Introduction to representation theory

December 4, 2008

1 Basic notions of representation theory

1.1 What is representation theory?

To say it in one sentence, representation theory is an exciting area of mathematics which studies representations of associative algebras. Representation theory has a wide variety of applications, ranging from physics (elementary particles) and chemistry (atoms, molecules) to probability (card shuffles) and number theory (Fermat's last theorem).

Representation theory was born in 1896 in the work of the German mathematician F. G. Frobenius. This work was triggered by a letter to Frobenius by R. Dedekind. In this letter Dedekind made the following observation: take the multiplication table of a finite group G and turn it into a matrix X_G by replacing every entry g of this table by a variable x_g. Then the determinant of X_G factors into a product of irreducible polynomials in x_g, each of which occurs with multiplicity equal to its degree. Dedekind checked this surprising fact in a few special cases, but could not prove it in general. So he gave this problem to Frobenius. In order to find a solution of this problem (which we will explain below), Frobenius created representation theory of finite groups.

The general content of representation theory can be very briefly summarized as follows.

An **associative algebra** over a field k is a vector space A over k equipped with an associative bilinear multiplication $a, b \to ab$, $a, b \in A$. We will always consider associative algebras with unit, i.e., with an element 1 such that $1 \cdot a = a \cdot 1 = a$ for all $a \in A$. A basic example of an associative algebra is the algebra $\mathrm{End}V$ of linear operators from a vector space V to itself. Other important examples include algebras defined by generators and relations, such as group algebras and univeral enveloping algebras of Lie algebras.

A **representation** of an associative algebra A (also called a left A-module) is a vector space V equipped with a homomorphism $\rho : A \to \mathrm{End}V$, i.e., a linear map preserving the multiplication and unit.

A **subrepresentation** of a representation V is a subspace $U \subset V$ which is invariant under all operators $\rho(a)$, $a \in A$. Also, if V_1, V_2 are two representations of A then the **direct sum** $V_1 \oplus V_2$ has an obvious structure of a representation of A.

A nonzero representation V of A is said to be **irreducible** if its only subrepresentations are 0 and V itself, and **indecomposable** if it cannot be written as a direct sum of two nonzero subrepresentations. Obviously, irreducible implies indecomposable, but not vice versa.

Typical problems of representation theory are as follows:

1. Classify irreducible representations of a given algebra A.

2. Classify indecomposable representations of A.

3. Do 1 and 2 restricting to finite dimensional representations.

As mentioned above, the algebra A is often given to us by generators and relations. For example, the universal enveloping algebra U of the Lie algebra $sl(2)$ is generated by h, e, f with defining relations

$$he - eh = 2e, \quad hf - fh = -2f, \quad ef - fe = h. \tag{1}$$

This means that the problem of finding, say, N-dimensional representations of A reduces to solving a bunch of nonlinear algebraic equations with respect to a bunch of unknown N by N matrices, for example system (1) with respect to unknown matrices h, e, f.

It is really striking that such, at first glance hopelessly complicated, systems of equations can in fact be solved completely by methods of representation theory! For example, we will prove the following theorem.

Theorem 1.1. *Let $k = \mathbb{C}$ be the field of complex numbers. Then:*

(i) The algebra U has exactly one irreducible representation V_d of each dimension, up to equivalence; this representation is realized in the space of homogeneous polynomials of two variables x, y of degree $d - 1$, and defined by the formulas

$$\rho(h) = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}, \quad \rho(e) = x\frac{\partial}{\partial y}, \quad \rho(f) = y\frac{\partial}{\partial x}.$$

(ii) Any indecomposable finite dimensional representation of U is irreducible. That is, any finite dimensional representation of U is a direct sum of irreducible representations.

As another example consider the representation theory of quivers.

A **quiver** is a finite oriented graph Q. A **representation** of Q over a field k is an assignment of a k-vector space V_i to every vertex i of Q, and of a linear operator $A_h : V_i \to V_j$ to every directed edge h going from i to j (loops and multiple edges are allowed). We will show that a representation of a quiver Q is the same thing as a representation of a certain algebra P_Q called the path algebra of Q. Thus one may ask: what are indecomposable finite dimensional representations of Q?

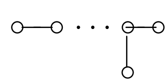
More specifically, let us say that Q is **finite** if it has finitely many indecomposable representations.

We will prove the following striking theorem, proved by P. Gabriel about 35 years ago:

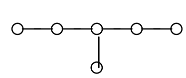
Theorem 1.2. *The finiteness property of Q does not depend on the orientation of edges. The connected graphs that yield finite quivers are given by the following list:*

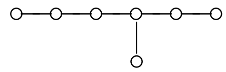
- A_n :

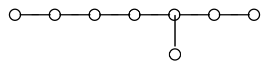
- D_n:



- E_6 :



- E_7 : 

- E_8 : 

The graphs listed in the theorem are called (simply laced) **Dynkin diagrams**. These graphs arise in a multitude of classification problems in mathematics, such as classification of simple Lie algebras, singularities, platonic solids, reflection groups, etc. In fact, if we needed to make contact with an alien civilization and show them how sophisticated our civilization is, perhaps showing them Dynkin diagrams would be the best choice!

As a final example consider the representation theory of finite groups, which is one of the most fascinating chapters of representation theory. In this theory, one considers representations of the group algebra $A = \mathbb{C}[G]$ of a finite group G – the algebra with basis $a_g, g \in G$ and multiplication law $a_g a_h = a_{gh}$. We will show that any finite dimensional representation of A is a direct sum of irreducible representations, i.e. the notions of an irreducible and indecomposable representation are the same for A (Maschke's theorem). Another striking result discussed below is the Frobenius' divisibility theorem: the dimension of any irreducible representation of A divides the order of G. Finally, we will show how to use representation theory of finite groups to prove Burnside's theorem: any finite group of order $p^a q^b$, where p, q are primes, is solvable. Note that this theorem does not mention representations, which are used only in its proof; a purely group-theoretical proof of this theorem (not using representations) exists but is much more difficult!

1.2 Algebras

Let us now begin a systematic discussion of representation theory.

Let k be a field. Unless stated otherwise, we will always assume that k is algebraically closed, i.e. any nonconstant polynomial with coefficients in k has a root in k. The main example is the field of complex numbers $\mathbb{C}$, but we will also consider fields of characteristic p, such as the algebraic closure $\overline{\mathbb{F}}_p$ of the finite field $\mathbb{F}_p$ of p elements.

Definition 1.3. An associative algebra over k is a vector space A over k together with a bilinear map $A \times A \to A$, $(a, b) \mapsto ab$, such that $(ab)c = a(bc)$.

Definition 1.4. A unit in an associative algebra A is an element $1 \in A$ such that $1a = a1 = a$.

Proposition 1.5. *If a unit exists, it is unique.*

Proof. Let $1, 1'$ be two units. Then $1 = 11' = 1'$. □

From now on, by an algebra A we will mean an associative algebra with a unit. We will also assume that $A \neq 0$.

Example 1.6. Here are some examples of algebras over k:

1. $A = k$.
2. $A = k[x_1, ..., x_n]$ – the algebra of polynomials in variables $x_1, ..., x_n$.

3. $A = \text{End}V$ – the algebra of endomorphisms of a vector space V over k (i.e., linear maps from V to itself). The multiplication is given by composition of operators.

4. The free algebra $A = k\langle x_1, ..., x_n\rangle$. The basis of this algebra consists of words in letters $x_1, ..., x_n$, and multiplication is simply concatenation of words.

5. The group algebra $A = k[G]$ of a group G. Its basis is $\{a_g, g \in G\}$, with multiplication law $a_g a_h = a_{gh}$.

Definition 1.7. An algebra A is commutative if $ab = ba$ for all $a, b \in A$.

For instance, in the above examples, A is commutative in cases 1 and 2, but not commutative in cases 3 (if $\dim V > 1$), and 4 (if $n > 1$). In case 5, A is commutative if and only if G is commutative.

Definition 1.8. A homomorphism of algebra $f : A \to B$ is a linear map such that $f(xy) = f(x)f(y)$ for all $x, y \in A$, and $f(1) = 1$.

1.3 Representations

Definition 1.9. A representation of an algebra A (also called a left A-module) is a vector space V together with a homomorphism of algebras $\rho : A \to \text{End}V$.

Similarly, a right A-module is a space V equipped with an antihomomorphism $\rho : A \to \text{End}V$; i.e., ρ satisfies $\rho(ab) = \rho(b)\rho(a)$ and $\rho(1) = 1$.

The usual abbreviated notation for $\rho(a)v$ is av for a left module and va for the right module. Then the property that ρ is an (anti)homomorphism can be written as a kind of associativity law: $(ab)v = a(bv)$ for left modules, and $(va)b = v(ab)$ for right modules.

Example 1.10. 1. $V = 0$.

2. $V = A$, and $\rho : A \to \text{End}A$ is defined as follows: $\rho(a)$ is the operator of left multiplication by a, so that $\rho(a)b = ab$ (the usual product). This representation is called the *regular* representation of A. Similarly, one can equip A with a structure of a right A-module by setting $\rho(a)b := ba$.

3. $A = k$. Then a representation of A is simply a vector space over k.

4. $A = k\langle x_1, ..., x_n\rangle$. Then a representation of A is just a vector space V over k with a collection of arbitrary linear operators $\rho(x_1), ..., \rho(x_n) : V \to V$ (explain why!).

Definition 1.11. A subrepresentation of a representation V of an algebra A is a subspace $W \subset V$ which is invariant under all operators $\rho(a) : V \to V$, $a \in A$.

For instance, 0 and V are always subrepresentations.

Definition 1.12. A representation $V \neq 0$ of A is irreducible (or simple) if the only subrepresentations of V are 0 and V.

Definition 1.13. Let V_1, V_2 be two representations over an algebra A. A homomorphism (or intertwining operator) $\phi : V_1 \to V_2$ is a linear operator which commutes with the action of A, i.e. $\phi(av) = a\phi(v)$ for any $v \in V_1$. A homomorphism ϕ is said to be an isomorphism of representations if it is an isomorphism of vector spaces.

Note that if a linear operator $\phi : V_1 \to V_2$ is an isomorphism of representations then so is the lienar operator $\phi^{-1} : V_2 \to V_1$ (check it!).

Two representations between which there exists an isomorphism are said to be isomorphic. For practical purposes, two isomorphic representations may be regarded as "the same", although there could be subtleties related to the fact that an isomorphism between two representations, when it exists, is not unique.

Definition 1.14. Let $V = V_1, V_2$ be representations of an algebra A. Then the space $V_1 \oplus V_2$ has an obvious structure of a representation of A, given by $a(v_1 \oplus v_2) = av_1 \oplus av_2$.

Definition 1.15. A nonzero representation V of an algebra A is said to be indecomposable if it is not isomorphic to a direct sum of two nonzero representations.

It is obvious that an irreducible representation is indecomposable. On the other hand, we will see below that the converse statement is false in general.

One of the main problems of representation theory is to classify irredicible and indecomposable representations of a given algebra up to isomorphism. This problem is usually hard and often can be solved only partially (say, for finite dimensional representations). Below we will see a number of examples in which this problem is partially or fully solved for specific algebras.

We will now prove our first result – Schur's lemma. Although it is very easy to prove, it is fundamental in the whole subject of representation theory.

Proposition 1.16. *(Schur's lemma) Let V_1, V_2 be irreducible representations of an algebra A over any field F (which need not be algebraically closed). Let $\phi : V_1 \to V_2$ be a nonzero homomorphism of representations. Then ϕ is an isomorphism.*

Proof. The kernel K of ϕ is a subrepresentation of V_1. Since $\phi \neq 0$, this subrepresentation cannot be V_1. So by irreducibility of V_1 we have $K = 0$. The image I of ϕ is a subrepresentation of V_2. Since $\phi \neq 0$, this subrepresentation cannot be 0. So by irreducibility of V_2 we have $I = V_2$. Thus ϕ is an isomorphism. □

Corollary 1.17. *(Schur's lemma for algebraically closed fields) Let V be a finite dimensional irreducible representation of an algebra A over an algebraically closed field k, and $\phi : V \to V$ is an intertwining operator. Then $\phi = \lambda \cdot \mathrm{Id}$ (the scalar operator).*

Proof. Let λ be an eigenvalue of ϕ (a root of the characteristic polynomial of ϕ). It exists since k is an algebraically closed field. Then the operator $\phi - \lambda\mathrm{Id}$ is an intertwining operator $V \to V$, which is not an isomorphism (since its determinant is zero). Thus by Schur's lemma this operator is zero, hence the result. □

Corollary 1.18. *Let A be a commutative algebra. Then every irreducible finite dimensional representation V of A is 1-dimensional.*

Remark. Note that a 1-dimensional representation of any algebra is automatically irreducible.

Proof. For any element $a \in A$, the operator $\rho(a) : V \to V$ is an intertwining operator. Indeed,

$$\rho(a)\rho(b)v = \rho(ab)v = \rho(ba)v = \rho(b)\rho(a)v$$

(the second equality is true since the algebra is commutative). Thus, by Schur's lemma, $\rho(a)$ is a scalar operator for any $a \in A$. Hence every subspace of V is a subrepresentation. So 0 and V are the only subspaces of V. This means that $\dim V = 1$ (since $V \neq 0$). □

Example 1.19. 1. $A = k$. Since representations of A are simply vector spaces, $V = A$ is the only irreducible and the only indecomposable representation.

2. $A = k[x]$. Since this algebra is commutative, the irreducible representations of A are its 1-dimensional representations. As we discussed above, they are defined by a single operator $\rho(x)$. In the 1-dimensional case, this is just a number from k. So all the irreducible representations of A are $V_\lambda = k$, $\lambda \in k$, in which the action of A defined by $\rho(x) = \lambda$. Clearly, these representations are pairwise non-isomorphic.

The classification of indecomposable representations is more interesting. To obtain it, recall that any linear operator on a finite dimensional vector space V can be brought to Jordan normal form. More specifically, recall that the Jordan block $J_{\lambda,n}$ is the operator on k^n which in the standard basis is given by the formulas $J_{\lambda,n}e_i = \lambda e_i + e_{i-1}$ for $i > 1$, and $J_{\lambda,n}e_1 = \lambda e_1$. Then for any linear operator $B : V \to V$ there exists a basis of V such that the matrix of B in this basis is a direct sum of Jordan blocks. This implies that all the indecomposable representations of A are $V_{\lambda,n} = k^n$, $\lambda \in k$, with $\rho(x) = J_{\lambda,n}$. The fact that these representations are indecomposable and pairwise non-isomorphic follows from the Jordan normal form theorem (which in particular says that the Jordan normal form of an operator is unique up to permutation of blocks).

This example shows that an indecomposable representation of an algebra need not be irreducible.

Problem 1.20. *Let V be a nonzero finite dimensional representation of an algebra A. Show that it has an irreducible subrepresentation. Then show by example that this does not always hold for infinite dimensional representations.*

Problem 1.21. *Let A be an algebra over a field k. The center $Z(A)$ of A is the set of all elements $z \in A$ which commute with all elements of A. For example, if A is commutative then $Z(A) = A$.*

(a) Show that if V is an irreducible finite dimensional representation of A then any element $z \in Z(A)$ acts in V by multiplication by some scalar $\chi_V(z)$. Show that $\chi_V : Z(A) \to k$ is a homomorphism. It is called the **central character** *of V.*

(b) Show that if V is an indecomposable finite dimensional representation of A then for any $z \in Z(A)$, the operator $\rho(z)$ by which z acts in V has only one eigenvalue $\chi_V(z)$, equal to the scalar by which z acts on some irreducible subrepresentation of V. Thus $\chi_V : Z(A) \to k$ is a homomorphism, which is again called the central character of V.

(c) Does $\rho(z)$ in (b) have to be a scalar operator?

Problem 1.22. *Let A be an associaive algebra, and V a representation of A. By $\mathrm{End}_A(V)$ one denotes the algebra of all morphisms of representations $V \to V$. Show that $\mathrm{End}_A(A) = A^{op}$, the algebra A with opposite multiplication.*

Problem 1.23. *Prove the following "Infinite dimensional Schur's lemma" (due to Dixmier): Let A be an algebra over $\mathbb{C}$ and V be an irreducible representation of A with at most countable basis. Then any homomorphism of representations $\phi : V \to V$ is a scalar operator.*

Hint. By the usual Schur's lemma, the alegbra $D := \mathrm{End}_A(V)$ is an algebra with division. Show that D is at most countably dimensional. Suppose ϕ is not a scalar, and consider the subfield $\mathbb{C}(\phi) \subset D$. Show that $\mathbb{C}(\phi)$ is a simple transcendental extension of $\mathbb{C}$. Derive from this that $\mathbb{C}(\phi)$ is uncountably dimensional and obtain a contradiction.

1.4 Ideals

A *left ideal* of an algebra A is a subspace $I \subseteq A$ such that $aI \subseteq I$ for all $a \in A$. Similarly, a *right ideal* of an algebra A is a subspace $I \subseteq A$ such that $Ia \subseteq I$ for all $a \in A$. A *two-sided ideal* is a subspace that is both a left and a right ideal.

Left ideals are the same as subrepresentations of the regular representation A. Right ideals are the same as subrepresentations of the regular representation of the opposite algebra A^{op}, in which the action of A is right multiplication.

Below are some examples of ideals:

- If A is any algebra, 0 and A are two-sided ideals. An algebra A is called *simple* if 0 and A are its only two-sided ideals.
- If $\phi : A \to B$ is a homomorphism of algebras, then $\ker \phi$ is a two-sided ideal of A.
- If S is any subset of an algebra A, then the two-sided ideal *generated* by S is denoted $\langle S \rangle$ and is the span of elements of the form asb, where $a, b \in A$ and $s \in S$. Similarly we can define $\langle S \rangle_\ell = \mathrm{span}\{as\}$ and $\langle S \rangle_r = \mathrm{span}\{sb\}$, the left, respectively right, ideal generated by S.

1.5 Quotients

Let A be an algebra and I a two-sided ideal in A. Then A/I is the set of (additive) cosets of I. Let $\pi : A \to A/I$ be the quotient map. We can define multiplication in A/I by $\pi(a) \cdot \pi(b) := \pi(ab)$. This is well-defined because if $\pi(a) = \pi(a')$ then

$$\pi(a'b) = \pi(ab + (a' - a)b) = \pi(ab) + \pi((a' - a)b) = \pi(ab)$$

because $(a' - a)b \in Ib \subseteq I = \ker \pi$, as I is a right ideal; similarly, if $\pi(b) = \pi(b')$ then

$$\pi(ab') = \pi(ab + a(b' - b)) = \pi(ab) + \pi(a(b' - b)) = \pi(ab)$$

because $a(b' - b) \in aI \subseteq I = \ker \pi$, as I is also a left ideal. Thus multiplication in A/I is well-defined, and A/I is an algebra.

Similarly, if V is a representation of A, and $W \subset V$ is a subrepresentation, then V/W is also a representation. Indeed, let $\pi : V \to V/W$ be the quotient map, and set $\rho_{V/W}(a)\pi(x) := \pi(\rho_V(a)x)$.

Above we noted the equivalence of left ideals of A and subrepresentations of the regular representation of A. Thus, if I is a left ideal in A, then A/I is a representation of A.

Problem 1.24. *Let $A = k[x_1, ..., x_n]$ and $I \neq A$ be any ideal in A containing all homogeneous polynomials of degree $\geq N$. Show that A/I is an indecomposable representation of A.*

Problem 1.25. *Let $V \neq 0$ be a representation of A. We say that a vector $v \in V$ is cyclic if it generates V, i.e., $Av = V$. A representation admitting a cyclic vector is said to be cyclic. Show that*

(a) V is irreducible if and only if all nonzero vectors of V are cyclic.

