D}, \Delta)$ = Abstract pseudodifferential operators.

Assume the zeta-type functions

$$\zeta(s) = \mathrm{Trace}(T(I + \Delta)^{-s})$$

have meromorphic extensions (as in the classical case) and form

$$\tau(T) = \frac{1}{\mathrm{order}(\Delta)} \mathrm{Res}_{s=0}\Big(\mathrm{Trace}(T(I + \Delta)^{-s})\Big).$$

Suppose given $D \in \mathcal{D}$, $D^2 = \Delta$ and $D = \left(\begin{smallmatrix} 0 & D_- \\ D_+ & 0 \end{smallmatrix}\right)$.

If $A \subseteq \Psi_0(\mathcal{D}, \Delta)$ is comprised of even-order operators commuting with D modulo lower order terms then we get

$$\mathrm{Index}_{\varepsilon,D} \colon K_0(A) \to \mathbb{Z}.$$

Theorem (Connes and Moscovici). *The formula*

$$\Phi_n(a^0,\dots,a^n) = \sum_{k\geq 0} c_{nk}\tau(\varepsilon a_0[D,a^1]^{(k_1)}[D,a^2]^{(k_2)}\dots[D,a^n]^{(k_n)}\Delta^{-\frac{n+2k}{2}})$$

is an index cocycle in the (b,B)*-bicomplex.* $\square$

$$\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \vdots \\
b\uparrow & & b\uparrow & & b\uparrow & & b\uparrow \\
\mathrm{Hom}(A^{\otimes^4},\mathbb{C}) & \xrightarrow{B} & \mathrm{Hom}(A^{\otimes^3},\mathbb{C}) & \xrightarrow{B} & \mathrm{Hom}(A^{\otimes^2},\mathbb{C}) & \xrightarrow{B} & \mathrm{Hom}(A,\mathbb{C}) \\
b\uparrow & & b\uparrow & & b\uparrow & & \\
\mathrm{Hom}(A^{\otimes^3},\mathbb{C}) & \xrightarrow{B} & \mathrm{Hom}(A^{\otimes^2},\mathbb{C}) & \xrightarrow{B} & \mathrm{Hom}(A,\mathbb{C}) & & \\
b\uparrow & & b\uparrow & & & & \\
\mathrm{Hom}(A^{\otimes^2},\mathbb{C}) & \xrightarrow{B} & \mathrm{Hom}(A,\mathbb{C}) & & & & \\
b\uparrow & & & & & & \\
\mathrm{Hom}(A,\mathbb{C}) & & & & & &
\end{array}$$

The (b,B)-bicomplex.

Classical Case

M^{2m}	Spin Manifold
D	Dirac Operator
$\tau\colon \Psi(M,\Delta) \to \mathbb{C}$	Wodzicki Residue

A priori there are many terms in the C-M formula (e.g. 8 for $\dim(M) = 4$). However:

Theorem. *In the classical case, the $k \neq 0$ terms in the Connes-Moscovici formula vanish.* □

Moreover:

Theorem. *In the classical case*

$$\tau(\varepsilon a_0[D, a^1][D, a^2] \ldots [D, a^n]\Delta^{-\frac{n}{2}}) \\ = \text{constant} \cdot \int_M a^0 da^1 \ldots da^n \wedge \widehat{A}(M).$$

□

This follows from Getzler's approach to the Atiyah Singer Theorem.

Typical Case (Complexity Estimate)

In the *simplest* case of interest to Connes and Moscovici one has

- $\Gamma \subseteq \mathrm{Diffeo}^+(\mathbb{R})$
- $A = C_c^\infty(\mathbb{R}^2) \rtimes \Gamma$ (the crossed product algebra), $g(x,t) = \big(g(x), t + \log(g'(x))\big)$.
- $D = \begin{pmatrix} e^t \frac{\partial}{\partial x} & \frac{\partial^2}{\partial t^2} \\ \frac{\partial^2}{\partial t^2} & -e^t \frac{\partial}{\partial x} \end{pmatrix}$ (roughly speaking).

A typical generator of A looks like $f \cdot g$, and

$$[D, f \cdot g] = f[D, g] + [D, f]g$$

2 terms 3 terms

The terms are of the form $f \cdot g$, or worse, and (by my rough count)

$$[\Delta, f \cdot g] = f[\Delta, g] + [\Delta, f]g$$

4 terms 9 terms

Thus $[D, f \cdot g]^{(1)}$ has say 65 terms. The full CM formula has $\gg 500$ terms!