(b) V is cyclic if and only if it is isomorphic to A/I, where I is a left ideal in A.

(c) Give an example of an indecomposable representation which is not cyclic.

Hint. Let $A = \mathbb{C}[x, y]/I_2$, where I_2 is the ideal spanned by homogeneous polynomials of degree ≥ 2 (so A has a basis $1, x, y$). Let $V = A^*$ be the space of linear functionals on A, with the action of A given by $(\rho(a)f)(b) = f(ba)$. Show that V provides a required example.

1.6 Algebras defined by generators and relations

A representation V of A is said to be *generated* by a subset S of V if V is the span of $\{as \mid a \in A, s \in S\}$.

If $f_1, \ldots, f_m$ are elements of the free algebra $k\langle x_1, \ldots, x_n\rangle$, we say that the algebra $A := k\langle x_1, \ldots, x_n\rangle/\langle\{f_1, \ldots, f_m\}\rangle$ is *generated by* $x_1, \ldots, x_n$ *with defining relations* $f_1 = 0, \ \ldots, \ f_m = 0$.

1.7 Examples of algebras

Throughout the following examples G will denote a group, and k a field.

1. The group algebra $k[G]$, whose basis is $\{e_g \mid g \in G\}$, and where multiplication is defined by $e_g e_h = e_{gh}$. A *representation of a group* G *over a field* k is a homomorphism of groups $\rho : G \to GL(V)$, where V is some vector space over k. In fact, a representation of G over k is the "same thing" as a representation of $k[G]$.

2. The Weyl algebra, $k\langle x, y\rangle/\langle yx - xy - 1\rangle$.

3. The q-Weyl algebra over k, generated by x, x^{-1}, y, y^{-1} with defining relations $yx = qxy$ and $xx^{-1} = x^{-1}x = yy^1 = y^{-1}y = 1$.

Proposition. (i) A basis for the Weyl algebra A is $\{x^i y^j, i, j \geq 0\}$.

(ii) A basis for the q-Weyl algebra A_q is $\{x^i y^j, i, j \in \mathbb{Z}\}$.

Proof. (i) First let us show that the elements $x^i y^j$ are a spanning set for A. To do this, note that any word in x, y can be ordered to have all the x on the left of the y, at the cost of interchanging some x and y. Since $yx - xy = 1$, this will lead to error terms, but these terms will be sums of monomials that have a smaller number of letters x, y than the original word. Therefore, continuing this process, we can order everything and represent any word as a linear combination of $x^i y^j$.

The proof that $x^i y^j$ are linearly independent is based on representation theory. Namely, let a be a variable, and $E = t^a k[a][t, t^{-1}]$ (here t^a is just a formal symbol, so really $E = k[a][t, t^{-1}]$). Then E is a representation of A with action given by $xf = tf$ and $yf = \frac{df}{dt}$ (where $\frac{d(t^{a+n})}{dt} := (a+n)t^{a+n-1}$). Suppose now that we have a nontrivial linear relation $\sum c_{ij} x^i y^j = 0$. Then the operator

$$L = \sum c_{ij} t^i \left(\frac{d}{dt}\right)^j$$

acts by zero in E. Let us write L as

$$L = \sum_{j=0}^{r} Q_j(t) \left(\frac{d}{dt}\right)^j,$$

where $Q_r \neq 0$. Then we have

$$Lt^a = \sum_{j=0}^{r} Q_j(t) a(a-1)...(a-j+1)t^{a-j}.$$

This must be zero, so we have $\sum_{j=0}^{r} Q_j(t) a(a-1)...(a-j+1)t^{-j} = 0$ in $k[a][t, t^{-1}]$. Taking the leading term in a, we get $Q_r(t) = 0$, a contradiction.

(ii) Any word in x, y, x^{-1}, y^{-1} can be ordered at the cost of multiplying it by a power of q. This easily implies both the spanning property and the linear independence.

Remark. The proof of (i) shows that the Weyl algebra A can be viewed as the algebra of polynomial differential operators in one variable t.

The proof of (i) also brings up the notion of a faithful representation.

Definition. A representation $\rho : A \to \mathrm{End}\, V$ is *faithful* if ρ is injective.

For example, $k[t]$ is a faithful representation of the Weyl algebra, if k has characteristic zero (check it!), but not in characteristic p, where $(d/dt)^p Q = 0$ for any polynomial Q. However, the representation $E = t^a k[a][t, t^{-1}]$, as we've seen, is faithful in any characteristic.

Problem 1.26. *Let A be the Weyl algebra, generated over an algebraically closed field k by two generators x, y with the relation $yx - xy - 1 = 0$.*

(a) If $\mathrm{char} k = 0$, what are the finite dimensional representations of A? What are the two-sided ideals in A?

Hint. For the first question, use the fact that for two square matrices A, B, $Tr(AB) = Tr(BA)$. For the second question, show that any nonzero two-sided ideal in A contains a nonzero polynomial in x, and use this to characterize this ideal.

(b) Suppose for the rest of the problem that $\mathrm{char} k = p$. What is the center of A?

Hint. Show that x^p and y^p are central elements.

(c) Find all irreducible finite dimensional representations of A.

Hint. Let V be an irreducible finite dimensional representation of A, and v be an eigenvector of y in V. Show that $\{v, xv, x^2v, ..., x^{p-1}v\}$ is a basis of V.

Problem 1.27. *Let q be a nonzero complex number, and A be the q-Weyl algebra over $\mathbb{C}$ generated by $x^{\pm 1}$ and $y^{\pm 1}$ with defining relations $xx^{-1} = x^{-1}x = 1, yy^{-1} = y^{-1}y = 1$, and $xy = qyx$.*

(a) What is the center of A for different q? If q is not a root of unity, what are the two-sided ideals in A?

(b) For which q does this algebra have finite dimensional representations?

Hint. Use determinants.

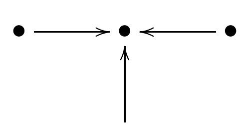
(c) Find all finite dimensional irreducible representations of A for such q.

Hint. This is similar to part (c) of the previous problem.

1.8 Quivers

Definition 1.28. A **quiver** Q is a directed graph, possibly with self-loops and/or multiple edges between two vertices.

Example 1.29.



We denote the set of vertices of the quiver Q as I, and the set of edges as E. For an edge $h \in E$, let h', h'' denote the source and target, respectively, of h.

$$\underset{h'}{\bullet} \xrightarrow[h]{} \underset{h''}{\bullet}$$

Definition 1.30. A representation of a quiver Q is an assignment to each vertex $i \in I$ of a vector space V_i and to each edge $h \in E$ of a linear map $x_h : V_{h'} \longrightarrow V_{h''}$.

It turns out that the theory of representations of quivers is a part of the theory of representations of algebras in the sense that for each quiver Q, there exists a certain algebra P_Q, called the path algebra of Q, such that a representation of the quiver Q is "the same" as a representation of the algebra P_Q. We shall first define the path algebra of a quiver and then justify our claim that representations of these two objects are "the same".

Definition 1.31. The **path algebra** P_Q of a quiver Q is the algebra whose basis is formed by oriented paths in Q, including the trivial paths p_i, $i \in I$, corresponding to the vertices of Q, and multiplication is concatenation of paths: ab is the path obtained by first tracing b and then a. If two paths cannot be concatenated, the product is defined to be zero.

Remark 1.32. It is easy to see that for a finite quiver $\sum\limits_{i \in I} p_i = 1$, so P_Q is an algebra with unit.

Problem 1.33. *Show that the algebra P_Q is generated by p_i for $i \in I$ and a_h for $h \in E$ with the defining relations:*

1. $p_i^2 = p_i$, $p_i p_j = 0$ *for* $i \neq j$
2. $a_h p_{h'} = a_h$, $a_h p_j = 0$ *for* $j \neq h'$
3. $p_{h''} a_h = a_h$, $p_i a_h = 0$ *for* $i \neq h''$

We now justify our statement that a representation of a quiver is the same thing as a representation of the path algebra of a quiver.

Let $\mathbf{V}$ be a representation of the path algebra P_Q. From this representation, we can construct a representation of Q as follows: let $V_i = p_i \mathbf{V}$, and for any edge h, let $x_h = a_h|_{p_{h'}\mathbf{V}} : p_{h'}\mathbf{V} \longrightarrow p_{h''}\mathbf{V}$ be the operator corresponding to the one-edge path h.

Similarly, let (V_i, x_h) be a representation of a quiver Q. From this representation, we can construct a representation of the path algebra P_Q: let $\mathbf{V} = \bigoplus_i V_i$, let $p_i : \mathbf{V} \to V_i \to \mathbf{V}$ be the projection onto V_i, and for any path $p = h_1...h_m$ let $a_p = x_{h_1}...x_{h_m} : V_{h'_m} \to V_{h''_1}$ be the composition of the operators corresponding to the edges occuring in p.

It is clear that the above assignments $\mathbf{V} \mapsto (p_i \mathbf{V})$ and $(V_i) \mapsto \bigoplus_i V_i$ are inverses of each other. Thus, we have a bijection between isomorphism classes of representations of the algebra P_Q and of the quiver Q.

Remark 1.34. In practice, it is generally easier to consider a representation of a quiver as in Definition 1.30. The above serves to show, as stated before, that the theory of representations of quivers is a part of the larger theory of representations of algebras.

We lastly define several previous concepts in the context of quivers representations.

Definition 1.35. A subrepresentation of a representation (V_i, x_h) of a quiver Q is a representation (W_i, x'_h) where $W_i \subseteq V_i$ for all $i \in I$ and where $x_h(W_{h'}) \subseteq W_{h''}$ and $x'_h = x_h|_{W_{h'}} : W_{h'} \longrightarrow W_{h''}$ for all $h \in E$.

Definition 1.36. The direct sum of two representations (V_i, x_h) and (W_i, y_h) is the representation $(V_i \oplus W_i, x_h \oplus y_h)$.

As with representations of algebras, a nonzero representation (V_i) of a quiver Q is said to be irreducible if its only subrepresentations are (0) and (V_i) itself, and indecomposable if it is not isomorphic to a direct sum of two nonzero representations.

Definition 1.37. Let (V_i, x_h) and (W_i, y_h) be representations of the quiver Q. A homomorphism $\varphi : (V_i) \longrightarrow (W_i)$ of quiver representations is a collection of maps $\varphi_i : V_i \longrightarrow W_i$ such that $y_h \circ \varphi_{h'} = \varphi_{h''} \circ x_h$ for all $h \in E$.

Problem 1.38. *Let A be a $\mathbb{Z}_+$-graded algebra, i.e., $A = \oplus_{n \geq 0} A[n]$, and $A[n] \cdot A[m] \subset A[n+m]$. If $A[n]$ is finite dimensional, it is useful to consider the Hilbert series $h_A(t) = \sum \dim A[n] t^n$ (the generating function of dimensions of $A[n]$). Often this series converges to a rational function, and the answer is written in the form of such function. For example, if $A = k[x]$ and $deg(x^n) = n$ then*

$$h_A(t) = 1 + t + t^2 + ... + t^n + ... = \frac{1}{1-t}$$

Find the Hilbert series of:

(a) $A = k[x_1, ..., x_m]$ (where the grading is by degree of polynomials);

(b) $A = k < x_1, ..., x_m >$ (the grading is by length of words);

(c) A is the exterior (=Grassmann) algebra $\wedge_k[x_1, ..., x_m]$, generated over some field k by $x_1, ..., x_m$ with the defining relations $x_i x_j + x_j x_i = 0$ and $x_i^2 = 0$ for all i, j (grading is by degree).[1]

(d) A is the path algebra P_Q of a quiver Q.

Hint. The closed answer is written in terms of the adjacency matrix M_Q of Q.

1.9 Lie algebras

Let $\mathfrak{g}$ be a vector space over a field k, and let $[\,,] : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$ be a skew-symmetric bilinear map. (So $[a, b] = -[b, a]$.) If k is of characteristic 2, we also require that $[x, x] = 0$ for all x (a requirement equivalent to $[a, b] = -[b, a]$ in fields of other characteristics).

Definition 1.39. $(\mathfrak{g}, [\,,])$ is a **Lie algebra** if $[\,,]$ satisfies the Jacobi identity

$$\big[[a, b], c\big] + \big[[b, c], a\big] + \big[[c, a], b\big] = 0. \tag{2}$$

Example 1.40. Some examples of Lie algebras are:

1. Any space $\mathfrak{g}$ with $[\,,] = 0$ (abelian Lie algebra).

2. Any associative algebra A with $[a, b] = ab - ba$.

3. Any subspace U of an associative algebra A such that $[a, b] \in U$ for all $a, b \in U$.

4. The space $\mathrm{Der}(A)$ of derivations of an algebra A, i.e. linear maps $D : A \to A$ which satisfies the Leibniz rule:
$$D(ab) = D(a)b + aD(b).$$

[1]The relation $x_i^2 = 0$ follows from $x_i x_j + x_j x_i = 0$ if the characteristic of the ground field is not equal to 2.

Remark 1.41. Derivations are important because they are the "infinitesimal version" of automorphisms. For example, assume that $g(t)$ is a differentiable family of automorphisms of a finite dimensional algebra A over $\mathbb{R}$ or $\mathbb{C}$ parametrized by $t \in (-\epsilon, \epsilon)$ such that $g(0) = \mathrm{Id}$. Then $D := g'(0) : A \to A$ is a derivation (check it!). Conversely, if $D : A \to A$ is a derivation, then e^{tD} is a 1-parameter family of automorphisms (give a proof!).

This provides a motivation for the notion of a Lie algebra. Namely, we see that Lie algebras arise as spaces of infinitesimal automorphisms (=derivations) of associative algebras. In fact, they similarly arise as spaces of derivations of any kind of linear algebraic structures, such as Lie algebras, Hopf algebras, etc., and for this reason play a very important role in algebra.

Here are a few more concrete examples of Lie algebras:

1. $\mathbb{R}^3$ with $[u, v] = u \times v$, the cross-product of u and v.
2. $sl(n)$, the set of $n \times n$ matrices with trace 0.
   For example, $sl(2)$ has the basis
   $$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$
   with relations $[e, f] = h$, $[h, f] = -2f$, $[h, e] = 2e$.
3. The Heisenberg Lie algebra $\mathcal{H}$ of matrices $\left(\begin{smallmatrix} 0 & * & * \\ 0 & 0 & * \\ 0 & 0 & 0 \end{smallmatrix}\right)$
   It has the basis
   $$x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \qquad y = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad c = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$
   with relations $[y, x] = c$ and $[y, c] = [x, c] = 0$.
4. The algebra $\mathrm{aff}(1)$ of matrices $\left(\begin{smallmatrix} * & * \\ 0 & 0 \end{smallmatrix}\right)$
   Its basis consists of $X = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and $Y = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$, with $[X, Y] = Y$.
5. $so(n)$, the space of skew-symmetric $n \times n$ matrices, with $[a, b] = ab - ba$.

Definition 1.42. Let $\mathfrak{g}_1, \mathfrak{g}_2$ be Lie algebras. A homomorphism $\varphi : \mathfrak{g}_1 \longrightarrow \mathfrak{g}_2$ of Lie algebras is a linear map such that $\varphi([a, b]) = [\varphi(a), \varphi(b)]$.

Definition 1.43. A representation of a Lie algebra $\mathfrak{g}$ is a vector space V with a homomorphism of Lie algebras $\rho : \mathfrak{g} \longrightarrow \operatorname{End} V$.

Example 1.44. Some examples of representations of Lie algebras are:

1. V=0
2. Any vector space V with $\rho = 0$ (the trivial representation).
3. The adjoint representation $V = \mathfrak{g}$ with $\rho(a)(b) = [a, b] \overset{def}{=} ab - ba$
   That this is a representation follows from Equation (2).

It turns out that a representation of a Lie algebra $\mathfrak{g}$ is the same as a representation of a certain associative algebra $\mathcal{U}(\mathfrak{g})$. Thus, as with quivers, we can view the theory of representations of Lie algebras as part of the theory of representations of associative algebras.

Definition 1.45. Let $\mathfrak{g}$ be a Lie algebra with basis x_i and $[,]$ defined by $[x_i, x_j] = \sum_k c_{ij}^k x_k$. The **universal enveloping algebra** $\mathcal{U}(\mathfrak{g})$ is the associative algebra generated by the x_i's with the relations $x_i x_j - x_j x_i = \sum_k c_{ij}^k x_k$.

Remark. This is not a very good definition since it depends on the choice of a basis. Later we will give an equivalent definition which will be basis-independent.

Example 1.46. The associative algebra $\mathcal{U}(sl(2))$ is the algebra generated by e, f, h with relations

$$he - eh = 2e \qquad hf - fh = -2f \qquad ef - fe = h.$$

Example 1.47. The algebra $\mathcal{U}(\mathcal{H})$, where $\mathcal{H}$ is the Heisenberg Lie algebra, is the algebra generated by x, y, c with the relations

$$yx - xy = c \qquad yc - cy = 0 \qquad xc - cx = 0.$$

Note that the Weyl algebra is the quotient of $\mathcal{U}(\mathcal{H})$ by the relation $c = 1$.

1.10 Tensor products

In this subsection we recall the notion of tensor product of vector spaces, which will be extensively used below.

Definition 1.48. The tensor product $V \otimes W$ of vector spaces V and W over a field k is the quotient of the space $V * W$ whose basis is given by formal symbols $v \otimes w$, $v \in V$, $w \in W$, by the subspace spanned by the elements

$$(v_1+v_2)\otimes w - v_1 \otimes w - v_2 \otimes w, v \otimes (w_1 + w_2) - v \otimes w_1 - v \otimes w_2, (av) \otimes w - a(v \otimes w), v \otimes (aw) - a(v \otimes w),$$

where $v \in V, w \in W, a \in k$.

This allows one to define the tensor product of any number of vector spaces, $V_1 \otimes ... \otimes V_n$. Note that this tensor product is associative, in the sense that $(V_1 \otimes V_2) \otimes V_3$ can be naturally identified with $V_1 \otimes (V_2 \otimes V_3)$.

In particular, people often consider tensor products of the form $V^{\otimes n} = V \otimes ... \otimes V$ (n times) for a given vector space V, and, more generally, $E := V^{\otimes n} \otimes (V^*)^{\otimes m}$. This space is called the space of tensors of type (m, n) on V. For instance, tensors of type $(0, 1)$ are vectors, of type $(1, 0)$ - linear functionals (covectors), of type $(1, 1)$ - linear operators, of type $(2, 0)$ - bilinear forms, of type $(2, 1)$ - algebra structures, etc.

If V is finite dimensional with basis e_i, $i = 1, ..., N$, and e^i is the dual basis of V^*, then a basis of E is the set of vectors

$$e_{i_1} \otimes ... \otimes e_{i_n} \otimes e^{j_1} \otimes ... \otimes e^{j_m},$$

and a typical element of E is

$$\sum_{i_1,...,i_n,j_1,...,j_m=1}^{N} T_{i_1...i_n}^{j_1...j_m} e_{i_1} \otimes ... \otimes e_{i_n} \otimes e^{j_1} \otimes ... \otimes e^{j_m},$$

where T is a multidimensional table of numbers.

Physicists define tensors as such multidimenstional tables, which change according to a certain rule when the basis of V is changed. Here it is important to distinguish upper an lower indices, since lower indices of T correspond to V and upper ones to V^*. The physicists don't write the sum sign, but remember that one should sum over indices that repeat twice - once as an upper index and once as lower. This convention is called the *Einstein summation*, and it also stipulates that if an index appears once, then there is no summation over it, while no index is supposed to appear more than once as an upper index and more than once as a lower index.

One can also define the tensor product of linear maps. Namely, if $A : V \to V'$ and $B : W \to W'$ are linear maps, then one can define the linear map $A \otimes B : V \otimes W \to V' \otimes W'$ given by the formula $(A \otimes B)(v \otimes w) = Av \otimes Bw$ (check that this is well defined!)

The most important properties of tensor products are summarized in the following problem.

Problem 1.49. *(a) Let U be any k-vector space. Construct a natural bijection between bilinear maps $V \times W \to U$ and linear maps $V \otimes W \to U$.*

(b) Show that if $\{v_i\}$ is a basis of V and $\{w_j\}$ is a basis of W then $\{v_i \otimes w_j\}$ is a basis of $V \otimes W$.

(c) Construct a natural isomorphism $V^* \otimes W \to \mathrm{Hom}(V, W)$ in the case when V is finite dimensional ("natural" means that the isomorphism is defined without choosing bases).

(d) Let V be a vector space over a field k. Let S^nV be the quotient of $V^{\otimes n}$ (n-fold tensor product of V) by the subspace spanned by the tensors $T - s(T)$ where $T \in V^{\otimes n}$, and s is some transposition. Also let $\wedge^n V$ be the the quotient of $V^{\otimes n}$ by the tensors T such that $s(T) = T$ for some transposition s. These spaces are called the n-th symmetric, respectively exterior, power of V. If $\{v_i\}$ is a basis of V, can you construct a basis of $S^nV, \wedge^n V$? If $\mathrm{dim}V = m$, what are their dimensions?

(e) If k has characteristic zero, find a natural identification of S^nV with the space of $T \in V^{\otimes n}$ such that $T = sT$ for all transposition s, and of $\wedge^n V$ with the space of $T \in V^{\otimes n}$ such that $T = -sT$ for all transpositions s.

(f) Let $A : V \to W$ be a linear operator. Then we have an operator $A^{\otimes n} : V^{\otimes n} \to W^{\otimes n}$, and its symmetric and exterior powers $S^nA : S^nV \to S^nW$, $\wedge^n A : \wedge^n V \to \wedge^n W$ which are defined in an obvious way. Suppose $V = W$ and has dimension N, and assume that the eigenvalues of A are $\lambda_1, ..., \lambda_N$. Find $Tr(S^nA), Tr(\wedge^n A)$.

(g) Show that $\wedge^N A = det(A)\mathrm{Id}$, use this equality to give a one-line proof of the fact that $\det(AB) = \det(A)\det(B)$.

Remark. Note that a similar definition to the above can be used to define the tensor product $V \otimes_A W$, where A is any ring, V is a right A-module, and W is a left A-module. Namely, $V \otimes_A W$ is the abelian group which is the quotient of the group $V * W$ freely generated by formal symbols $v \otimes w$, $v \in V$, $w \in W$, modulo the relations

$$(v_1 + v_2) \otimes w - v_1 \otimes w - v_2 \otimes w, v \otimes (w_1 + w_2) - v \otimes w_1 - v \otimes w_2, (va) \otimes w - (v \otimes aw), a \in A.$$

1.11 The tensor algebra

The notion of tensor product allows us to give more conceptual definitions of the free algebra, polynomial algebra, exterior algebra, and universal enveloping algebra of a Lie algebra.

Namely, given a vector space V, define its *tensor algebra* TV over a field k to be $TV = \oplus_{n\geq 0} V^{\otimes n}$, with multiplication defined by $a \cdot b := a \otimes b$, $a \in V^{\otimes n}$, $b \in V^{\otimes m}$. Observe that a choice of a basis $x_1, ..., x_N$ in V defines an isomorphism of TV with the free algebra $k < x_1, ..., x_n >$.

Also, one can make the following definition.

Definition 1.50. (i) The symmetric algebra SV of V is the quotient of TV by the ideal generated by $v \otimes w - w \otimes v$, $v, w \in V$.

(ii) The exterior algebra $\wedge V$ of V is the quotient of TV by the ideal generated by $v \otimes v$, $v \in V$.

(iii) If V is a Lie algebra, the universal enveloping algebra $U(V)$ of V is the quotient of TV by the ideal generated by $v \otimes w - w \otimes v - [v, w]$, $v, w \in V$.

It is easy to see that a choice of a basis $x_1, ..., x_N$ in V identifies SV with the polynomial algebra $k[x_1, ..., x_N]$, $\wedge V$ with the exterior algebra $\wedge_k(x_1, ..., x_N)$, and the universal enveloping algebra $U(V)$ with one defined previously.

Also, it is easy to see that we have decompositions $SV = \oplus_{n\geq 0} S^n V$, $\wedge V = \oplus_{n\geq 0} \wedge^n V$.

1.12 Hilbert's third problem

Problem 1.51. *It is known that if A and B are two polygons of the same area then A can be cut by finitely many straight cuts into pieces from which one can make B. David Hilbert asked in 1900 whether it is true for polyhedra in 3 dimensions. In particular, is it true for a cube and a regular tetrahedron of the same volume?*

The answer is "no", as was found by Dehn in 1901. The proof is very beautiful. Namely, to any polyhedron A let us attach its "Dehn invariant" $D(A)$ in $V = \mathbb{R} \otimes (\mathbb{R}/\mathbb{Q})$ (the tensor product of $\mathbb{Q}$-vector spaces). Namely,

$$D(A) = \sum_a l(a) \otimes \frac{\beta(a)}{\pi},$$

where a runs over edges of A and $l(a), \beta(a)$ are the length of a and the angle at a.

(a) Show that if you cut A into B and C by a straight cut, then $D(A) = D(B) + D(C)$.

(b) Show that $\alpha = \arccos(1/3)/\pi$ is not a rational number.

Hint. Assume that $\alpha = 2m/n$, for integers m, n. Deduce that roots of the equation $x + x^{-1} = 2/3$ are roots of unity of degree n. Conclude that $x^k + x^{-k}$ has denominator 3^k and get a contradiction.

(c) Using (a) and (b), show that the answer to Hilbert's question is negative. (Compute the Dehn invariant of the regular tetrahedron and the cube).

1.13 Tensor products and duals of representations of Lie algebras

Definition 1.52. The tensor product of two representations V, W of a Lie algebra $\mathfrak{g}$ is the space $V \otimes W$ with $\rho_{V\otimes W}(x) = \rho_V(x) \otimes Id + Id \otimes \rho_W(x)$.

Definition 1.53. The dual representation V^* to a representation V of a Lie algebra $\mathfrak{g}$ is the dual space V^* to V with $\rho_{V^*}(x) = -\rho_V(x)^*$.

It is easy to check that these are indeed representations.

Problem 1.54. *Let V, W, U be finite dimensional representations of a Lie algebra $\mathfrak{g}$. Show that the space $Hom_{\mathfrak{g}}(V \otimes W, U)$ is isomorphic to $Hom_{\mathfrak{g}}(V, U \otimes W^*)$.*

1.14 Representations of $sl(2)$

This subsection is devoted to the representation theory of $sl(2)$, which is of central importance in many areas of mathematics. It is useful to study this topic by solving the following sequence of exercises, which every mathematician should do, in one form or another.

Problem 1.55. *According to the above, a representation of $sl(2)$ is just a vector space V with a triple of operators E, F, H such that $HE - EH = 2E, HF - FH = -2F, EF - FE = H$ (the corresponding map ρ is given by $\rho(e) = E, \rho(f) = F$, $\rho(h) = H$.*

Let V be a finite dimensional representation of $sl(2)$ (the ground field in this problem is $\mathbb{C}$).

(a) Take eigenvalues of H and pick one with the biggest real part. Call it λ. Let $\bar{V}(\lambda)$ be the generalized eigenspace corresponding to λ. Show that $E|_{\bar{V}(\lambda)} = 0$.

(b) Let W be any representation of $sl(2)$ and $w \in W$ be a nonzero vector such that $Ew = 0$. For any $k > 0$ find a polynomial $P_k(x)$ of degree k such that $E^k F^k w = P_k(H)w$. (First compute $EF^k w$, then use induction in k).

(c) Let $v \in \bar{V}(\lambda)$ be a generalized eigenvector of H with eigenvalue λ. Show that there exists $N > 0$ such that $F^N v = 0$.

(d) Show that H is diagonalizable on $\bar{V}(\lambda)$. (Take N to be such that $F^N = 0$ on $\bar{V}(\lambda)$, and compute $E^N F^N v$, $v \in \bar{V}(\lambda)$, by (b). Use the fact that $P_k(x)$ does not have multiple roots).

(e) Let N_v be the smallest N satisfying (c). Show that $\lambda = N_v - 1$.

(f) Show that for each $N > 0$, there exists a unique up to isomorphism irreducible representation of sl(2) of dimension N. Compute the matrices E, F, H in this representation using a convenient basis. (For V finite dimensional irreducible take λ as in (a) and $v \in V(\lambda)$ an eigenvector of H. Show that $v, Fv, ..., F^\lambda v$ is a basis of V, and compute matrices of all operators in this basis.)

Denote the $\lambda + 1$-dimensional irreducible representation from (f) by V_λ. Below you will show that any finite dimensional representation is a direct sum of V_λ.

(g) Show that the operator $C = EF + FE + H^2/2$ (the so-called Casimir operator) commutes with E, F, H and equals $\frac{\lambda(\lambda+2)}{2} Id$ on V_λ.

Now it will be easy to prove the direct sum decomposition. Namely, assume the contrary, and let V a reducible representation of the smallest dimension, which is not a direct sum of smaller representations.

(h) Show that C has only one eigenvalue on V, namely $\frac{\lambda(\lambda+2)}{2}$ for some nonnegative integer λ. (use that the generalized eigenspace decomposition of C must be a decomposition of representations).

(i) Show that V has a subrepresentation $W = V_\lambda$ such that $V/W = nV_\lambda$ for some n (use (h) and the fact that V is the smallest which cannot be decomposed).

(j) Deduce from (i) that the eigenspace $V(\lambda)$ of H is $n+1$-dimensional. If $v_1, ..., v_{n+1}$ is its basis, show that $F^j v_i$, $1 \le i \le n+1$, $0 \le j \le \lambda$ are linearly independent and therefore form a basis of V (establish that if $Fx = 0$ and $Hx = \mu x$ then $Cx = \frac{\mu(\mu-2)}{2} x$ and hence $\mu = -\lambda$).

(k) Define $W_i = span(v_i, Fv_i, ..., F^\lambda v_i)$. Show that V_i are subrepresentations of V and derive a contradiction with the fact that V cannot be decomposed.

(l) (Jacobson-Morozov Lemma) Let V be a finite dimensional complex vector space and $A : V \to V$ a nilpotent operator. Show that there exists a unique, up to an isomorphism, representation of $sl(2)$ on V such that $E = A$. (Use the classification of the representations and Jordan normal form

theorem)

(m) (Clebsch-Gordan decomposition) Find the decomposition into irreducibles of the representation $V_\lambda \otimes V_\mu$ of $sl(2)$.

Hint. For a finite dimensional representation V of $sl(2)$ it is useful to introduce the character $\chi_V(x) = Tr(e^{xH})$, $x \in \mathbb{C}$. Show that $\chi_{V\oplus W}(x) = \chi_V(x) + \chi_W(x)$ and $\chi_{V\otimes W}(x) = \chi_V(x)\chi_W(x)$. Then compute the character of V_λ and of $V_\lambda \otimes V_\mu$ and derive the decomposition. This decomposition is of fundamental importance in quantum mechanics.

(n) Let $V = \mathbb{C}^M \otimes \mathbb{C}^N$, and $A = J_M(0) \otimes Id_N + Id_M \otimes J_N(0)$, where $J_n(0)$ is the Jordan block of size n with eigenvalue zero (i.e., $J_n(0)e_i = e_{i-1}$, $i = 2, ..., n$, and $J_n(0)e_1 = 0$). Find the Jordan normal form of A using (l),(m).

1.15 Problems on Lie algebras

Problem 1.56. *(***Lie's Theorem***) Recall that the commutant $K(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ is the linear span of elements $[x, y]$, $x, y \in \mathfrak{g}$. This is an ideal in $\mathfrak{g}$ (i.e. it is a subrepresentation of the adjoint representation). A finite dimensional Lie algebra $\mathfrak{g}$ over a field k is said to be solvable if there exists n such that $K^n(\mathfrak{g}) = 0$. Prove the Lie theorem: if $k = \mathbb{C}$ and V is a finite dimensional irreducible representation of a solvable Lie algebra $\mathfrak{g}$ then V is 1-dimensional.*

Hint. Prove the result by induction in dimension. By the induction assumption, $K(\mathfrak{g})$ has a common eigenvector v in V, that is there is a linear function $\chi : K(\mathfrak{g}) \to \mathbb{C}$ such that $av = \chi(a)v$ for any $a \in K(\mathfrak{g})$. Show that $\mathfrak{g}$ preserves common eigenspaces of $K(\mathfrak{g})$ (for this you will need to show that $\chi([x, a]) = 0$ for $x \in \mathfrak{g}$ and $a \in K(\mathfrak{g})$. To prove this, consider the smallest vector subspace U containing v and invariant under x. This subspace is invariant under $K(\mathfrak{g})$ and any $a \in K(\mathfrak{g})$ acts with trace $\dim(U)\chi(a)$ in this subspace. In particular $0 = \mathrm{Tr}([x, a]) = \dim(U)\chi([x, a])$.).

Problem 1.57. *Classify irreducible finite dimensional representations of the two dimensional Lie algebra with basis X, Y and commutation relation $[X, Y] = Y$. Consider the cases of zero and positive characteristic. Is the Lie theorem true in positive characteristic?*

Problem 1.58. *(hard!) For any element x of a Lie algebra $\mathfrak{g}$ let $ad(x)$ denote the operator $\mathfrak{g} \to \mathfrak{g}, y \mapsto [x, y]$. Consider the Lie algebra $\mathfrak{g}_n$ generated by two elements x, y with the defining relations $ad(x)^2(y) = ad(y)^{n+1}(x) = 0$.*

(a) Show that the Lie algebras $\mathfrak{g}_1, \mathfrak{g}_2, \mathfrak{g}_3$ are finite dimensional and find their dimensions.

(b) (harder!) Show that the Lie algebra $\mathfrak{g}_4$ has infinite dimension. Construct explicitly a basis of this algebra.

2 General results of representation theory

2.1 Subrepresentations in semisimple representations

Let A be an algebra.

Definition 2.1. A semisimple (or completely reducible) representation of A is a direct sum of irreducible representations.

Example. Let V be an irreducible representation of A of dimension n. Then $Y = \mathrm{End}(V)$, with action of A by left multiplication, is a semisimple representation of A, isomorphic to nV (the direct sum of n copies of V). Indeed, any basis $v_1, ..., v_n$ of V gives rise to an isomorphism of representations $\text{End}(V) \to nV$, given by $x \to (xv_1, ..., xv_n)$.

Remark. Note that by Schur's lemma, any semisimple representation V of A is canonically identified with $\oplus_X \text{Hom}_A(X, V) \otimes X$, where X runs over all irreducible representations of A. Indeed, we have a natural map $f : \oplus_X \text{Hom}(X, V) \otimes X \to V$, given by $g \otimes x \to g(x)$, $x \in X$, $g \in \text{Hom}(X, V)$, and it is easy to verify that this map is an isomorphism.

We'll see now how Schur's lemma allows us to classify subrepresentations in finite dimensional semisimple representations.

Proposition 2.2. *Let $V_i, 1 \leq i \leq m$ be irreducible finite dimensional pairwise nonisomorphic representations of A, and W be a subrepresentation of $V = \oplus_{i=1}^m n_i V_i$. Then W is isomorphic to $\oplus_{i=1}^m r_i V_i$, $r_i \leq n_i$, and the inclusion $\phi : W \to V$ is a direct sum of inclusions $\phi_i : r_i V_i \to n_i V_i$ given by multiplication of a row vector of elements of V_i (of length r_i) by a certain r_i-by-n_i matrix X_i with linearly independent rows: $\phi(v_1, ..., v_{r_i}) = (v_1, ..., v_{r_i}) X_i$.*

Proof. The proof is by induction in $n := \sum_{i=1}^m n_i$. The base of induction ($n = 1$) is clear. To perform the induction step, let us assume that W is nonzero, and fix an irreducible subrepresentation $P \subset W$. Such P exists (Problem 1.20). [2] Now, by Schur's lemma, P is isomorphic to V_i for some i, and the inclusion $\phi|_P : P \to V$ factors through $n_i V_i$, and upon identification of P with V_i is given by the formula $v \mapsto (vq_1, ..., vq_{n_i})$, where $q_l \in k$ are not all zero.

Now note that the group $G_i = GL_{n_i}(k)$ of invertible n_i-by-n_i matrices over k acts on $n_i V_i$ by $(v_1, ..., v_{n_i}) \to (v_1, ..., v_{n_i}) g_i$ (and by the identity on $n_j V_j$, $j \neq i$), and therefore acts on subrepresentations of V, preserving the property we need to establish: namely, under the action of g_i, the matrix X_i goes to $X_i g_i$, while $X_j, j \neq i$ don't change. Take $g_i \in G_i$ such that $(q_1, ..., q_{n_i}) g_i = (1, 0, ..., 0)$. Then $W g_i$ contains the first summand V_i of $n_i V_i$ (namely, it is $P g_i$), hence $W g_i = V_i \oplus W'$, where $W' \subset n_1 V_1 \oplus ... \oplus (n_i - 1) V_i \oplus n_m V_m$ is the kernel of the projection of $W g_i$ to the first summand V_i. Thus the required statement follows from the induction assumption. $\square$

Remark 2.3. In Proposition 2.2, it is not important that k is algebraically closed, nor it matters that V is finite dimensional. If these assumptions are dropped, the only change needed is that the entries of the matrix X_i are no longer in k but in $D_i = \text{End}_A(V_i)$, which is, as we know, a division algebra. The proof of this generalized version of Proposition 2.2 is the same as before (check it!).

2.2 The density theorem

Let A be an algebra over a field k.

Corollary 2.4. *Let V be an irreducible finite dimensional representation of A, and $v_1, ..., v_n \in V$ be any linearly independent vectors. Then for any $w_1, ..., w_n \in V$ there exists an element $a \in A$ such that $av_i = w_i$.*

Proof. Assume the contrary. Then the image of the map $A \to nV$ given by $a \to (av_1, ..., av_n)$ is a proper subrepresentation, so by Proposition 2.2 it corresponds to an r-by-n matrix X, $r <$

[2] Another proof of the existence of P, which does not use the finite dimensionality of V, is by induction in n. Namely, if W itself is not irreducible, let K be the kernel of the projection of W to the first summand V_1. Then K is a subrepresentation of $(n_1 - 1)V_1 \oplus ... \oplus n_m V_m$, which is nonzero since W is not irreducible, so K contains an irreducible subrepresentation by the induction assumption.

n. Thus there exist vectors $u_1, ..., u_r \in V$ such that $(u_1, ..., u_r)X = (v_1, ..., v_n)$. Let $(q_1, ..., q_n)$ be a nonzero vector such that $X(q_1, ..., q_n)^T = 0$ (it exists because $r < n$). Then $\sum q_i v_i = (u_1, ..., u_r)X(q_1, ..., q_n)^T = 0$ - a contradiction. □

Theorem 2.5. *(the Density Theorem). (i) Let V be an irreducible finite dimensional representation of A. Then the map $\rho : A \to \mathrm{End}V$ is surjective.*

(ii) Let $V = V_1 \oplus ... \oplus V_r$, where V_i are irreducible pairwise nonisomorphic finite dimensional representations of A. Then the map $\oplus_{i=1}^r \rho_i : A \to \oplus_{i=1}^r \mathrm{End}(V_i)$ is surjective.