Bi-Algebras

We are going to generalize the construction of cyclic cocycles from Lie algebra cycles . . .

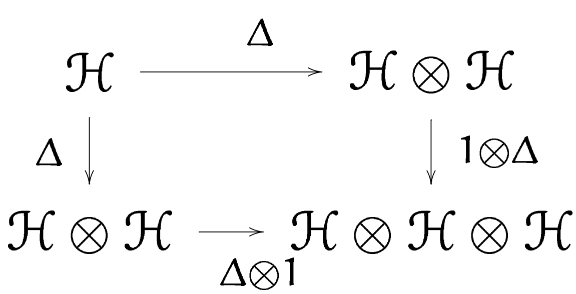
Definition. A *bi-algebra* is an associative algebra $\mathcal{H}$ with unit, equipped with algebra homomorphisms

$$\Delta\colon \mathcal{H} \to \mathcal{H} \otimes \mathcal{H} \qquad \text{(comultiplication)}$$

and

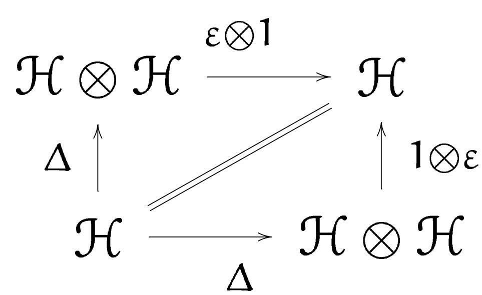
$$\varepsilon\colon \mathcal{H} \to \mathbb{C} \qquad \text{(co-unit)}$$

such that the following diagrams commute:



Co-associativity

and



Co-unit Property

Example. Let $\mathfrak{g}$ be a Lie algebra and let $\mathcal{H}$ be its enveloping algebra. For $X \in \mathfrak{g}$ define

$$\Delta(X) = X \otimes 1 + 1 \otimes X \quad \text{and} \quad \varepsilon(X) = 0.$$

Since for example $\Delta\colon \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}$ is a Lie algebra map we obtain $\Delta\colon \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$.

Example. Let G be a discrete group and $\mathcal{H} = \mathbb{C}[G]$. Define $\Delta(g) = g \otimes g$ and $\varepsilon(g) = 1$.

Example. Let G be a group and let $\mathcal{H} = \mathcal{F}(G)$ be a suitable algebra of functions on G. Define

$$\Delta\colon \mathcal{H} \to \mathcal{H} \otimes \mathcal{H} \quad \text{and} \quad \varepsilon\colon \mathcal{H} \to \mathbb{C}$$

by $\Delta(f)(g_1, g_2) = f(g_1 g_2)$ and $\varepsilon(f) = f(e)$.

Remark. On a finite group we can take $\mathcal{F} =$ all functions. On an algebraic group we can take $\mathcal{F} =$ regular coordinate functions.

These will combine to form our main examples.

Actions of Bi-Algebras

Sweedler Notation. Write $\Delta h = \sum h_1 \otimes h_2$.

Example. With this notation,

$$(\Delta \otimes 1)\big(\Delta(h)\big) = \sum h_{11} \otimes h_{12} \otimes h_2,$$

and by co-associativity,

$$\sum h_{11} \otimes h_{12} \otimes h_2 = \sum h_1 \otimes h_{21} \otimes h_{22}.$$

Definition. An *action* of $\mathcal{H}$ on an associative algebra A is a unital homomorphism $\mathcal{H} \to \mathrm{End}_{\mathbb{C}}(A)$ for which

- $h(1_A) = \varepsilon(h)1_A$, and
- $h(a_1a_2) = \sum h_1(a_1)h_2(a_2)$.

Example. If $h \in \mathcal{H}$ is *group-like*, meaning $\Delta(h) = h \otimes h$, then h acts as an automorphism. If h is *primitive*, meaning $\Delta(h) = h \otimes 1 + 1 \otimes h$, then h acts as a derivation.

Example. If $\mathcal{H}$ is an enveloping algebra then from $\Delta(X) = X \otimes 1 + 1 \otimes X$, for $X \in \mathfrak{g}$, we get

$$X(ab) = X(a)b + aX(b)$$

Thus actions of $\mathcal{H}$ correspond to actions of $\mathfrak{g}$ by derivations.