Proof. (i) Let B be the image of A in $\mathrm{End}(V)$. Then $B \subset \mathrm{End}(V)$. We want to show that $B = \mathrm{End}(V)$. Let $c \in \mathrm{End}(V)$, $v_1, ..., v_n$ be a basis of V, and (c_{ij}) the matrix of c in this basis. Let $w_j = \sum v_i c_{ij}$. By Corollary 2.4, there exists $a \in A$ such that $av_i = w_i$. Then a maps to c, so $c \in B$, and we are done.

(ii) Let B_i be the image of A in $\mathrm{End}(V_i)$, and B be the image of A in $\oplus_{i=1}^r \mathrm{End}(V_i)$. Recall that as a representation of A, $\oplus_{i=1}^r \mathrm{End}(V_i)$ is semisimple: it is isomorphic to $\oplus_{i=1}^r d_i V_i$, where $d_i = \dim V_i$. Then by Proposition 2.2, $B = \oplus_i B_i$. On the other hand, (i) implies that $B_i = \mathrm{End}(V_i)$. Thus (ii) follows. □

2.3 Representations of direct sums of matrix algebras

In this section we consider representations of algebras $A = \bigoplus_i \mathrm{Mat}_{d_i}(k)$ for any field k.

Theorem 2.6. *Let $A = \bigoplus_{i=1}^r \mathrm{Mat}_{d_i}(k)$. Then the irreducible representations of A are $V_1 = k^{d_1}, \ldots, V_r = k^{d_r}$, and any finite dimensional representation of A is a direct sum of copies of $V_1, \ldots, V_r$.*

In order to prove Theorem 2.6, we shall need the notion of a dual representation.

Definition 2.7. (Dual representation) Let V be a representation of any algebra A. Then the dual representation V^* is the representation of the opposite algebra A^{op} (or, equivalently, right A-module) with the action

$$(f \cdot a)(v) := f(av).$$

Proof of Theorem 2.6. First, the given representations are clearly irreducible, as for any $v \neq 0, w \in V_i$, there exists $a \in A$ such that $av = w$. Next, let X be an n-dimensional representation of A. Then, X^* is an n-dimensional representation of A^{op}. But $(\mathrm{Mat}_{d_i}(k))^{\mathrm{op}} \cong \mathrm{Mat}_{d_i}(k)$ with isomorphism $\varphi(X) = X^T$, as $(BC)^T = C^T B^T$. Thus, $A \cong A^{\mathrm{op}}$ and X^* may be viewed as an n-dimensional representation of A. Define

$$\phi : \underbrace{A \oplus \cdots \oplus A}_{n \text{ copies}} \longrightarrow X^*$$

by

$$\phi(a_1, \ldots, a_n) = a_1 y_1 + \cdots + a_n y_n$$

where $\{y_i\}$ is a basis of X^*. ϕ is clearly surjective, as $k \subset A$. Thus, the dual map $\phi^* : X \longrightarrow A^{n*}$ is injective. But $A^{n*} \cong A^n$ as representations of A. Hence, $\mathrm{Im}\, \phi^* \cong X$ is a subrepresentation of A^n. Next, $\mathrm{Mat}_{d_i}(k) = d_i V_i$, so $A = \oplus_{i=1}^r d_i V_i$, $A^n = \oplus_{i=1}^r n d_i V_i$, as a representation of A. Hence by Proposition 2.2, $X = \oplus_{i=1}^r m_i V_i$, as desired. □

2.4 Filtrations

Let A be an algebra. Let V be a representation of A. A (finite) *filtration* of V is a sequence of subrepresentations $0 = V_0 \subset V_1 \subset ... \subset V_n = V$.

Lemma 2.8. *Any finite dimensional representation V of an algebra A admits a finite filtration $0 = V_0 \subset V_1 \subset ... \subset V_n = V$ such that the successive quotients V_i/V_{i-1} are irreducible.*

Proof. The proof is by induction in $\dim(V)$. The base is clear, and only the induction step needs to be justified. Pick an irreducible subrepresentation $V_1 \subset V$, and consider the representation $U = V/V_1$. Then by the induction assumption U has a filtration $0 = U_0 \subset U_1 \subset ... \subset U_{n-1} = U$ such that U_i/U_{i-1} are irreducible. Define V_i for $i \geq 2$ to be the preimages of U_{i-1} under the tautological projection $V \to V/V_1 = U$. Then $0 = V_0 \subset V_1 \subset V_2 \subset ... \subset V_n = V$ is a filtration of V with the desired property. □

2.5 Finite dimensional algebras

Definition 2.9. The **radical** of a finite dimensional algebra A is the set of all elements of A which act by 0 in all irreducible representations of A. It is denoted $\mathrm{Rad}(A)$.

Proposition 2.10. *$Rad(A)$ is a two-sided ideal.*

Proof. Easy. □

Proposition 2.11. *Let A be a finite dimensional algebra.*

(i) Let I be a nilpotent two-sided ideal in A, i.e. $I^n = 0$ for some n. Then $I \subset Rad(A)$.

(ii) $Rad(A)$ is a nilpotent ideal. Thus, $Rad(A)$ is the largest nilpotent two-sided ideal in A.

Proof. (i) Let V be an irreducible representation of A. Let $v \in V$. Then $Iv \subset V$ is a subrepresentation. If $Iv \neq 0$ then $Iv = V$ so there is $x \in I$ such that $xv = v$. Then $x^n \neq 0$, a contradiction. Thus $Iv = 0$, so I acts by 0 in V and hence $I \subset \mathrm{Rad}(A)$.

(ii) Let $0 = A_0 \subset A_1 \subset ... \subset A_n = A$ be a filtration of the regular representation of A by subrepresentations such that A_{i+1}/A_i are irreducible. It exists by Lemma 2.8. Let $x \in \mathrm{Rad}(A)$. Then x acts on A_{i+1}/A_i by zero, so x maps A_{i+1} to A_i. This implies that $\mathrm{Rad}(A)^n = 0$, as desired. □

Theorem 2.12. *A finite dimensional algebra A has only finitely many irreducible representations V_i up to isomorphism, these representations are finite dimensional, and*

$$A/Rad(A) \cong \bigoplus_i \mathrm{End}\, V_i.$$

Proof. First, for any irreducible representation V of A, and for any nonzero $v \in V$, $Av \subseteq V$ is a finite dimensional subrepresentation of V. (It is finite dimensional as A is finite dimensional.) As V is irreducible and $Av \neq 0$, $V = Av$ and V is finite dimensional.

Next, suppose we have non-isomorphic irreducible representations $V_1, V_2, \ldots, V_r$. By Theorem 2.5, the homomorphism

$$\bigoplus_i \rho_i : A \longrightarrow \bigoplus_i \mathrm{End}\, V_i$$

is surjective. So $r \leq \sum \dim \operatorname{End} V_i \geq r \leq \dim A$. Thus, A has only finitely many non-isomorphic irreducible representations (at most $\dim A$).

Next, let $V_1, V_2, \ldots, V_r$ be all non-isomorphic irreducible finite dimensional representations of A. By Theorem 2.5, the homomorphism

$$\bigoplus_i \rho_i : A \longrightarrow \bigoplus_i \operatorname{End} V_i$$

is surjective. The kernel of this map, by definition, is exactly $\operatorname{Rad}(A)$. □

Corollary 2.13. $\sum_i (\dim V_i)^2 \leq \dim A$, *where the* V_i*'s are the irreducible representations of* A.

Proof. As $\dim \operatorname{End} V_i = (\dim V_i)^2$, Theorem 2.12 implies that $\dim A - \dim \operatorname{Rad}(A) = \sum_i \dim \operatorname{End} V_i = \sum_i (\dim V_i)^2$. As $\dim \operatorname{Rad}(A) \geq 0$, $\sum_i (\dim V_i)^2 \leq \dim A$. □

Example 2.14. 1. Let $A = k[x]/(x^n)$. This algebra has a unique irreducible representation, which is a 1-dimensional space k, in which x acts by zero. So the radical $\operatorname{Rad}(A)$ is the ideal (x).

2. Let A be the algebra of upper triangular n by n matrices. It is easy to check that the irreducible representations of A are V_i, $i = 1, ..., n$, which are 1-dimensional, and any matrix x acts by x_{ii}. So the radical $\operatorname{Rad}(A)$ is the ideal of strictly upper triangular matrices (as it is a nilpotent ideal and contains the radical). A similar result holds for block-triangular matrices.

Definition 2.15. A finite dimensional algebra A is said to be **semisimple** if $\operatorname{Rad}(A) = 0$.

Proposition 2.16. *For a finite dimensional algebra A, the following are equivalent:*

1. *A is semisimple.*
2. $\sum_i (\dim V_i)^2 = \dim A$, *where the* V_i*'s are the irreducible representations of* A.
3. $A \cong \bigoplus_i \operatorname{Mat}_{d_i}(k)$ *for some* d_i.
4. *Any finite dimensional representation of A is completely reducible (that is, isomorphic to a direct sum of irreducible representations).*
5. *A is a completely reducible representation of A.*

Proof. As $\dim A - \dim \operatorname{Rad}(A) = \sum_i (\dim V_i)^2$, clearly $\dim A = \sum_i (\dim V_i)^2$ if and only if $\operatorname{Rad}(A) = 0$. Thus, (1) ⇔ (2).

Next, by Theorem 2.12, if $\operatorname{Rad}(A) = 0$, then clearly $A \cong \bigoplus_i \operatorname{Mat}_{d_i}(k)$ for $d_i = \dim V_i$. Thus, (1) ⇒ (3). Conversely, if $A \cong \bigoplus_i \operatorname{Mat}_{d_i}(k)$, then by Theorem 2.6, $\operatorname{Rad}(A) = 0$, so A is semisimple. Thus (3) ⇒ (1).

Next, (3) ⇒ (4) by Theorem 2.6. Clearly (4) ⇒ (5). To see that (5) ⇒ (3), let $A = \bigoplus_i n_i V_i$. Consider $\operatorname{End}_A(A)$ (endomorphisms of A as a representation of A). As the V_i's are pairwise non-isomorphic, by Schur's lemma, no copy of V_i in A can be mapped to a distinct V_j. Also, by Schur, $\operatorname{End}_A(V_i) = k$. Thus, $\operatorname{End}_A(A) \cong \bigoplus_i \operatorname{Mat}_{n_i}(k)$. But $\operatorname{End}_A(A) \cong A^{\mathrm{op}}$ by Problem 1.22, so $A^{\mathrm{op}} \cong \bigoplus_i \operatorname{Mat}_{n_i}(k)$. Thus, $A \cong (\bigoplus_i \operatorname{Mat}_{n_i}(k))^{\mathrm{op}} = \bigoplus_i \operatorname{Mat}_{n_i}(k)$, as desired. □

2.6 Characters of representations

Let A be an algebra and V a finite-dimensional representation of A with action ρ. Then the *character* of V is the linear function $\chi_V : A \to k$ given by

$$\chi_V(a) = \mathrm{tr}|_V(\rho(a)).$$

If $[A, A]$ is the span of commutators $[x, y] := xy - yx$ over all $x, y \in A$, then $[A, A] \subseteq \ker \chi_V$. Thus, we may view the character as a mapping $\chi_V : A/[A, A] \to k$.

Exercise. Show that if $W \subset V$ are finite dimensional representations of A, then $\chi_V = \chi_W + \chi_{V/W}$.

Theorem 2.17. *(1) Characters of (distinct) irreducible finite-dimensional representations of A are linearly independent.*

(2) If A is a finite-dimensional semisimple algebra, then the characters form a basis of $(A/[A, A])^*$.

Proof. (1) If $V_1, \ldots, V_r$ are nonisomorphic irreducible finite-dimensional representations of A, then $\rho_{V_1} \oplus \cdots \oplus \rho_{V_r} : A \to \mathrm{End}\, V_1 \oplus \cdots \oplus \mathrm{End}\, V_r$ is surjective by the density theorem, so $\chi_{V_1}, \ldots, \chi_{V_r}$ are linearly independent. (Indeed, if $\sum \lambda_i \chi_{V_i}(a) = 0$ for all $a \in A$, then $\sum \lambda_i \mathrm{Tr}(M_i) = 0$ for all $M_i \in \mathrm{End}_k V_i$. But each $\mathrm{tr}(M_i)$ can range independently over k, so it must be that $\lambda_1 = \cdots = \lambda_r = 0$.)

(2) First we prove that $[\mathrm{Mat}_d(k), \mathrm{Mat}_d(k)] = sl_d(k)$, the set of all matrices with trace 0. It is clear that $[\mathrm{Mat}_d(k), \mathrm{Mat}_d(k)] \subseteq sl_d(k)$. If we denote by E_{ij} the matrix with 1 in the ith row of the jth column and 0's everywhere else, we have $[E_{ij}, E_{jm}] = E_{im}$ for $i \neq m$, and $[E_{i,i+1}, E_{i+1,i}] = E_{ii} - E_{i+1,i+1}$. Now $\{E_{im}\} \cup \{E_{ii} - E_{i+1,i+1}\}$ forms a basis in $sl_d(k)$, and indeed $[\mathrm{Mat}_d(k), \mathrm{Mat}_d(k)] = sl_d(k)$, as claimed.

By semisimplicity, we can write $A = \mathrm{Mat}_{d_1}(k) \oplus \cdots \oplus \mathrm{Mat}_{d_r}(k)$. Then $[A, A] = sl_{d_1}(k) \oplus \cdots \oplus sl_{d_r}(k)$, and $A/[A, A] \cong k^r$. By Theorem 2.6, there are exactly r irreducible representations of A (isomorphic to $k^{d_1}, \ldots, k^{d_r}$, respectively), and therefore r linearly independent characters on the r-dimensional vector space $A/[A, A]$. Thus, the characters form a basis. ☐

2.7 The Jordan-Hölder theorem

We will now state and prove two important theorems about representations of finite dimensional algebras - the Jordan-Hölder theorem and the Krull-Schmidt theorem.

Theorem 2.18. *(Jordan-Hölder theorem). Let V be a finite dimensional representation of A, and $0 = V_0 \subset V_1 \subset ... \subset V_n = V$, $0 = V_0' \subset ... \subset V_m' = V$ be filtrations of V, such that the representations $W_i := V_i/V_{i-1}$ and $W_i' := V_i'/V_{i-1}'$ are irreducible for all i. Then $n = m$, and there exists a permutation σ of $1, ..., n$ such that $W_{\sigma(i)}$ is isomorphic to W_i'.*

Proof. **First proof** (for k of characteristic zero). The character of V obviously equals the sum of characters of W_i, and also the sum of characters of W_i'. But by Theorem 2.17, the characters of irreducible representations are linearly independent, so the multiplicity of every irreducible representation W of A among W_i and among W_i' are the same. This implies the theorem.

Second proof (general). The proof is by induction on $\dim V$. The base of induction is clear, so let us prove the induction step. If $W_1 = W_1'$ (as subspaces), we are done, since by the induction assumption the theorem holds for V/W_1. So assume $W_1 \neq W_1'$. In this case $W_1 \cap W_1' = 0$ (as

W_1, W_1' are irreducible), so we have an embedding $f : W_1 \oplus W_1' \to V$. Let $U = V/(W_1 \oplus W_1')$, and $0 = U_0 \subset U_1 \subset ... \subset U_p = U$ be a filtration of U with simple quotients $Z_i = U_i/U_{i-1}$ (it exists by Lemma 2.8). Then we see that:

1) V/W_1 has a filtration with successive quotients $W_1', Z_1, ..., Z_p$, and another filtration with successive quotients $W_2, ..., W_n$.

2) V/W_1' has a filtration with successive quotients $W_1, Z_1, ..., Z_p$, and another filtration with successive quotients $W_2', ..., W_n'$.

By the induction assumption, this means that the collection of irreducible modules with multiplicities $W_1, W_1', Z_1, ..., Z_p$ coincides on one hand with $W_1, ..., W_n$, and on the other hand, with $W_1', ..., W_m'$. We are done. □

The Jordan-Hölder theorem shows that the number n of terms in a filtration of V with irreducible successive quotients does not depend on the choice of a filtration, and depends only on V. This number is called the *length* of V. It is easy to see that n is also the maximal length of a filtration of V in which all the inclusions are strict.

2.8 The Krull-Schmidt theorem

Theorem 2.19. *(Krull-Schmidt theorem) Any finite dimensional representation of A can be uniquely (up to order of summands) decomposed into a direct sum of indecomposable representations.*

Proof. It is clear that a decomposition of V into a direct sum of indecomposable representations exists, so we just need to prove uniqueness. We will prove it by induction on $\dim V$. Let $V = V_1 \oplus ... \oplus V_m = V_1' \oplus ... \oplus V_n'$. Let $i_s : V_s \to V$, $i_s' : V_s' \to V$, $p_s : V \to V_s$, $p_s' : V \to V_s'$ be the natural maps associated to these decompositions. Let $\theta_s = p_1 i_s' p_s' i_1 : V_1 \to V_1$. We have $\sum_{s=1}^{n} \theta_s = 1$. Now we need the following lemma.

Lemma 2.20. *Let W be a finite dimensional indecomposable representation of A. Then*

(i) Any homomorphism $\theta : W \to W$ is either an isomorphism of nilpotent;

(ii) If $\theta_s : W \to W$, $s = 1, ..., n$ are nilpotent homomorphisms, then so is $\theta := \theta_1 + ... + \theta_n$.

Proof. (i) Generalized eigenspaces of θ are subrepresentations of V, and V is their direct sum. Thus, θ can have only one eigenvalue λ. If λ is zero, θ is nilpotent, otherwise it is an isomorphism.

(ii) The proof is by induction in n. The base is clear. To make the induction step ($n-1$ to n), assume that θ is not nilpotent. Then by (i) θ is an isomorphism, so $\sum_{i=1}^{n} \theta^{-1}\theta_i = 1$. The morphisms $\theta^{-1}\theta_i$ are not isomorphisms, so they are nilpotent. Thus $1 - \theta^{-1}\theta_n = \theta^{-1}\theta_1 + ... + \theta^{-1}\theta_{n-1}$ is an isomorphism, which is a contradiction with the induction assumption. □

By the lemma, we find that for some s, θ_s must be an isomorphism; we may assume that $s = 1$. In this case, $V_1' = \mathrm{Im} p_1' i_1 \oplus \mathrm{Ker}(p_1 i_1')$, so since V_1' is indecomposable, we get that $f := p_1' i_1 : V_1 \to V_1'$ and $g := p_1 i_1' : V_1' \to V_1$ are isomorphisms.

Let $B = \oplus_{j>1} V_j$, $B' = \oplus_{j>1} V_j'$; then we have $V = V_1 \oplus B = V_1' \oplus B'$. Consider the map $h : B \to B'$ defined as a composition of the natural maps $B \to V \to B'$ attached to these decompositions. We claim that h is an isomorphism. To show this, it suffices to show that $\mathrm{Ker} h = 0$ (as h is a map between spaces of the same dimension). Assume that $v \in \mathrm{Ker} h \subset B$. Then $v \in V_1'$.

On the other hand, the projection of v to V_1 is zero, so $gv = 0$. Since g is an isomorphism, we get $v = 0$, as desired.

Now by the induction assumption, $m = n$, and $V_j = V'_{\sigma(j)}$ for some permutation σ of $2, ..., n$. The theorem is proved. $\square$

2.9 Problems

Problem 2.21. Extensions of representations. *Let A be an algebra, and V, W be a pair of representations of A. We would like to classify representations U of A such that V is a subrepresentation of U, and $U/V = W$. Of course, there is an obvious example $U = V \oplus W$, but are there any others?*

Suppose we have a representation U as above. As a vector space, it can be (non-uniquely) identified with $V \oplus W$, so that for any $a \in A$ the corresponding operator $\rho_U(a)$ has block triangular form

$$\rho_U(a) = \begin{pmatrix} \rho_V(a) & f(a) \\ 0 & \rho_W(a) \end{pmatrix},$$

where $f : A \to \mathrm{Hom}_k(W, V)$.

(a) What is the necessary and sufficient condition on $f(a)$ under which $\rho_U(a)$ is a representation? Maps f satisfying this condition are called (1-)cocycles (of A with coefficients in $\mathrm{Hom}_k(W, V)$). They form a vector space denoted $Z^1(W, V)$.

(b) Let $X : W \to V$ be a linear map. The coboundary of X, dX, is defined to be the function $A \to \mathrm{Hom}_k(W, V)$ given by $dX(a) = \rho_V(a)X - X\rho_W(a)$. Show that dX is a cocycle, which vanishes iff X is a homomorphism of representations. Thus coboundaries form a subspace $B^1(W, V) \subset Z^1(W, V)$, which is isomorphic to $\mathrm{Hom}_k(W, V)/\mathrm{Hom}_A(W, V)$. The quotient $Z^1(W, V)/B^1(W, V)$ is denoted $\mathrm{Ext}^1(W, V)$.

(c) Show that if $f, f' \in Z^1(W, V)$ and $f - f' \in B^1(W, V)$ then the corresponding extensions U, U' are isomorphic representations of A. Conversely, if $\phi : U \to U'$ is an isomorphism such that

$$\phi(a) = \begin{pmatrix} 1_V & * \\ 0 & 1_W \end{pmatrix}$$

then $f - f' \in B^1(V, W)$. Thus, the space $\mathrm{Ext}^1(W, V)$ "classifies" extensions of W by V.

(d) Assume that W, V are finite dimensional irreducible representations of A. For any $f \in Ext^1(W, V)$, let U_f be the corresponding extension. Show that U_f is isomorphic to $U_{f'}$ as representations if and only if f and f' are proportional. Thus isomorphism classes (as representations) of nontrivial extensions of W by V (i.e., those not isomorphic to $W \oplus V$) are parametrized by the projective space $\mathbb{P}\mathrm{Ext}^1(W, V)$. In particular, every extension is trivial iff $\mathrm{Ext}^1(W, V) = 0$.

Problem 2.22. *(a) Let $A = \mathbf{C}[x_1, ..., x_n]$, and V_a, V_b be one-dimensional representations in which x_i act by a_i and b_i, respectively ($a_i, b_i \in \mathbf{C}$). Find $\mathrm{Ext}^1(V_a, V_b)$ and classify 2-dimensional representations of A.*

(b) Let B be the algebra over $\mathbf{C}$ generated by $x_1, ..., x_n$ with the defining relations $x_i x_j = 0$ for all i, j. Show that for $n > 1$ the algebra B has infinitely many non-isomorphic indecomposable representations.

Problem 2.23. *Let Q be a quiver without oriented cycles, and P_Q the path algebra of Q. Find irreducible representations of P_Q and compute Ext^1 between them. Classify 2-dimensional representations of P_Q.*

Problem 2.24. *Let A be an algebra, and V a representation of A. Let $\rho : A \to \mathrm{End}V$. A formal deformation of V is a formal series*

$$\tilde{\rho} = \rho_0 + t\rho_1 + ... + t^n\rho_n + ...,$$

where $\rho_i : A \to \mathrm{End}(V)$ are linear maps, $\rho_0 = \rho$, and $\tilde{\rho}(ab) = \tilde{\rho}(a)\tilde{\rho}(b)$.

If $b(t) = 1 + b_1 t + b_2 t^2 + ...$, where $b_i \in \mathrm{End}(V)$, and $\tilde{\rho}$ is a formal deformation of ρ, then $b\tilde{\rho}b^{-1}$ is also a deformation of ρ, which is said to be isomorphic to $\tilde{\rho}$.

(a) Show that if $\mathrm{Ext}^1(V, V) = 0$, then any deformation of ρ is trivial, i.e. isomorphic to ρ.

(b) Is the converse to (a) true? (consider the algebra of dual numbers $A = k[x]/x^2$).

Problem 2.25. The Clifford algebra. *Let V be a finite dimensional complex vector space equipped with a symmetric bilinear form $(,)$. The Clifford algebra $\mathrm{Cl}(V)$ is the quotient of the tensor algebra TV by the ideal generated by the elements $v \otimes v - (v, v)1$, $v \in V$. More explicitly, if $x_i, 1 \le i \le N$ is a basis of V and $(x_i, x_j) = a_{ij}$ then $\mathrm{Cl}(V)$ is generated by x_i with defining relations*

$$x_i x_j + x_j x_i = 2a_{ij}, x_i^2 = a_{ii}.$$

Thus, if $(,) = 0$, $\mathrm{Cl}(V) = \wedge V$.

(i) Show that if $(,)$ is nondegenerate then $\mathrm{Cl}(V)$ semisimple, and has one irreducible representation of dimension 2^n if $\dim V = 2n$ (so in this case $\mathrm{Cl}(V)$ is a matrix algebra), and two such representations if $\dim(V) = 2n+1$ (i.e. in this case $\mathrm{Cl}(V)$ is a direct sum of two matrix algebras).

Hint. In the even case, pick a basis $a_1, ..., a_n, b_1, ..., b_n$ of V in which $(a_i, a_j) = (b_i, b_j) = 0$, $(a_i, b_j) = \delta_{ij}/2$, and construct a representation of $\mathrm{Cl}(V)$ on $S := \wedge(a_1, ..., a_n)$ in which b_i acts as "differentiation" with respect to a_i. Show that S is irreducible. In the odd case the situation is similar, except there should be an additional basis vector c such that $(c, a_i) = (c, b_i) = 0$, $(c, c) = 1$, and the action of c on S may be defined either by $(-1)^{\mathrm{degree}}$ or by $(-1)^{\mathrm{degree}+1}$, giving two representations S_+, S_- (why are they non-isomorphic?). Show that there is no other irreducible representations by finding a spanning set of $\mathrm{Cl}(V)$ with $2^{\dim V}$ elements.

(ii) Show that $\mathrm{Cl}(V)$ is semisimple if and only if $(,)$ is nondegenerate. If $(,)$ is degenerate, what is $\mathrm{Cl}(V)/Rad(\mathrm{Cl}(V))$?

2.10 Representations of tensor products

Let A, B be algebras. Then $A \otimes B$ is also an algebra, with multiplication $(a_1 \otimes b_1)(a_2 \otimes b_2) = a_1 a_2 \otimes b_1 b_2$. The following theorem describes irreducible finite dimensional representations of $A \otimes B$ in terms of irreducible finite dimensional representations of A and those of B.

Theorem 2.26. *(i) Let V be an irreducible finite dimensional representation of A and W an irreducible finite dimensional representation of B. Then $V \otimes W$ is an irreducible representation of $A \otimes B$.*

(ii) Any irreducible finite dimensional representation M of $A \otimes B$ has the form (i) for unique V and W.

Remark 2.27. Part (ii) of the theorem typically fails for infinite dimensional representations; e.g. it fails when A is the Weyl algebra in characteristic zero. Part (i) also may fail. E.g. let $A = B = V = W = \mathbb{C}(x)$. Then (i) fails, as $A \otimes B$ is not a field.

Proof. (i) By the density theorem, the maps $A \to \mathrm{End}\,V$ and $B \to \mathrm{End}\,W$ are surjective. Therefore, the map $A \otimes B \to \mathrm{End}\,V \otimes \mathrm{End}\,W = \mathrm{End}(V \otimes W)$ is surjective. Therefore, $V \otimes W$ is irreducible.

(ii) First we show the existence of V and W. Let A', B' be the images of A, B in $\mathrm{End}\,M$. Then A', B' are finite dimensional algebras, and M is a representation of $A' \otimes B'$, so we may assume without loss of generality that A and B are finite dimensional.

In this case, we claim that $\mathrm{Rad}(A \otimes B) = \mathrm{Rad}(A) \otimes B + A \otimes \mathrm{Rad}(B)$. Indeed, denote the latter by J. Then J is a nilpotent ideal in $A \otimes B$, as $\mathrm{Rad}(A)$ and $\mathrm{Rad}(B)$ are nilpotent. On the other hand, $(A \otimes B)/J = A/\mathrm{Rad}(A) \otimes B/\mathrm{Rad}B$, which is a product of two semisimple algebras, hence semisimple. This implies $J \supset \mathrm{Rad}(A \otimes B)$. Altogether, by Proposition 2.11, we see that $J = \mathrm{Rad}(A \otimes B)$, proving the claim.

Thus, we see that

$$(A \otimes B)/\mathrm{Rad}(A \otimes B) = A/\mathrm{Rad}(A) \otimes B/\mathrm{Rad}(B).$$

Now, M is an irreducible representation of $(A \otimes B)/\mathrm{Rad}(A \otimes B)$, so it is clearly of the form $M = V \otimes W$, where V is an irreducible representation of $A/\mathrm{Rad}(A)$ and W is an irreducible representation of $B/\mathrm{Rad}(B)$, and V, W are uniquely determined by M (as all of the algebras invovled are direct sums of matrix algebras). □

3 Representations of finite groups: basic results

3.1 Maschke's Theorem

Theorem 3.1. *(Maschke) Let G be a finite group and k a field whose characteristic does not divide $|G|$. Then:*

(i) The algebra $k[G]$ is semisimple.

(ii) One has $k[G] = \oplus_i \mathrm{End} V_i$, where V_i are the irreducible representations of G. In particular, the regular representations $k[G]$ decomposes into irreducibles as $\oplus_i \dim(V_i)V_i$, and therefore one has

$$|G| = \sum_i \dim(V_i)^2.$$

(the "sum of squares formula").

Proof. By Proposition 2.16, (i) implies (ii), and to prove (ii), it is sufficient to show that if V is a finite-dimensional representation of G and $W \subset V$ is any subrepresentation, then there exists a subrepresentation $W' \subset V$ such that $V = W \oplus W'$ as representations.

Choose any complement $\hat{W}$ of W in V. (Thus $V = W \oplus \hat{W}$ as *vector spaces*, but not necessarily as *representations*.) Let P be the projection along $\hat{W}$ onto W, i.e., the operator on V defined by $P|_W = \mathrm{Id}$ and $P|_{\hat{W}} = 0$. Let

$$\overline{P} := \frac{1}{|G|} \sum_{g \in G} \rho(g) P \rho(g^{-1}),$$

where $\rho(g)$ is the action of g on V, and let

$$W' = \ker \overline{P}.$$

Now $\overline{P}|_W = \text{Id}$ and $\overline{P}(V) \subseteq W$, so $\overline{P}^2 = \overline{P}$, so $\overline{P}$ is a projection along W'. Thus, $V = W \oplus W'$ as vector spaces.

Moreover, for any $h \in G$ and any $y \in W'$,

$$\overline{P}\rho(h)y = \frac{1}{|G|}\sum_{g\in G}\rho(g)P\rho(g^{-1}h)y = \frac{1}{|G|}\sum_{\ell\in G}\rho(h\ell)P\rho(\ell^{-1})y = \rho(h)\overline{P}y = 0,$$

so $\rho(h)y \in \ker \overline{P} = W'$. Thus, W' is invariant under the action of G and is therefore a subrepresentation of V. Thus, $V = W \oplus W'$ is the desired decomposition into subrepresentations. □

Proposition 3.2. *Conversely, if $k[G]$ is semisimple, then the characteristic of k does not divide $|G|$.*

Proof. Write $k[G] = \bigoplus_{i=1}^r \text{End } V_i$, where the V_i are irreducible representations and $V_1 = k$ is the trivial one-dimensional representation. Then

$$k[G] = k \oplus \bigoplus_{i=2}^r \text{End } V_i = k \oplus \bigoplus_{i=2}^r d_i V_i,$$

where $d_i = \dim V_i$. By Schur's Lemma,

$$\text{Hom}_{k[G]}(k, k[G]) = k\Lambda$$
$$\text{Hom}_{k[G]}(k[G], k) = k\epsilon,$$

for nonzero homomorphisms $\epsilon : k[G] \to k$ and $\Lambda : k \to k[G]$ unique up to scaling. We can take ϵ such that $\epsilon(g) = 1$ for all $g \in G$, and Λ such that $\Lambda(1) = \sum_{g\in G} g$. Then

$$\epsilon \circ \Lambda(1) = \epsilon\left(\sum_{g\in G} g\right) = \sum_{g\in G} 1 = |G|.$$

If $|G| = 0$, then Λ has no left inverse, a contradiction. □

Example 3.3. If $G = \mathbb{Z}/p\mathbb{Z}$ and k has characteristic p, then every irreducible representation of G over k is trivial (so $k[\mathbb{Z}/p\mathbb{Z}]$ indeed is not semisimple). Indeed, an irreducible representation of this group is a 1-dimensional space, on which the generator acts by a p-th root of unity, and every p-th roiot of unity in k equals 1, as $x^p - 1 = (x-1)^p$ over k.

Problem 3.4. *Let G be a group of order p^n. Show that every irreducible representation of G over a field k of characteristic p is trivial.*

3.2 Characters

If V is a finite-dimensional representation of a finite group G, then its character is defined by the formula $\chi_V(g) = \text{tr}|_V(\rho(g))$. Obviously, $\chi_V(g)$ is simply the restriction of the character $\chi_V(a)$ of V as a representation of the algebra $A = k[G]$ to the basis $G \subset A$, so it carries exactly the same information. The character is a *central* or *class function*: $\chi_V(g)$ depends only on the conjugacy class of g; i.e., $\chi_V(hgh^{-1}) = \chi_V(g)$.

Theorem 3.5. *If the characteristic of k does not divide $|G|$, characters of irreducible representations of G form a basis in the space $F_c(G, k)$ of class functions on G.*

Proof. By the Maschke theorem, $k[G]$ is semisimple, so by Theorem 2.17, the characters are linearly independent and are a basis of $(A/[A,A])^*$, where $A = k[G]$. It suffices to note that, as vector spaces over k,

$$\begin{aligned}(A/[A,A])^* &\cong \{\varphi \in \mathrm{Hom}_k(k[G],k) \mid gh - hg \in \ker\varphi \ \forall g,h \in G\}\\ &\cong \{f \in \mathrm{Fun}(G,k) \mid f(gh) = f(hg) \ \forall g,h \in G\},\end{aligned}$$

which is precisely $F_c(G,k)$. □

Corollary 3.6. *The number of irreducible representations of G equals the number of conjugacy classes of G (if $|G| \neq 0$ in k).*

Corollary 3.7. *Any representation of G is determined by its character; namely, $\chi_V = \chi_W$ implies $V \cong W$ if k has characteristic 0.*

3.3 Examples

The following are examples of representations of finite groups over $\mathbb{C}$.

1. Finite abelian groups $G = \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_k}$. Let $G^\vee$ be the set of irreducible representations of G. Every element of G forms a conjugacy class, so $|G^\vee| = |G|$. Recall that all irreducible representations over $\mathbb{C}$ (and algebraically closed fields in general) of commutative algebras and groups are one-dimensional. Thus, $G^\vee$ is an abelian group: if $\rho_1, \rho_2 : G \to \mathbb{C}^\times$ are irreducible representations then so are $\rho_1(g)\rho_2(g)$ and $\rho_1(g)^{-1}$. $G^\vee$ is called the *dual* or *character group* of G.

   For given $n \geq 1$, define $\rho : \mathbb{Z}_n \to \mathbb{C}^\times$ by $\rho(m) = e^{2\pi i m/n}$. Then $\mathbb{Z}_n^\vee = \{\rho^k : k = 0, \ldots, n-1\}$, so $\mathbb{Z}_n^\vee \cong \mathbb{Z}_n$. In general,

   $$(G_1 \times G_2 \times \cdots \times G_n)^\vee = G_1^\vee \times G_2^\vee \times \cdots \times G_n^\vee,$$

   so $G^\vee \cong G$ for any finite abelian group G. This isomorphism is, however, noncanonical: the particular decomposition of G as $\mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_k}$ is not unique as far as which elements of G correspond to $\mathbb{Z}_{n_1}$, etc. is concerned. On the other hand, $G \cong (G^\vee)^\vee$ is a canonical isomorphism, given by $\varphi : G \to (G^\vee)^\vee$, where $\varphi(g)(\chi) = \chi(g)$.

2. The symmetric group S_3. In S_n, conjugacy classes are based on cycle decomposition sizes: two permutations are conjugate iff they have the same number of cycles of each length. For S_3, there are 3 conjugacy classes, so there are 3 different irreducible representations over $\mathbb{C}$. If their dimensions are d_1, d_2, d_3, then $d_1^2 + d_2^2 + d_3^2 = 6$, so S_3 must have two 1-dimensional and one 2-dimensional representations. The 1-dimensional representations are the trivial representation $\mathbb{C}_+$ given by $\rho(\sigma) = 1$ and the sign representation $\mathbb{C}_-$ given by $\rho(\sigma) = (-1)^\sigma$.

   The 2-dimensional representation can be visualized as representing the symmetries of the equilateral triangle with vertices 1, 2, 3 at the points $(\cos 120°, \sin 120°)$, $(\cos 240°, \sin 240°)$, $(1, 0)$ of the coordinate plane, respectively. Thus, for example,

   $$\rho((12)) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \rho((123)) = \begin{pmatrix} \cos 120° & -\sin 120° \\ \sin 120° & \cos 120° \end{pmatrix}.$$

   To show that this representation is irreducible, consider any subrepresentation V. V must be the span of a subset of the eigenvectors of $\rho((12))$, which are the nonzero multiples of $(1,0)$ and $(0,1)$. V must also be the span of a subset of the eigenvectors of $\rho((123))$, which are different vectors. Thus, V must be either $\mathbb{C}^2$ or 0.

3. The quaternion group $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$, with defining relations

$$i = jk = -kj, \qquad j = ki = -ik, \qquad k = ij = -ji, \qquad -1 = i^2 = j^2 = k^2.$$

The 5 conjugacy classes are $\{1\}, \{-1\}, \{\pm i\}, \{\pm j\}, \{\pm k\}$, so there are 5 different irreducible representations, the sum of the squares of whose dimensions is 8, so their dimensions must be 1, 1, 1, 1, and 2.

The center $Z(Q_8)$ is $\{\pm 1\}$, and $Q_8/Z(Q_8) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. The four 1-dimensional irreducible representations of $\mathbb{Z}_2 \times \mathbb{Z}_2$ can be "pulled back" to Q_8. That is, if $q : Q_8 \to Q_8/Z(Q_8)$ is the quotient map, and ρ any representation of $Q_8/Z(Q_8)$, then $\rho \circ q$ gives a representation of Q_8.

The 2-dimensional representation is $V = \mathbb{C}^2$, given by $\rho(-1) = -\mathrm{Id}$ and

$$\rho(i) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \rho(j) = \begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix}, \qquad \rho(k) = \begin{pmatrix} 0 & -\sqrt{-1} \\ -\sqrt{-1} & 0 \end{pmatrix}.$$

These are the Pauli matrices, which arise in quantum mechanics.

Exercise. Show that the 2-dimensional irreducible representation of Q_8 can be realized in the space of functions $f : Q_8 \to \mathbb{C}$ such that $f(gi) = \sqrt{-1}f(g)$ (the action of G is by right multiplication, $g \circ f(x) = f(xg)$).