Example. If $\mathcal{H} = \mathbb{C}[G]$ then actions of $\mathcal{H}$ on A correspond to actions of G by algebra automorphisms.

Example. Actions of $\mathcal{F}(G)$ correspond to 'coactions'. An important instance is $A = B \bowtie G$ and

$$h(a)(g) = h(g)a(g)$$

(think of A as functions $a\colon G \to B$ with twisted convolution multiplication).

Remark. If G is abelian then

$$\mathbb{C}[\widehat{G}] \cong \mathcal{F}(G) \qquad \text{(Fourier duality).}$$

Actions of $\mathcal{H} = \mathcal{F}(G)$ correspond to actions of $\widehat{G}$.

Construction of Bi-Algebras

$$G = G_1 \cdot G_2$$

G finite

Identify G/G_2 with G_1 and $G_1 \backslash G$ with G_2 to form

$$\mathcal{H} = \mathcal{F}(G_1) \bowtie G_2 \qquad A = G_1 \bowtie \mathcal{F}(G_2).$$

There is a natural algebra homomorphism

$$\mathcal{H} \to \mathrm{End}_{\mathbb{C}}(A).$$

If G_2 is normal then there is a natural coproduct on $\mathcal{H}$, assembled from the coproducts on $\mathcal{F}(G/G_2)$ and $\mathbb{C}[G_2]$:

$$\Delta(f \cdot g_2) = \Delta(f) \cdot \Delta(g_2).$$

The action of $\mathcal{H}$ on A is a bi-algebra action.

Amazing Fact. There is *always* a coproduct:

$$\Delta(g_2) = \sum_{g_1} q(g_2 g_1) \otimes p(g_1) \cdot g_2.$$

More on this *matched pair construction* next lecture.

Antipodes

Definition. A *Hopf algebra* is a bi-algebra for which there is a linear map $S\colon \mathcal{H} \to \mathcal{H}$ such that

$$\sum S(h_1)h_2 = \varepsilon(h) = \sum h_1 S(h_2),$$

for every $h \in \mathcal{H}$. Terminology: S = *Antipode*.

Example. For enveloping algebras, $S(X) = -X$.

Example. For group algebras, $S(g) = g^{-1}$.

Example. For $\mathcal{F}(G)$, $S(f)(g) = f(g^{-1})$.

Lemma. *The antipode S is unique, supposing it exists at all.* □

Lemma. *The antipode is anti-multiplicative and anti-co-multiplicative:*[1]

$$S(hk) = S(k)S(h) \quad \textit{and} \quad \Delta(S(h)) = \sum S(h_2) \otimes S(h_1).$$

Warning. It is not true that $S^2 = 1$.

[1] If that is a word.

Invariant Traces

To construct cyclic cocycles from Lie algebra cycles we also required a trace . . .

Definition. A functional $\tau\colon \mathcal{A} \to \mathbb{C}$ is *invariant* if

$$\tau(h(a)) = \varepsilon(h)\tau(a) \qquad \forall a \in A, \forall h \in \mathcal{H}.$$

Example.

$$h \in \mathcal{H} \text{ group-like} \quad \Rightarrow \tau(h(a)) = \tau(a)$$
$$h \in \mathcal{H} \text{ primitive} \quad \Rightarrow \quad \tau(h(a)) = 0$$

Example. Let $\mathcal{A} = \mathbb{C}[G]$. The canonical trace $\tau\colon \mathbb{C}[G] \to \mathbb{C}$,

$$\tau(f) = f(e),$$

is invariant for the action of $\mathcal{H} = \mathcal{F}(G)$,

$$h(f)(g) = h(g)f(g).$$

Cyclic Cocycles from Hopf Algebras

We want to construct cyclic cocycles from the correspondence

$$\underbrace{h^1 \otimes \cdots \otimes h^n}_{\text{Element of } \mathcal{H} \otimes \cdots \otimes \mathcal{H}} \quad \leftrightarrow \quad \underbrace{\tau\big(a^0 h^1(a^1) \ldots h^n(a^n)\big)}_{\text{Multi-linear functional on } \mathcal{A}}.$$

We have:

$$1 \otimes h^1 \otimes \cdots \otimes h^n \leftrightarrow \tau\big(a^0 a^1 h^1(a^2) \ldots h^n(a^{n+1})\big)$$
$$h^1 \otimes \cdots \otimes \Delta h^i \otimes \cdots \otimes h^n \leftrightarrow \tau\big(a^0 h^1(a^1) \ldots h^i(a^i a^{i+1}) \ldots h^n(a^{n+1})\big)$$
$$h^1 \otimes \cdots \otimes h^n \otimes 1 \leftrightarrow \tau\big(a^{n+1} a^0 h^1(a^1) \ldots h^n(a^n)\big)$$

Conclusion. Using Δ and the unit 1 we can construct a complex from $\mathcal{H}$, mapping to the Hochschild complex of $\mathcal{A}$.