4. The symmetric group S_4. The order is 24, and there are 5 conjugacy classes: $e, (12), (123), (1234), (12)(34)$. Thus the sum of the squares of the dimensions of 5 irreducible representations is 24. As with S_3, there are two of dimension 1: the trivial and sign representations, $\mathbb{C}_+$ and $\mathbb{C}_-$. The other three must then have dimensions 2, 3, and 3. Because $S_3 \cong S_4/\mathbb{Z}_2 \times \mathbb{Z}_2$, where $\mathbb{Z}_2 \times \mathbb{Z}_2$ is $\{e, (12)(34), (13)(24), (14)(23)\}$, the 2-dimensional representation of S_3 can be pulled back to the 2-dimensional representation of S_4, which we will call $\mathbb{C}^2$.

We can consider S_4 as the group of rotations of a cube acting by permuting the interior diagonals (or, equivalently, on a regular octahedron permuting pairs of opposite faces); this gives the 3-dimensional representation $\mathbb{C}^3_+$.

The last 3-dimensional representation is $\mathbb{C}^3_-$, the product of $\mathbb{C}^3_+$ with the sign representation, or equivalently the permutation group of a regular tetrahedron. $\mathbb{C}^3_+$ and $\mathbb{C}^3_-$ are different, for if g is a transposition, $\det g|_{\mathbb{C}^3_+} = 1$ while $\det g|_{\mathbb{C}^3_-} = (-1)^3 = -1$. Note that another realization of $\mathbb{C}^3_-$ is by action of S_4 by symmetries (not necessarily rotations) of the regular tetrahedron.

3.4 Duals and tensor products of representations

If V is a representation of a group G, then V^* is also a representation, via

$$\rho_{V^*}(g) = (\rho_V(g)^*)^{-1} = (\rho_V(g)^{-1})^* = \rho_V(g^{-1})^*.$$

The character is $\chi_{V^*}(g) = \chi_V(g^{-1})$.

We have $\chi_V(g) = \sum \lambda_i$, where the λ_i are the eigenvalues of g in V. These eigenvalues must be roots of unity because $\rho(g)^{|G|} = \rho(g^{|G|}) = \rho(e) = \mathrm{Id}$. Thus for complex representations

$$\chi_{V^*}(g) = \chi_V(g^{-1}) = \sum \lambda_i^{-1} = \sum \overline{\lambda_i} = \overline{\sum \lambda_i} = \overline{\chi_V(g)}.$$

In particular, $V \cong V^*$ as *representations* (not just as vector spaces) iff $\chi_V(g) \in \mathbb{R}$ for all $g \in G$.

If V, W are representations of G, then $V \otimes W$ is also a representation, via

$$\rho_{V\otimes W}(g) = \rho_V(g) \otimes \rho_W(g).$$

Therefore, $\chi_{V\otimes W}(g) = \chi_V(g)\chi_W(g)$.

It is an interesting problem to decompose $V \otimes W$ into the direct sum of irreducible representations.

3.5 Orthogonality of characters

We define a positive definite Hermitian inner product on $\mathrm{F}_c(G, \mathbb{C})$ (the space of central functions) by

$$(f_1, f_2) = \frac{1}{|G|} \sum_{g\in G} f_1(g)\overline{f_2(g)}.$$

The following theorem says that characters of irreducible representations of G form an orthonormal basis of $F_c(G, \mathbb{C})$ under this inner product.

Theorem 3.8. *For any representations V, W*

$$(\chi_V, \chi_W) = \dim Hom(W, V),$$

and

$$(\chi_V, \chi_W) = \begin{cases} 1, & \text{if } V \cong W, \\ 0, & \text{if } V \ncong W \end{cases}$$

if V, W are irreducible.

Proof. By the definition

$$\begin{aligned}(\chi_V, \chi_W) &= \frac{1}{|G|} \sum_{g\in G} \chi_V(g)\overline{\chi_W(g)} = \frac{1}{|G|} \sum_{g\in G} \chi_V(g)\chi_{W^*}(g) \\ &= \frac{1}{|G|} \sum_{g\in G} \chi_{V\otimes W^*}(g) = \mathrm{tr}\,|_{V\otimes W^*}(P),\end{aligned}$$

where $P = \frac{1}{|G|}\sum_{g\in G} g \in Z(\mathbb{C}[G])$. (Here $Z(\mathbb{C}[G])$ denotes the center of $\mathbb{C}[G]$). If X is an irreducible representation of G then

$$P|_X = \begin{cases} \mathrm{Id}, & \text{if } X = \mathbb{C}, \\ 0, & X \neq \mathbb{C}. \end{cases}$$

Therefore, for any representation X the operator $P|_X$ is the G-invariant projector onto the subspace X^G of G-invariants in X. Thus,

$$\begin{aligned}\mathrm{tr}\,|_{V\otimes W^*}(P) &= \dim \mathrm{Hom}_G(\mathbb{C}, V \otimes W^*) \\ &= \dim(V \otimes W^*)^G = \dim \mathrm{Hom}_G(W, V).\end{aligned}$$

□

Here is another "orthogonality formula" for characters, in which summation is taken over irreducible representations rather than group elements.

Theorem 3.9. *Let $g, h \in G$, and let Z_g denote the centralizer of g in G. Then*

$$\sum_V \chi_V(g)\overline{\chi_V(h)} = \begin{cases} |Z_g| \text{ if } g \text{ is conjugate to } h \\ 0, \text{ otherwise} \end{cases}$$

where the summation is taken over all irreducible representations of G.

Proof. As noted above, $\overline{\chi_V(h)} = \chi_{V^*}(h)$, so the left hand side equals (using Maschke's theorem):

$$\sum_V \chi_V(g)\chi_{V^*}(h) = \mathrm{tr}|_{\mathbb{C}[G]}(x \mapsto gxh^{-1}).$$

If g and h are not conjugate, this trace is clearly zero, since the matrix of the operator $x \to gxh^{-1}$ in the basis of group elements has zero diagonal entries. On the other hand, if g and h are in the same conjugacy class, the trace is equal to the number of elements x such that $x = gxh^{-1}$, i.e., the order of the centralizer Z_g of g. We are done. □

Remark. Another proof of this result is as follows. Consider the matrix U whose rows are labeled by irreducible representations of G and columns by conjugacy classes, with entries $U_{V,g} = \chi_V(g)/\sqrt{|Z_g|}$. Note that G/Z_g is the conjugacy class of g, thus $|G|/|Z_g|$ is the number of elements conjugate to G. Thus, by Theorem 3.8, the rows of the matrix U are orthonomal. This means that U is unitary and hence its columns are orthonormal, which implies the statement.

3.6 Unitary representations. Another proof of Maschke's theorem for complex representations

Definition 3.10. A unitary finite dimensional representation of a group G is a representation of G on a complex finite dimensional vector space V over $\mathbb{C}$ equipped with a G-invariant positive definite Hermitian form[3] $(,)$, i.e. such that $\rho_V(g)$ are unitary operators: $(\rho_V(g)v, \rho_V(g)w) = (v, w)$.

Theorem 3.11. *If G is finite, then any finite dimensional representation of G has a unitary structure. If the representation if irreducible, this structure is unique up to scaling by a positive real number.*

Proof. Take any positive definite form B on V and define another form $\overline{B}$ as follows:

$$\overline{B}(v, w) = \sum_{g \in G} B(\rho_V(g)v, \rho_V(g)w)$$

Then $\overline{B}$ is a positive definite Hermitian form on V, and $\rho_V(g)$ are unitary operators. If V is an irreducible representation and B_1, B_2 are two positive definite Hermitian forms on V, then $B_1(v, w) = B_2(Av, w)$ for some homomorphism $A : V \to V$ (since any positive definite Hermitian form is nondegenerate). By Schur's lemma, $A = \lambda \mathrm{Id}$, and clearly $\lambda > 0$. □

Theorem 3.11 implies that if V is a finite dimensional representation of a finite group G, then the *complex conjugate representation* $\overline{V}$ (i.e. the same space V with the same addition and the same action of G, but complex conjugate action of scalars) is isomorphic to the dual representation V^*. Indeed, a homomorphism of representations $\overline{V} \to V^*$ is obviously the same thing as an invariant Hermitian form on V, and an isomorphism is the same thing as a nondegenerate Hermitian form. So one can use a unitary structure on V to define an isomorphism $\overline{V} \to V^*$.

[3]We agree that Hermitian forms are linear in the first argument and antilinear in the second one.

Theorem 3.12. *A finite dimensional unitary representation V of any group G is completely reducible.*

Proof. Let W be a subrepresentation of V. Let $W^{\perp}$ be the orthogonal complement of W in V under the Hermitian inner product. Then $W^{\perp}$ is a subrepresentation of W, and $V = W \oplus W^{\perp}$. This implies that V is completely reducible. □

Remark 3.13. This implies that not any finite dimensional representation of a group admits a unitary structure (as there exist finite dimensional representations, e.g. for $G = \mathbb{Z}$, which are indecomposable but not irreducible).

3.7 Orthogonality of matrix elements

Let V be an irreducible representation of G, and $v_1, v_2, \ldots, v_n$ be an orthonormal basis of V under the Hermitian form. The matrix elements of V are $T_{ij}^{V}(x) = (\rho_V(x)v_i, v_j)$.

Proposition 3.14. *(1) Matrix elements of nonisomorphic representations are orthogonal in $F(G, \mathbb{C})$ under the form $(f, g) = \frac{1}{|G|}\sum_{x\in G} f(x)\overline{g(x)}$.*

(2) $(t_{ij}^{V}, t_{i'j'}^{V}) = \delta_{ii'}\delta_{jj'} \cdot \frac{1}{\dim V}$

Thus, matrix elements of irreducible representations of G form an orthogonal basis of $F(G, \mathbb{C})$.

Proof. Let V and W be two irreducible representations of G. Take $\{v_i\}$ to be an orthonormal basis of V and $\{w_i\}$ to be an orthonormal basis of W under their positive definite invariant Hermitian forms. Let $w_i^* \in W^*$ be the linear function on W defined by taking the inner product with w_i: $w_i^*(u) = (u, w_i)$. Then for $x \in G$ we have $(xw_i^*, w_j^*) = \overline{(xw_i, w_j)}$. Therefore, putting $P = \sum_{x\in G} x$, we have

$$\sum_{x\in G}\langle xv_i, v_j\rangle\overline{\langle xw_{i'}, w_{j'}\rangle} = \sum_{x\in G}\langle xv_i, v_j\rangle\langle xw_{i'}^*, w_{j'}^*\rangle = \langle P(v_i \otimes w_{i'}^*), v_j \otimes w_{j'}^*\rangle$$

If $V \neq W$, this is zero, since P projects to the trivial representation, which does not occur in $V \otimes W^*$. If $V = W$, we need to consider $\langle P(v_i \otimes v_{i'}^*), v_j \otimes v_{j'}^*\rangle$. We have a G-invariant decomposition

$$\begin{aligned} V \otimes V^* &= \mathbb{C} \oplus L \\ \mathbb{C} &= \operatorname{span}(\sum v_k \otimes v_k^*) \\ L &= \operatorname{span}(\sum_{a:\sum_k a_{kk}=0} a_{kl} v_k \otimes v_l^*), \end{aligned}$$

and P projects to the first summand. The projection of $v_i \otimes v_{i'}^*$ to $\mathbb{C} \subset \mathbb{C} \oplus L$ is

$$\frac{\delta_{ii'}}{\dim V}\sum v_k \otimes v_k^*$$

This shows that

$$\langle P(v_i \otimes v_{i'}), v_j \otimes v_{j'}^*\rangle = \frac{\delta_{ii'}\delta_{jj'}}{\dim V}$$

which finishes the proof of (1) and (2). The last statement follows immediately from the sum of squares formula. □

3.8 Character tables, examples

The characters of all the irreducible representations of a finite group can be arranged into a character table, with conjugacy classes of elements as the columns, and characters as the rows. More specifically, the first row in a character table lists representatives of conjugacy classes, the second one the numbers of elements in the conjugacy classes, and the other rows are the values of the characters on the conjugacy classes. Due to Theorems 3.8 and 3.9 the rows and columns of a character table are orthonormal with respect to the appropriate inner products.

Note that in any character table, the row corresponding to the trivial representation consists of ones, and the column corresponding to the neutral element consists of the dimensions of the representations.

Here is, for example, the character table of S_3 :

S_3	Id	(12)	(123)
#	1	3	2
$\mathbb{C}_+$	1	1	1
$\mathbb{C}_-$	1	-1	1
$\mathbb{C}^2$	2	0	-1

It is obtained by explicitly computing traces in the irreducible representations.

For another example consider A_4, the group of even permutations of 4 items. There are three one-dimensional representations (as A_4 has a normal subgroup $\mathbb{Z}_2 \oplus \mathbb{Z}_2$, and $A_4/\mathbb{Z}_2 \oplus \mathbb{Z}_2 = \mathbb{Z}_3$. Since there are four conjugacy classes in total, there is one more irreducible representation of dimension 3. Finally, the character table is

A_4	Id	(123)	(132)	(12)(34)
#	1	4	4	3
$\mathbb{C}$	1	1	1	1
$\mathbb{C}_\epsilon$	1	ϵ	ϵ^2	1
$\mathbb{C}_{\epsilon^2}$	1	ϵ^2	ϵ	1
$\mathbb{C}^3$	3	0	0	-1

where $\epsilon = \exp(\frac{2\pi i}{3})$.

The last row can be computed using the orthogonality of rows. Another way to compute the last row is to note that $\mathbb{C}^3$ is the representation of A_4 by rotations of the regular tetrahedron: in this case $(123), (132)$ are the rotations by 120^0 and 240^0 around a perpendicular to a face of the tetrahedron, while $(12)(34)$ is the rotation by 180^0 around an axis perpendicular to two opposite edges.

Example 3.15. The following three character tables are of Q_8, S_4, and A_5 respectively.

Q_8	1	-1	i	j	k
#	1	1	2	2	2
$\mathbb{C}_{++}$	1	1	1	1	1
$\mathbb{C}_{+-}$	1	1	1	-1	-1
$\mathbb{C}_{-+}$	1	1	-1	1	-1
$\mathbb{C}_{--}$	1	1	-1	-1	1
$\mathbb{C}^2$	2	-2	0	0	0

S_4	Id	(12)	(12)(34)	(123)	(1234)
#	1	6	3	8	6
$\mathbb{C}_+$	1	1	1	1	1
$\mathbb{C}_-$	1	-1	1	1	-1
$\mathbb{C}^2$	2	0	2	-1	0
$\mathbb{C}^3_+$	3	-1	-1	0	1
$\mathbb{C}^3_-$	3	1	-1	0	-1

A_5	Id	(123)	(12)(34)	(12345)	(13245)
#	1	20	15	12	12
$\mathbb{C}$	1	1	1	1	1
$\mathbb{C}^3_+$	3	0	-1	$\frac{1+\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$
$\mathbb{C}^3_-$	3	0	-1	$\frac{1-\sqrt{5}}{2}$	$\frac{1+\sqrt{5}}{2}$
$\mathbb{C}^4$	4	1	0	-1	-1
$\mathbb{C}^5$	5	-1	1	0	0

Indeed, the computation of the characters of the 1-dimensional representations is straightforward.

The character of the 2-dimensional representation of Q_8 is obtained from the explicit formula for this representation, or by using the orthogonality.

For S_4, the 2-dimensional irreducible representation is obtained from the 2-dimensional irreducible representation of S_3 via the surjective homomorphism $S_4 \to S_3$, which allows to obtain its character from the character table of S_3.

The character of the 3-dimensional representation $\mathbb{C}^3_+$ is computed from its geometric realization by rotations of the cube. Namely, by rotating the cube, S_4 permutes the main diagonals. Thus (12) is rotaion by 180^0 around an axis that is perpendicular to two opposite edges, (12)(34) is the rotation by 180^0 around an axis that is perpendicular to two opposite faces, (123) is the rotation around a main diagonal by 120^0, and (1234) is the rotation by 90^0 around an axis that is perpendicular to two opposite faces; this allows us to compute the traces easily, using the fact that the trace of a rotation by the angle ϕ in $\mathbb{R}^3$ is $1 + 2\cos\phi$. Now the character of $\mathbb{C}^3_-$ is found by multiplying the character of $\mathbb{C}^3_+$ by the character of the sign representation.

Finally, we explain how to obtain the character table of A_5 (even permutations of 5 items). The group A_5 is the group of rotations of the regular icosahedron. Thus it has a 3-dimensional "rotation representation" $\mathbb{C}^3_+$, in which (12)(34) is the rotation by 180^0 around an axis perpendicular to two opposite edges, (123) is the rotation by 120^0 around an axis perpendicular to two opposite faces, and (12345), (13254) are the rotations by 72^0, respectively 144^0, around axes going through two opposite vertices. The character of this representation is computed from this description in a straightforward way.

Another representation of A_5, which is also 3-dimensional, is $\mathbb{C}^3_+$ twisted by the automorphism of A_5 given by coinjugation by (12) inside S_5. This representation is denoted by $\mathbb{C}^3_-$. It has the same character as $\mathbb{C}^3_+$, except that the conjugacy classes (12345) and (13245) are interchanged.

There are two remaining irreducible representations, and by the sum of squares formula their dimensions are 4 and 5. So we call them $\mathbb{C}^4$ and $\mathbb{C}^5$.

The representation $\mathbb{C}^4$ is realized on the space of functions on the set $\{1, 2, 3, 4, 5\}$ with zero sum of values (where A_5 acts by permutations). The character of this representation is equal to the character of the 5-dimensional permutation representation minus the character of the 1-dimensional trivial representation (constant functions). The former at an element g equals to the number of items among 1,2,3,4,5 which are fixed by g.

The representation $\mathbb{C}^5$ is realized on the space of functions on pairs of opposite vertices of the icosahedron which has zero sum of values. The character of this representation is computed similarly to the character of $\mathbb{C}^4$, or from the orthogonality formula.

3.9 Computing tensor product multiplicities and restriction multiplicities using character tables

Character tables allow us to compute the tensor product multiplicities N_{ij}^k using

$$V_i \otimes V_j = \sum N_{ij}^k V_k, \quad N_{ij}^k = (\chi_i \chi_j, \chi_k)$$

Example 3.16. The following tables represent computed tensor product multiplicities of irreducible representations of S_3, S_4, and A_5 respectively.

S_3	$\mathbb{C}_+$	$\mathbb{C}_-$	$\mathbb{C}^2$
$\mathbb{C}_+$	$\mathbb{C}_+$	$\mathbb{C}_-$	$\mathbb{C}^2$
$\mathbb{C}_-$		$\mathbb{C}_+$	$\mathbb{C}^2$
$\mathbb{C}^2$			$\mathbb{C}_+ \oplus \mathbb{C}_- \oplus \mathbb{C}^2$

S_4	$\mathbb{C}_+$	$\mathbb{C}_-$	$\mathbb{C}^2$	$\mathbb{C}^3_+$	$\mathbb{C}^3_-$
$\mathbb{C}_+$	$\mathbb{C}_+$	$\mathbb{C}_-$	$\mathbb{C}^2$	$\mathbb{C}^3_+$	$\mathbb{C}^3_-$
$\mathbb{C}_-$		$\mathbb{C}_+$	$\mathbb{C}^2$	$\mathbb{C}^3_-$	$\mathbb{C}^3_+$
$\mathbb{C}^2$			$\mathbb{C}_+ \oplus \mathbb{C}_- \oplus \mathbb{C}^2$	$\mathbb{C}^3_+ \oplus \mathbb{C}^3_-$	$\mathbb{C}^3_+ \oplus \mathbb{C}^3_-$
$\mathbb{C}^3_+$				$\mathbb{C}_+ \oplus \mathbb{C}^2 \oplus \mathbb{C}^3_+ \oplus \mathbb{C}^3_-$	$\mathbb{C}_- \oplus \mathbb{C}^2 \oplus \mathbb{C}^3_+ \oplus \mathbb{C}^3_-$
$\mathbb{C}^3_-$					$\mathbb{C}_+ \oplus \mathbb{C}^2 \oplus \mathbb{C}^3_+ \oplus \mathbb{C}^3_-$

A_5	$\mathbb{C}$	$\mathbb{C}^3_+$	$\mathbb{C}^3_-$	$\mathbb{C}^4$	$\mathbb{C}^5$
$\mathbb{C}$	$\mathbb{C}$	$\mathbb{C}^+_3$	$\mathbb{C}^3_-$	$\mathbb{C}^4$	$\mathbb{C}^5$
$\mathbb{C}^3_+$		$\mathbb{C} \oplus \mathbb{C}^5 \oplus \mathbb{C}^3_+$	$\mathbb{C}^4 \oplus \mathbb{C}^5$	$\mathbb{C}^3_- \oplus \mathbb{C}^4 \oplus \mathbb{C}^5$	$\mathbb{C}^3_+ \oplus \mathbb{C}^3_- \oplus \mathbb{C}^4 \oplus \mathbb{C}^5$
$\mathbb{C}^3_-$			$\mathbb{C} \oplus \mathbb{C}^5 \oplus \mathbb{C}^3_+$	$\mathbb{C}^3_+ \oplus \mathbb{C}^4 \oplus \mathbb{C}^5$	$\mathbb{C}^3_+ \oplus \mathbb{C}^3_- \oplus \mathbb{C}^4 \oplus \mathbb{C}^5$
$\mathbb{C}^4$				$\mathbb{C}^3_+ \oplus \mathbb{C}^3_- \oplus \mathbb{C} \oplus \mathbb{C}^4 \oplus \mathbb{C}^5$	$\mathbb{C}^3_+ \oplus \mathbb{C}^3_- \oplus 2\mathbb{C}^5 \oplus \mathbb{C}^4$
$\mathbb{C}^5$					$\mathbb{C} \oplus \mathbb{C}^3_+ \oplus \mathbb{C}^3_- \oplus 2\mathbb{C}^4 \oplus 2\mathbb{C}^5$

3.10 Problems

Problem 3.17. *Let G be the group of symmetries of a regular N-gon (it has $2N$ elements).*

(a) Describe all irreducible complex representations of this group (consider the cases of odd and even N)

(b) Let V be the 2-dimensional complex representation of G obtained by complexification of the standard representation on the real plane (the plane of the polygon). Find the decomposition of $V \otimes V$ in a direct sum of irreducible representations.

Problem 3.18. *Let G be the group of 3 by 3 matrices over $\mathbb{F}_p$ which are upper triangular and have ones on the diagonal, under multiplication (its order is of course p^3). It is called the Heisenberg group. For any complex number z such that $z^p = 1$ we define a representation of G on the space V of complex functions on $\mathbb{F}_p$, by*

$$(\rho \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} f)(x) = f(x-1),$$

$$(\rho \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} f)(x) = z^x f(x).$$

(note that z^x makes sense since $z^p = 1$).

(a) Show that such a representation exists and is unique, and compute $\rho(g)$ for all $g \in G$.

(b) Denote this representation by R_z. Show that R_z is irreducible if and only if $z \neq 1$.

(c) Classify all 1-dimensional representations of G. Show that R_1 decomposes into a direct sum of 1-dimensional representations, where each of them occurs exactly once.

(d) Use (a)-(c) and the "sum of squares" formula to classify all irreducible representations of G.

Problem 3.19. *Let V be a finite dimensional complex vector space, and $GL(V)$ be the group of invertible linear transformations of V. Then S^nV and Λ^mV ($m \leq dim(V)$) are representations of $GL(V)$ in a natural way. Show that they are irreducible representations.*

Hint: Choose a basis $\{e_i\}$ in V. Find a diagonal element H of $GL(V)$ such that $\rho(H)$ has distinct eigenvalues. (where ρ is one of the above representations). This shows that if W is a subrepresentation, then it is spanned by a subset S of a basis of eigenvectors of $\rho(H)$. Use the invariance of W under the operators $\rho(1+E_{ij})$ (where E_{ij} is defined by $E_{ij}e_k = \delta_{jk}e_i$) for all $i \neq j$ to show that if the subset S is nonempty, it is necessarily the entire basis.

Problem 3.20. *Recall that the adjacency matrix of a graph Γ (without multiple edges) is the matrix in which the ij-th entry is 1 if the vertices i and j are connected with an edge, and zero otherwise. Let Γ be a finite graph whose automorphism group is nonabelian. Show that the adjacency matrix of Γ must have repeated eigenvalues.*

Problem 3.21. *Let I be the set of vertices of a regular icosahedron ($|I| = 12$). Let $F(I)$ be the space of complex functions on I. Recall that the group $G = A_5$ of even permutations of 5 items acts on the icosahedron, so we have a 12-dimensional representation of G on $F(I)$.*

(a) Decompose this representation in a direct sum of irreducible representations (i.e., find the multiplicities of occurrence of all irreducible representations).

(b) Do the same for the representation of G on the space of functions on the set of faces and the set of edges of the icosahedron.

Problem 3.22. *Let F be a finite field with q elements, and G be the group of inhomogeneous linear transformations, $x \to ax + b$, over F (i.e., $a \in F^\times, b \in F$). Find all irreducible complex representations of G, and compute their characters. Compute the tensor products of irreducible representations.*

Hint. Let V be the representation of G on the space of functions on F with sum of all values equal to zero. Show that V is an irreducible representation of G.

Problem 3.23. *Let $G = SU(2)$ (unitary 2 by 2 matrices with determinant 1), and $V = \mathbb{C}^2$ the standard 2-dimensional representation of $SU(2)$. We consider V as a real representation, so it is 4-dimensional.*

(a) Show that V is irreducible (as a real representation).

(b) Let $\mathbb{H}$ be the subspace of $\mathrm{End}_{\mathbb{R}[G]}(V)$ consisting of endomorphisms of V as a real representation. Show that $\mathbb{H}$ is 4-dimensional and closed under multiplication. Show that every nonzero element in $\mathbb{H}$ is invertible, i.e. $\mathbb{H}$ is an algebra with division.

(c) Find a basis $1, i, j, k$ of $\mathbb{H}$ such that 1 is the unit and $i^2 = j^2 = k^2 = -1$, $ij = -ji = k$, $jk = -kj = i$, $ki = -ik = j$. Thus we have that Q_8 is a subgroup of the group $\mathbb{H}^\times$ of invertible elements of $\mathbb{H}$ under multiplication.

The algebra $\mathbb{H}$ is called the quaternion algebra.

(d) For $q = a+bi+cj+dk$, $a, b, c, d \in \mathbb{R}$, let $\bar{q} = a-bi-cj-dk$, and $||q||^2 = q\bar{q} = a^2+b^2+c^2+d^2$. Show that $\overline{q_1q_2} = \bar{q}_2\bar{q}_1$, and $||q_1q_2|| = ||q_1|| \cdot ||q_2||$.

(e) Let G be the group of quaternions of norm 1. Show that this group is isomorphic to $SU(2)$. (Thus $SU(2)$ is the 3-dimensional sphere).

(f) Consider the action of G on the space $V \subset \mathbb{H}$ spanned by i, j, k, by $x \to qxq^{-1}$, $q \in G$, $x \in V$. Since this action preserves the norm on V, we have a homomorphism $h : SU(2) \to SO(3)$, where $SO(3)$ is the group of rotations of the three-dimensional Euclidean space. Show that this homomorphism is surjective and that its kernel is $\{1, -1\}$.

Problem 3.24. *It is known that the classification of finite subgroups of $SO(3)$ is as follows:*

1) the cyclic group $\mathbb{Z}/n\mathbb{Z}$, $n \geq 1$, generated by a rotation by $2\pi/n$ around an axis;

2) the dihedral group D_n of order $2n$, $n \geq 2$ (the group of rotational symmetries in 3-space of a plane containing a regular n-gon[4];

3) the group of rotations of the regular tetrahedron (A_4).

4) the group of rotations of the cube or regular octahedron (S_4).

5) the group of rotations of a regular dodecahedron or icosahedron (A_5).

(a) Derive this classification.

Hint. Let G be a finite subgroup of $SO(3)$. Consider the action of G on the unit sphere. A point of the sphere preserved by some nontrivial element of G is called a pole. Show that every nontrivial element of G fixes a unique pair of opposite poles, and that the subgroup of G fixing a particular pole P is cyclic, or some order m (called the order of P). Thus the orbit of P has n/m elements, where $n = |G|$. Now let $P_1, ..., P_k$ be the poles representing all orbits of G on the set of poles, and $m_1, ..., m_k$ be their orders. By counting nontrivial elements of G, show that

$$2(1 - \frac{1}{n}) = \sum_i (1 - \frac{1}{m_i}).$$

Then find all possible m_i and n that can satisfy this equation and classify the corresponding groups.

(b) Using this classification, classify finite subgroups of $SU(2)$ (use the homomorphism $SU(2) \to SO(3)$).

Problem 3.25. *Find the characters and tensor products of irreducible complex representations of the Heisenberg group from Problem 3.18.*

Problem 3.26. *Let G be a finite group, and V a complex representation of G which is faithful, i.e. the corresponding map $G \to GL(V)$ is injective. Show that any irreducible representation of G occurs inside S^nV (and hence inside $V^{\otimes n}$) for some n.*

Hint. Show that there exists a vector $u \in V^*$ whose stabilizer in G is 1. Now define the map $SV \to F(G, \mathbb{C})$ sending a polynomial f on V^* to the function f_u on G given by $f_u(g) = f(gu)$. Show that this map is surjective and use this to deduce the desired result.

[4] A regular 2-gon is just a line segment.

Problem 3.27. *This problem is about an application of representation theory to physics (elasticity theory). We first describe the physical motivation and then state the mathematical problem.*

Imagine a material which occupies a certain region U in the physical space $V = \mathbb{R}^3$ (a space with a positive definite inner product). Suppose the material is deformed. This means, we have applied a diffeomorphism (=change of coordinates) $g : U \to U'$. The question in elasticity theory is how much stress in the material this deformation will cause.

For every point P, let $A_P : V \to V$ be defined by $A_P = dg(P)$. A_P is nondegenerate, so it has a polar decomposition $A_P = D_P O_P$, where O_P is orthogonal and D_P is symmetric. The matrix O_P characterizes the rotation part of A_P (which clearly produces no stress), and D_P is the distortion part, which actually causes stress. If the deformation is small, D_P is close to 1, so $D_P = 1 + d_P$, where d_P is a small symmetric matrix, i.e. an element of S^2V. This matrix is called the deformation tensor at P.

Now we define a stress tensor, which characterizes stress. Let v be a small nonzero vector in V, and σ a small disk perpendicular to v centered at P of area $||v||$. Let F_v be the force with which the part of the material on the v-side of σ acts on the part on the opposite side. It is easy to deduce from Newton's laws that F_v is linear in v, so there exists a linear operator $S_P : V \to V$ such that $F_v = S_P v$. It is called the stress tensor.

An elasticity law is an equation $S_P = f(d_P)$, where f is a function. The simplest such law is a linear law (Hooke's law): $f : S^2V \to End(V)$ is a linear function. In general, such a function is defined by $9 \cdot 6 = 54$ parameters, but we will show there are actually only two essential ones – the compression modulus K and the shearing modulus μ. For this purpose we will use representation theory.

Recall that the group $SO(3)$ of rotations acts on V, so S^2V, $End(V)$ are representations of this group. The laws of physics must be invariant under this group (Galileo transformations), so f must be a homomorphism of representations.

(a) Show that $End(V)$ admits a decomposition $\mathbb{R} \oplus V \oplus W$, where $\mathbb{R}$ is the trivial representation, V is the standard 3-dimensional representation, and W is a 5-dimensional representation of SO(3). Show that $S^2V = \mathbb{R} \oplus W$

(b) Show that V and W are irreducible, even after complexification. Deduce using Schur's lemma that S_P is always symmetric, and for $x \in \mathbb{R}, y \in W$ one has $f(x + y) = Kx + \mu y$ for some real numbers K, μ.

In fact, it is clear from physics that K, μ are positive.

4 Representations of finite groups: further results

4.1 Frobenius-Schur indicator

Suppose that G is a finite group and V is an irreducible representation of G over $\mathbb{C}$. We say that V is

- of complex type, if $V \ncong V^*$,
- of real type, if V has a nondegenerate symmetric form invariant under G,
- of quaternionic type, if V has a nondegenerate skew form invariant under G.

Problem 4.1. *(a) Show that* $\mathrm{End}_{\mathbb{R}[G]} V$ *is* $\mathbb{C}$ *for* V *of complex type,* $\mathrm{Mat}_2(\mathbb{R})$ *for* V *of real type, and* $\mathbb{H}$ *for* V *of quaternionic type, which motivates the names above.*

Hint. Show that the complexification $V_{\mathbb{C}}$ *of* V *decomposes as* $V \oplus V^*$. *Use this to compute the dimension of* $\mathrm{End}_{\mathbb{R}[G]} V$ *in all three cases. Using the fact that* $\mathrm{End}_{\mathbb{R}[G]} V$ *is a division algebra, prove the result in the complex case. In the remaining two cases, let* B *be the invariant bilinear form on* V, *and* $(,)$ *the invariant positive Hermitian form (they are defined up to a nonzero complex scalar and a positive real scalar, respectively), and define the operator* $j : V \to V$ *such that* $B(v, w) = (v, jw)$. *Show that* j *is complex antilinear* ($ji = -ij$), *and* $j^2 = \lambda \cdot Id$, *where* λ *is a real number, positive in the real case and negative in the quaternionic case (if* B *is renormalized,* j *multiplies by a nonzero complex number* z *and* j^2 *by* $z\bar{z}$, *as* j *is antilinear). Thus* j *can be normalized so that* $j^2 = 1$ *for the real case, and* $j^2 = -1$ *in the quaternionic case. Deduce the claim from this.*

(b) Show that V *is of real type if and only if* V *is the complexification of a representation* $V_{\mathbb{R}}$ *over the field of real numbers.*

Example 4.2. For $\mathbb{Z}/n\mathbb{Z}$ all irreducible representations are complex, except the trivial one and, if n is even, the "sign" representation, $m \to (-1)^m$, which are real. For S_3 all three irreducible representations $\mathbb{C}_+, \mathbb{C}_-, \mathbb{C}^2$ are real. For S_4 there are five irreducible representations $\mathbb{C}_+$, $\mathbb{C}_-$, $\mathbb{C}^2$, $\mathbb{C}^3_+$, $\mathbb{C}^3_-$, which are all real. Similarly, all five irreducible representations of A_5 – $\mathbb{C}$, $\mathbb{C}^3_+$, $\mathbb{C}^3_-$, $\mathbb{C}^4$, $\mathbb{C}^5$ are real. As for Q_8, its one-dimensional representations are real, and the two-dimensional one is quaternionic.

Definition 4.3. The Frobenius-Schur indicator $FS(V)$ of an irreducible representation V is 0 if it is of complex type, 1 if it is of real type, and -1 if it is of quaternionic type.

Theorem 4.4. *(Frobenius-Schur) The number of involutions (=elements of order 2) in* G *is equal to* $\sum_V \dim(V) FS(V)$, *i.e. the sum of dimensions of all real representations of* G *minus the sum of dimensions of its quaternionic representations.*

Proof. Let $A : V \to V$ have eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. We have

$$\mathrm{Tr}|_{S^2V}(A \otimes A) = \sum_{i \le j} \lambda_i \lambda_j$$

$$\mathrm{Tr}|_{\Lambda^2V}(A \otimes A) = \sum_{i < j} \lambda_i \lambda_j$$

Thus,

$$\mathrm{Tr}|_{S^2V}(A \otimes A) - \mathrm{Tr}|_{\Lambda^2V}(A \otimes A) = \sum_{1 \le i \le n} \lambda_i^2 = \mathrm{Tr}(A^2).$$

Thus for $g \in G$ we have

$$\chi_V(g^2) = \chi_{S^2V}(g) - \chi_{\Lambda^2V}(g)$$

Therefore,

$$\chi_V(\sum_{g \in G} g^2) = |G| \begin{cases} 1, & \text{if } V \text{ is real} \\ -1, & \text{if } V \text{ is quaternionic} \\ 0, & \text{if } V \text{ is complex} \end{cases}$$

Finally, the number of involutions in G equals

$$\frac{1}{|G|} \sum_V \dim V \chi_V(\sum_{g \in G} g^2) = \sum_{\text{real } V} \dim V - \sum_{\text{quat } V} \dim V.$$

□

Corollary 4.5. *Assume that all representations of a finite group G are defined over real numbers (i.e. all complex representations of G are obtained by complexifying real representations). Then the sum of dimensions of irreducible representations of G equals the number of involutions in G.*

4.2 Frobenius determinant

Enumerate the elements of a finite group G as follows: $g_1, g_2, \ldots, g_n$. Introduce n variables indexed with the elements of G :

$$x_{g_1}, x_{g_2}, \ldots, x_{g_n}.$$

Definition 4.6. Consider the matrix X_G with entries $a_{ij} = x_{g_i g_j}$. The determinant of X_G is some polynomial of degree n of $x_{g_1}, x_{g_2}, \ldots, x_{g_n}$ that is called *the Frobenius determinant.*

The following theorem, discovered by Dedekind and proved by Frobenius, became the starting point for creation of representation theory.

Theorem 4.7.

$$\det X_G = \prod_{j=1}^{r} P_j(\mathbf{x})^{\deg P_j}$$

for some pairwise non-proportional irreducible polynomials $P_j(\mathbf{x})$, where r is the number of conjugacy classes of G.

We will need the following simple lemma.

Lemma 4.8. *Let Y be a $n \times n$ matrix with entries y_{ij}. Then $\det Y$ is an irreducible polynomial of $\{y_{ij}\}$.*

Proof. Let $\det Y = q_1 q_2 \ldots q_k$, be the factorization of $\det Y$ into irreducible polynomials (it is defined uniquely up to scaling and permutation of factors). Since $\det Y$ has degree 1 with respect to each row and each column of Y, by uniqueness of factorization all q_i must be homogeneous with respect to each row and each column (of degree either 0 or 1). Now consider the factor q_1. It is homogeneous of degree 1 in some row. This means that it depends on all columns, so is homogeneous of degree 1 in all columns. Thus $q_1 = \det Y$, as desired. □

Now we are ready to proceed to the proof Theorem 4.7.

Proof. Let $V = \mathbb{C}[G]$ be the regular representation of G. Consider the operator-valued polynomial

$$L(\mathbf{x}) = \sum_{g \in G} x_g \rho(g),$$

where $\rho(g) \in \mathrm{End}V$ is induced by g. The action of $L(\mathbf{x})$ on an element $h \in G$ is

$$L(\mathbf{x})h = \sum_{g \in G} x_g \rho(g) h = \sum_{g \in G} x_g g h = \sum_{z \in G} x_{zh^{-1}} z$$

So the matrix of the linear operator $L(\mathbf{x})$ in the basis $g_1, g_2, \ldots, g_n$ is X_G with permuted columns and hence has the same determinant up to sign.

Further, by Maschke's theorem, we have

$$\det_V L(\mathbf{x}) = \prod_{i=1}^{r} (\det_{V_i} L(\mathbf{x}))^{\dim V_i}.$$

We set $P_i = \det_{V_i} L(\mathbf{x})$. Let $\{e_{im}\}$ be bases of V_i and $E_{i,jk} \in \operatorname{End} V_i$ be the matrix units in these bases. Then $\{E_{i,jk}\}$ is a basis of $\mathbb{C}[G]$ and

$$L(\mathbf{x})|_{V_i} = \sum_{j,k} y_{i,jk} E_{i,jk},$$

where $y_{i,jk}$ are new coordinates on $\mathbb{C}[G]$ related to x_g by a linear transformation. Then

$$P_i(\mathbf{x}) = \det|_{V_i} L(\mathbf{x}) = \det(y_{i,jk})$$

Hence, P_i are irreducible (by Lemma 4.8) and not proportional to each other. The theorem is proved. □

4.3 Algebraic numbers and algebraic integers

We are now passing to deeper results in representation theory of finite groups. These results require the theory of algebraic numbers, which we will now briefly review.