Cyclicity

Problem. When is $\tau\big(a^0 h^1(a^1)\dots h^n(a^n)\big)$ cyclic?

Take for example $n = 1$. We want to fill in the blank:

$$\begin{array}{ccc} h^1 & \leftrightarrow & \tau\big(a^0 h^1(a^1)\big) \\ ?? & \leftrightarrow & \tau\big(a^1 h^1(a^0)\big) \end{array}$$

And for $n = 2$,

$$\begin{array}{ccc} h^1 \otimes h^2 & \leftrightarrow & \tau\big(a^0 h^1(a^1) h^2(a^2)\big) \\ ?????? & \leftrightarrow & \tau\big(a^2 h^1(a^0) h^2(a^1)\big) \end{array}$$

Solution.

For $n = 1$: $\quad S(h^1)$

For $n = 2$: $\quad \Delta(S(h^1)) \cdot (h^2 \otimes 1)$

General case: $\quad \Delta^{n-1}(S(h^1)) \cdot (h^2 \otimes \cdots \otimes h^n \otimes 1)$.

Lemma. *If the Hopf algebra $\mathcal{H}$ acts on $\mathcal{A}$ and if τ is invariant then*

$$\tau\big(h(a)b\big) = \tau\big(aS(h)(b)\big),$$

for every $h \in \mathcal{H}$ and $a, b \in \mathcal{A}$.

Proof. From $h = \sum h_1 \varepsilon(h_2)$ (co-unit property) we get

$$\begin{aligned} h(a)b &= \sum h_1(a)\varepsilon(h_2)b && \text{(co-unit)} \\ &= \sum h_1(a)h_{21}\big(S(h_{22})(b)\big) && \text{(antipode)} \\ &= \sum h_{11}(a)h_{12}\big(S(h_2)(b)\big) && \text{(co-associativity)} \\ &= \sum h_1\big(aS(h_2)(b)\big) && \text{(action)} \end{aligned}$$

Taking traces we get

$$\begin{aligned} \tau(h(a)b) &= \sum \tau\big(h_1(aS(h_2)(b))\big) \\ &= \sum \varepsilon(h_1)\tau\big(aS(h_2)(b)\big) && \text{(invariance)} \\ &= \sum \tau\big(aS(\varepsilon(h_1)h_2)(b)\big) \\ &= \tau\big(aS(h)(b)\big) && \text{(co-unit)} \qquad \square \end{aligned}$$

Explanation: Cyclicity

We have

$$\mathcal{H} \xrightarrow{\Delta} \mathcal{H}\otimes\mathcal{H} \xrightarrow{\Delta} \mathcal{H}\otimes\mathcal{H}\otimes\mathcal{H} \xrightarrow{\Delta} \dots \xrightarrow{\Delta} \mathcal{H}\otimes\cdots\otimes\mathcal{H}$$

and so $\Delta^{n-1}(h) \in \mathcal{H}^{\otimes^n}$.

From the definition of action,

$$\Delta^{n-1}(h) \quad \leftrightarrow \quad \tau\big(a^0 h(a^1 \dots a^n)\big)$$

Therefore

$$\Delta^{n-1}(h^1)\cdot(h^2\otimes\cdots\otimes h^n\otimes 1) \\ \leftrightarrow \tau\big(a^0 S(h^1)\big(h^2(a^1)\dots h^n(a^{n-1})a^n\big)\big)$$

By the lemma and the trace property

$$\Delta^{n-1}(h^1)\cdot(h^2\otimes\cdots\otimes h^n\otimes 1) \\ \leftrightarrow \tau\big(h^1(a^0)h^2(a^1)\dots h^n(a^{n-1})a^n)\big) \\ = \tau\big(a^n h^1(a^0)h^2(a^1)\dots h^n(a^{n-1})\big).$$

Cyclic Cohomology Again

$A =$ Algebra over $\mathbb{C}$, $\qquad C^n = \mathrm{Hom}(A^{\otimes^{n+1}}, \mathbb{C})$

Define maps

$$\underset{i=0,\dots,n+1}{\delta_i\colon C^n \to C^{n+1}}, \quad \underset{i=1,\dots,n}{\sigma_i\colon C^n \to C^{n-1}} \qquad \gamma\colon C^n \to C^n$$

by the formulas

$$\delta^i\phi(a^0,\dots,a^{n+1}) = \phi(a^0,\dots,a^i a^{i+1},\dots,a^{n+1})$$
$$\sigma_i\phi(a^0,\dots,a^{n-1}) = \phi(a^0,\dots,a^{i-1},1,a^i,\dots,a^{n-1})$$
$$\gamma\phi(a^0,\dots,a^n) = \phi(a^n,a^0,\dots,a^{n-1}).$$

The δ_i and σ_i satisfy face-degeneracy relations, and in addition

$$\gamma\delta_i = \delta_{i-1}\gamma \qquad \tau\delta_0 = \delta_{n+1}$$
$$\gamma\sigma_j = \sigma_{j-1}\gamma \qquad \gamma\sigma_0 = \sigma_n\gamma^2$$
$$\gamma^{n+1} = 1$$

These relations define the *cyclic category*.

Cyclic Objects

The complexes to compute cyclic cohomology are constructed from δ, σ and γ. For example:

$$\lambda = (-1)^n \gamma$$

$$b = \sum_{i=0}^{n+1} (-1)^j \delta_j, \qquad B = \left(\sum_{i=0}^{n-1} \lambda^i \right) (\sigma_n \gamma)(1 - \lambda),$$

on the object C^n.

Definition. A *cyclic object* in the category of abelian groups is a functor from the cyclic category to abelian groups. Its *cyclic cohomology* is (for example) the cohomology of the (b, B)-bicomplex constructed as above.

Definition. The *cyclic cohomology* of a Hopf algebra $\mathcal{H}$ for which $S^2 = 1$ is the cyclic cohomology of the cyclic object obtained from the following operators . . .

$$\delta_0(h^1 \otimes \cdots \otimes h^n) = 1 \otimes h^1 \otimes \cdots \otimes h^n$$
$$\delta_i(h^1 \otimes \cdots \otimes h^n) = h^1 \otimes \cdots \otimes \Delta h^i \otimes \cdots \otimes h^n$$
$$\delta_{n+1}(h^1 \otimes \cdots \otimes h^n) = h^1 \otimes \cdots \otimes h^n \otimes 1$$

$$\sigma_i(h^1 \otimes \cdots \otimes h^n) = \varepsilon(h^i) h^1 \otimes \cdots \otimes \widehat{h^i} \otimes \cdots \otimes h^n$$

$$\gamma(h^1 \otimes \cdots \otimes h^n) = \Delta^{n-1}(S(h^1)) \cdot (h^2 \otimes \cdots \otimes h^n \otimes 1)$$

Theorem (Connes and Moscovici). *As long as $S^2 = 1$ these formulas do indeed define a cyclic object.* □

Definition. Let $\mathcal{H}$ be a Hopf algebra for which $S^2 = 1$. If $\mathcal{H}$ acts on an algebra A, and if τ is an invariant trace on A, then define

$$\tau^\natural \colon HC^*(\mathcal{H}) \longrightarrow HC^*(A)$$

by the correspondence

$$h^1 \otimes \cdots \otimes h^n \mapsto \tau\big(a^0 h^1(a^1) \ldots h^n(a^n)\big).$$

Example

$\mathfrak{g}$	Lie algebra
$\mathfrak{g} \otimes \mathcal{A} \to \mathcal{A}$	Action of $\mathfrak{g}$ by derivations
$\tau\colon \mathcal{A} \to \mathbb{C}$	Invariant trace
$\mathcal{H}$	Enveloping algebra

Theorem. *The Hopf algebra periodic cyclic cohomology of the enveloping algebra $\mathcal{H}$ is isomorphic to the Lie algebra homology of $\mathfrak{g}$ (with trivial coefficients),*

$$\mathrm{HP}^{\mathrm{even/odd}}(\mathcal{H}) = \mathrm{H}_{\mathrm{even/odd}}(\mathfrak{g}, \mathbb{C}),$$

in such a way that the characteristic map

$$\tau^{\natural}\colon \mathrm{HP}^{*}(\mathcal{H}) \to \mathrm{HC}^{*}(\mathcal{A}),$$

associates to the class of a Lie algebra cycle the cyclic cocycle constructed at the beginning of the lecture.

Sketch of the Proof. We shall use the (b, B)-bicomplex.

Step 1. The inclusion $\wedge^n \mathfrak{g} \subseteq \otimes^n \mathfrak{g} \subseteq \otimes^n \mathcal{H}$ gives

$$\mathrm{Kernel}(b) = \wedge^n \mathfrak{g} \oplus \mathrm{Image}(b).$$

(The definition of b does not invoke the Lie bracket $[\,,\,]$. In effect, we can assume $\mathfrak{g}$ is abelian.)

Step 2. The operator $B\colon \otimes^n \mathcal{H} \to \otimes^{n-1}\mathcal{H}$ maps $\wedge^n \mathfrak{g}$ to $\wedge^{n-1}\mathfrak{g}$ and coincides with the Chevalley-Eilenberg boundary map. (A direct computation.)

Step 3. The result follows from the first two steps, plus some bookkeeping. □

Remark. As Connes and Moscovici observe, the same argument is used to compute $HC^*(C^\infty(M))$.

A First Generalization

Unfortunately, in important examples $S^2 \neq 1$.

Definition. A *character* of $\mathcal{H}$ is an algebra homomorphism $\delta \colon \mathcal{H} \to \mathbb{C}$.

Definition. A trace $\tau \colon A \to \mathbb{C}$ is *δ-invariant* if

$$\tau(h(a)) = \delta(h)\tau(a)$$

for all $h \in \mathcal{H}$ and $a \in A$.

Lemma. *If τ is δ-invariant then*

$$\tau(h(a)b) = \tau\big(aS_\delta(h)(b)\big),$$

where

$$S_\delta(h) = \sum \delta(h_1)S(h_2). \qquad \square$$

Theorem. *Assume that* $S_\delta^2 = 1$. *The twisted cyclic operator*

$$\gamma(h^1 \otimes \cdots \otimes h^n) = \Delta^{n-1}(S_\delta(h^1))(h^2 \otimes \cdots \otimes h^n \otimes 1)$$

and the previous face and degeneracy operators constitute a cyclic object. □

Definition. Denote by $HC_\delta^*(\mathcal{H})$ the associated cyclic cohomology groups, and by

$$\tau^\natural \colon HC_\delta^*(\mathcal{H}) \longrightarrow HC^*(A)$$

the characteristic map associated to a δ-invariant trace.

Theorem. *Let* $\mathcal{H}$ *be the enveloping algebra of* $\mathfrak{g}$ *and let* δ *be a character of* $\mathcal{H}$. *Then* $S_\delta^2 = 1$ *and*

$$HP_\delta^{\text{even/odd}}(\mathcal{H}) = H_{\text{even/odd}}(\mathfrak{g}, \mathbb{C}_\delta). \qquad \square$$

Ultimate Generalization

It is to replace the trace property by a *modular condition*:

Definition. A *modular pair* for a Hopf algebra $\mathcal{H}$ consists of a character $\delta\colon \mathcal{H} \to \mathbb{C}$ and a group-like element $u \in \mathcal{H}$ such that $\delta(u) = 1$. The pair (δ, u) is *involutive* if

$$S_\delta^2 = \mathrm{Ad}(u)\colon \mathcal{H} \to \mathcal{H}.$$

The definition is suggested by the conditions

$$\tau(ab) = \tau(bu(a))$$
$$\tau(h(a)) = \delta(h)\tau(a)$$

on a linear functional $\tau\colon A \to \mathbb{C}$, which imply

$$\tau(h(a)b) = \tau(aS_\delta(h)(b))$$

as before.

Theorem. *The amended formulas*

$$\delta_{n+1}(h^1 \otimes \cdots \otimes h^n) = (h^1 \otimes \cdots \otimes h^n \otimes u)$$

and

$$\gamma(h^1 \otimes \cdots \otimes h^n) = \Delta^{n-1}(S_\delta(h^1))(h^2 \otimes \cdots \otimes h^n \otimes u)$$

determine a cyclic object. □

We obtain a characteristic map

$$\tau^\natural \colon HC^*_\delta(\mathcal{H}) \longrightarrow HC^*(A),$$

as before.

The present generalization treats the algebra and co-algebra structures of $\mathcal{H}$ more symmetrically than the previous generalization.

We shall consider examples in the next lecture (time permitting), but in our main examples we shall have $u = 1$.