Definition 4.9. $z \in \mathbb{C}$ is an **algebraic number** (respectively, an **algebraic integer**), if z is a root of a monic polynomial with rational (respectively, integer) coefficients.

Definition 4.10. $z \in \mathbb{C}$ is an **algebraic number**, (respectively an **algebraic integer**), if z is an eigenvalue of a matrix with rational (respectively, integer) entries.

Proposition 4.11. *Definitions (4.9) and (4.10) are equivalent.*

Proof. To show (4.10) ⇒ (4.9), notice that z is a root of the characteristic polynomial of the matrix (a monic polynomial with rational, respectively integer, coefficients).
To show (4.9) ⇒ (4.10), suppose z is a root of

$$p(x) = x^n + a_1 x^{n-1} + \ldots + a_{n-1}x + a_n.$$

Then the characteristic polynomial of the following matrix (called the **companion matrix**) is $p(x)$:

$$\begin{pmatrix} 0 & 0 & 0 & \ldots & 0 & -a_n \\ 1 & 0 & 0 & \ldots & 0 & -a_{n-1} \\ 0 & 1 & 0 & \ldots & 0 & -a_{n-1} \\ & & & \vdots & & \\ 0 & 0 & 0 & \ldots & 1 & -a_1 \end{pmatrix}.$$

Since z is a root of the characteristic polynomial of this matrix, it is its eigenvalue. □

The set of algebraic numbers is denoted by $\overline{\mathbb{Q}}$, and the set of algebraic integers by $\mathbb{A}$.

Proposition 4.12. *(i)* $\mathbb{A}$ *is a ring.*

(ii) $\overline{\mathbb{Q}}$ *is a field. Namely, it is an algebraic closure of the field of rational numbers.*

Proof. We will be using definition (4.10). Let α be an eigenvalue of

$$\mathcal{A} \in \mathrm{Mat}_n(\mathbb{C})$$

with eigenvector v, let β be an eigenvalue of

$$\mathcal{B} \in \mathrm{Mat}_m(\mathbb{C})$$

with eigenvector w. Then $\alpha \pm \beta$ is an eigenvalue of

$$\mathcal{A} \otimes \mathrm{Id}_m \pm \mathrm{Id}_n \otimes \mathcal{B},$$

and $\alpha\beta$ is an eigenvalue of

$$\mathcal{A} \otimes \mathcal{B}.$$

The corresponding eigenvector is in both cases $v \otimes w$. This shows that both $\mathbb{A}$ and $\overline{\mathbb{Q}}$ are rings. To show that the latter is a field, it suffices to note that if $\alpha \neq 0$ is a root of a polynomial $p(x)$ of degree d, then α^{-1} is a root of $x^d p(1/x)$. The last statement is easy, since a number α is algebraic if and only if it defines a finite extension of $\mathbb{Q}$. □

Proposition 4.13. $\mathbb{A} \cap \mathbb{Q} = \mathbb{Z}$.

Proof. We will be using definition (4.9). Let z be a root of

$$p(x) = x^n + a_1 x^{n-1} + \ldots + a_{n-1}x + a_n,$$

and suppose

$$z = \frac{p}{q} \in \mathbb{Q}, gcd(p,q) = 1.$$

Notice that the leading term of $p(x)$ will have q^n in the denominator, whereas all the other terms will have a lower power of q there. Thus, if

$$q \neq \pm 1,$$

then

$$p(z) \notin \mathbb{Z},$$

a contradiction. Thus,

$$z \in \mathbb{A} \cap \mathbb{Q} \Rightarrow z \in \mathbb{Z}.$$

The reverse inclusion follows because $n \in \mathbb{Z}$ is a root of $x - n$. □

Every algebraic number α has a **minimal polynomial** $p(x)$, which is the monic polynomial with rational coefficients of the smallest degree such that $p(\alpha) = 0$. Any other polynomial $q(x)$ with rational coefficients such that $q(\alpha) = 0$ is divisible by $p(x)$. Roots of $p(x)$ are called the **algebraic conjugates** of α; they are roots of any rational polynomial q such that $q(\alpha) = 0$.

Note that any algebraic conjugate of an algebraic integer is obviously also an algebraic integer. Therefore, by the Vieta theorem, the minimal polynomial of an algebraic integer has integer coefficients.

Below we will need the following lemma:

Lemma 4.14. *If $\alpha_1, ..., \alpha_m$ are algebraic numbers, then all algebraic conjugates to $\alpha_1 + ... + \alpha_m$ are of the form $\alpha_1' + ... + \alpha_m'$, where α_i' are some algebraic conjugates of α_i.*

Proof. It suffices to prove this for two summands. If α_i are eigenvalues of rational matrices A_i of smallest size (i.e. their characteristic polynomials are the minimal polynomials of α_i), then $\alpha_1 + \alpha_2$ is an eigenvalue of $A := A_1 \otimes \mathrm{Id} + \mathrm{Id} \otimes A_2$. Therefore, so is any algebraic conjugate to $\alpha_1 + \alpha_2$. But all eigenvalues of A are of the form $\alpha_1' + \alpha_2'$, so we are done. □

Problem 4.15. *Show that if V is an irreducible complex representation of a finite group G of dimension > 1 then there exists $g \in G$ such that $\chi_V(g) = 0$.*

Hint. Assume the contrary. Use orthonormality of characters to show that the arithmetic mean of the numbers $|\chi_V(g)|^2$ for $g \neq 1$ is < 1. Deduce that their product β satisfies $0 < \beta < 1$. Show that all conjugates of β satisfy the same inequalities (consider the Galois conjugates of the representation V). Then derive a contradiction.

4.4 Frobenius divisibility

Theorem 4.16. *Let G be a finite group, and let V be an irreducible (necessarily finite-dimensional) representation of G over $\mathbb{C}$. Then*

$$\dim V \text{ divides } |G|.$$

Proof. Let

$$C_1, C_2, \ldots, C_n$$

be the conjugacy classes of G, with

$$C_1 = \{e\}.$$

Let

$$p_{C_i} \in \mathbb{C}[G]$$

be defined for each conjugacy class as

$$p_{C_i} = \sum_{g \in C_i} g.$$

Since G acts transitively on each conjugacy class, every conjugate of p_{C_i} is equal to itself, i.e. p_{C_i} is a central element in $\mathbb{C}[G]$. By Schur's lemma, p_{C_i} acts on V by a scalar λ_i; therefore,

$$|C_i|\chi_V(g_{C_i}) = \mathrm{tr}\,(p_{C_i}) = \dim V \cdot \lambda_i.$$

Therefore,

$$\lambda_i = \chi_V(g_{C_i})\frac{|C_i|}{\dim V},$$

where g_{C_i} is a representative of C_i.

Proposition 4.17. *The number λ_i is an algebraic integer for all i.*

Proof. Notice that

$$p_{C_i} p_{C_j} = \sum_{g \in C_i, h \in C_j} gh = \sum_{u \in G} N(g, h, u)u,$$

where $N(g,h,u)$ is the number of ways to obtain u as gh for some $g \in C_i, h \in C_j$.
Thus,

$$p_{C_i} p_{C_j} = \sum_{i,j,k} N_{ij}^k p_{C_k},$$

where N_{ij}^k is the number of ways to obtain some element of C_k as gh for some $g \in C_i, h \in C_j$.
Therefore,

$$\lambda_i \lambda_j = \sum_{i,j,k} N_{ij}^k \lambda_k$$

Let $\vec{\lambda} = \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_n \end{pmatrix}$,

then

$$N_i(\vec{\lambda}) = \lambda_i \vec{\lambda},$$

where N_i is the matrix whose jk^{th} entry is N_{ij}^k. Since

$$\vec{\lambda} \neq 0,$$

λ_i is an eigenvalue of an integer matrix N_i, and by definition (4.10) an algebraic integer. □

Now, consider

$$\sum_i \lambda_i \overline{\chi_V(g_{C_i})}.$$

This is an algebraic integer, since λ_i was just proven to be an algebraic integer, and $\chi_V(g_{C_i})$ is a sum of roots of unity (it is the sum of eigenvalues of the matrix of $\rho(g_{C_i})$, and since

$$g_{C_i}^{|G|} = e$$

in G, the eigenvalues of $\rho(g_{C_i})$ are roots of unity), and $\mathbb{A}$ is a ring (4.12). On the other hand, from the definition of λ_i,

$$\sum_{C_i} \lambda_i \overline{\chi_V(g_{C_i})} = \sum_i \frac{|C_i| \chi_V(g_{C_i}) \overline{\chi_V(g_{C_i})}}{\dim V}.$$

Recalling that χ_V is a class function, this is equivalent to

$$\sum_{g \in G} \frac{\chi_V(g) \overline{\chi_V(g)}}{\dim V} = \frac{|G|(\chi_V, \chi_V)}{\dim V}.$$

Since V was an irreducible representation,

$$(\chi_V, \chi_V) = 1,$$

so

$$\sum_{C_i} \lambda_i \overline{\chi_V(g_{C_i})} = \frac{|G|}{\dim V}.$$

Since

$$\frac{|G|}{\dim V} \in \mathbb{Q}$$

and

$$\sum_{C_i} \lambda_i \overline{\chi_V(g_{C_i})} \in \mathbb{A},$$

by (4.13)

$$\frac{|G|}{\dim V} \in \mathbb{Z}.$$

□

4.5 Burnside's Theorem

Definition 4.18. A group G is called *solvable* if there exists a series of nested normal subgroups

$$\{e\} = G_1 \triangleleft G_2 \triangleleft \ldots \triangleleft G_n = G$$

where G_{i+1}/G_i is abelian for all $1 \leq i \leq n-1$.

Remark 4.19. These groups are called solvable because they first arose as Galois groups of polynomial equations which are solvable in radicals.

Theorem 4.20 (Burnside)**.** *Any group G of order $p^a q^b$, where p and q are prime and $a, b \geq 0$, is solvable.*

This famous result in group theory was proved by the British mathematician William Burnside in the late 19th century. Here is a proof of his theorem using Representation Theory.

Before proving Burnside's theorem we will prove several other results which may be of independent interest.

Theorem 4.21. *Let V be an irreducible representation of a finite group G and let C be a conjugacy class of G with* $\gcd(|C|, \dim(V)) = 1$.
Then for any $g \in C$, either $\chi_V(g) = 0$ or g acts as a scalar on V.

The proof will be based on the following lemma.

Lemma 4.22. *If $\epsilon_1, \epsilon_2 \ldots \epsilon_n$ are roots of unity such that $\dfrac{1}{n}(\epsilon_1 + \epsilon_2 + \ldots + \epsilon_n)$ is an algebraic integer, then either $\epsilon_1 = \ldots = \epsilon_n$ or $\epsilon_1 + \ldots + \epsilon_n = 0$.*

Proof. Let $a = \frac{1}{n}(\epsilon_1 + \ldots + \epsilon_n)$. Let $q = x^m + q_{m-1}x^{m-1} + \ldots + q_1 x + q_0$ be the minimal polynomial of a, and let $\{a_i\}, i = 1, \ldots, m$ be the set of all the conjugates of a.

By Lemma 4.14, $a_i = \frac{1}{n}(\epsilon'_1 + \epsilon'_2 + \ldots + \epsilon'_n)$, where ϵ'_i are conjugate to ϵ_i. Since conjugates of roots of unity are roots of unity, $|\epsilon'_i| = 1$. This means that $|\epsilon'_1 + \ldots + \epsilon'_n| \leq n$ and $|a_i| \leq 1$. . Thus, $|q_0| = \prod_{i=1}^m |a_i| \leq 1$. However, by our assumption, a is an algebraic integer and $q_0 \in \mathbb{Z}$. Therefore, either $q_0 = 0$ or $|q_0| = 1$.

Assume that $|q_0| = 1$. Then $|a_i| = 1$ for all i, and in particular, $|a| = |\frac{1}{n}(\epsilon_1 + \ldots + \epsilon_n)| = 1 = \frac{1}{n}(|\epsilon_1| + |\epsilon_2| + \ldots + |\epsilon_n|)$. This means that all ϵ_i have the same argument. It follows that $\epsilon_1 = \ldots = \epsilon_n$ since all ϵ_i have the same absolute value.

Otherwise, $q_0 = 0$. Then $x|q$ and since q is irreducible, $q = x$ and $a = 0$. □

Proof of theorem 4.21.

Let $\dim V = n$. Let $\epsilon_1, \epsilon_2, \ldots \epsilon_n$ be eigenvalues of $\rho(g)$. Since G is a finite group, $\rho(g)$ is diagonalizable and ϵ_i are roots of unity. We know that $\frac{1}{n}(|C|\chi_V(g)) \in \mathbb{A}$ and that $\chi_V(g) \in \mathbb{A}$. Since $GCD(n, |C|) = 1$, there are integers α, β such that $\alpha n + \beta|C| = 1$. Therefore,

$$\alpha\chi_V(g) + \beta\frac{|C|\chi_V(g)}{n} = \frac{\chi_V(g)}{n} \in \mathbb{A}.$$

However, since $\chi_V(g) = \epsilon_1 + \ldots + \epsilon_n$, we get that either $\epsilon_1 + \ldots + \epsilon_n = \chi_V(g) = 0$ or $\epsilon_1 = \ldots = \epsilon_n$ by 4.22. If $\epsilon_1 = \ldots = \epsilon_n$, then, since $\rho(g)$ is diagonalizable, it must be scalar. Otherwise, $\chi_V(G) = 0$. □

Proposition 4.23. *Let G be a finite, simple non-abelian group and let V be a non-trivial, irreducible representation of G. Then, if $g \in G$ acts by a scalar in V, $g = e$.*

Proof. Assume that $g \neq e$. Let N be the set of all $x \in G$ whose action in V is scalar. Clearly, $N \triangleleft G$ and $g \in N$. Since $g \neq e$, this means that $N \neq e$ and $N = G$.

Now let K be the kernel of $\rho : G \to \text{End}V$. Since ρ is a group homomorphism, $K \triangleleft G$ and because V is non-trivial, $K \neq G$ and $K = \{e\}$.

This means that ρ is an injection and $G \cong \text{Im}\rho$. But $\rho(x)$ is scalar for any $x \in G$, so G is commutative, which is a contradiction. □

We are now ready to prove another result in group theory which will later imply Burnside's Theorem.

Theorem 4.24. *Let G be a group and let C be a conjugacy class of order p^k where p is prime and $k > 0$. Then G has a proper normal subgroup.*

Proof. Assume the contrary, i.e. that G is simple.

Choose an element $g \in C$. Since $g \neq e$, by orthogonality of columns of the character table,

$$\sum_{V \in X} \dim V \chi_V(g) = 0.$$

We can divide X into three parts:

1. S, the set of irreducible representations whose dimension is divisible by p,
2. T, the set of non-trivial irreducible representations whose dimension is not divisible by p, and
3. I, the trivial representation.

Lemma 4.25. *If $V \in T$ then $\chi_V(g) = 0$.*

Proof. Since $\gcd(|C|, dim(V)) = 1$, by Theorem 4.21, either

1. $\chi_V(g) = 0$ or
2. g acts as a scalar in V, and by Proposition 4.23, $g = e$ which is a contradiction.

□

Also, if $V \in S$, we have $\frac{1}{p}\dim(V)\chi_V(g) \in \mathbb{A}$, so

$$a = \sum_{V \in S} \frac{1}{p} \dim(V)\chi_V(g) \in \mathbb{A}.$$

Therefore,

$$0 = \sum_{V \in S} \dim V \chi_V(g) + \sum_{V \in T} \dim V \chi_V(g) + \dim I \chi_I(g) = pa + 1.$$

This means that $a = \frac{-1}{p}$, which is not an algebraic integer, so we havea contradiction.

□

Now we can finally prove Burnside's theorem.

Assume that there exists a group of order $p^a q^b$ that is not solvable. We may assume that G has the smallest order among such groups. Since $|G| \neq 1$, either a or b must be non-zero.

We may assume without loss of generality that $b \neq 0$.

Lemma 4.26. *Let G be a group as above.*

(i) G is simple.

(ii) G has a trivial center (in particular, it is not abelian).

(iii) G has a conjugacy class C of order p^k.

Proof. (i) Assume that N is a non-trivial proper normal subgroup of G. Since $|N|$ divides $|G|$, $|N| = p^r q^s$ for some $r \leq a, s \leq b$.

Let $H = N/G$. Then $|H| = p^{a-r} q^{b-s}$. By our minimality assumption, both N and H are solvable, and there exist normal series

$$\{e\} = N_1 \lhd N_2 \lhd \ldots \lhd N_m = N \text{ and } \{e\} = H_1 \lhd H_2 \lhd \ldots \lhd H_n = H$$

with abelian quotients.

Let π be the canonical epimorphism $G \to G/N = H$. Then

$$\pi^{-1}(H_j) \lhd \pi^{-1}(H_{j+1}) \text{ and } \pi^{-1}(H_{j+1})/\pi^{-1}(H_j) = H_{j+1}/H_j$$

for any j.

Consider the normal series

$$\{e\} \lhd N_1 \lhd \ldots \lhd N_m = N = \pi^{-1}(e) = \pi^{-1}(H_1) \lhd \ldots \lhd \pi^{-1}(H_n) = G.$$

The quotient of any two consecutive subgroups of this series is either N_{i+1}/N_i or $\pi^{-1}(H_{j+1})/\pi^{-1}(H_j) = H_{j+1}/H_j$, all of which are abelian. Because of this, G is solvable, which is a contradiction. Therefore G is simple.

(ii) The center of G, $Z(G)$ is a normal subgroup of G. If $Z(G) = G$ then G is abelian and $\{e\} \lhd G$ is a normal series with abelian factors. Therefore, since G is simple, $Z(G) = \{e\}$.

(iii) Assume that G does not have a conjugacy class of order p^k. Let C be any conjugacy class. The order of C divides $|G|$, so $|C| = p^i q^j$. Because of our assumption, either $j \neq 0$ and q divides $|C|$ or $|C| = 1$ and C is central. But there is exactly one central element, e, so $|G| = \sum |C| = 1 \mathrm{mod} q$ where the sum is taken over all the conjugacy classes C. However, since $b \neq 0$, q divides $|G|$, which is a contradiction.

□

But by Theorem 4.24, this is impossible! Therefore, there are no groups of order $p^a q^b$ which are not solvable, and we have proven Burnside's Theorem.

4.6 Representations of products

Theorem 4.27. *Let G, H be finite groups, $\{V_i\}$ be the irreducible representations of G over a field k (of any characteristic), and $\{W_j\}$ be the irreducible representations of H over k. Then the irreducible representations of $G \times H$ over k are $\{V_i \otimes W_j\}$.*

Proof. This follows from Theorem 2.26. □

4.7 Induced Representations

Given a representation V of a group G and a subgroup $H < G$, there is a natural way to construct a representation of H. The restricted representation of V to H, $\mathrm{Res}_H^G V$ is the representation given by the vector space V and the action $\rho_{\mathrm{Res}_H^G V} = \rho_V|_H$.

There is also a natural, but more complicated way to construct a representation of a group G given a representation V of its subgroup H.

4.7.1 Definition

Definition 4.28. If G is a group, $H < G$, and V is a representation of H, then the *induced representation $Ind_H^G V$* is a representation of G with

$$\mathrm{Ind}_H^G V = \{f : G \to V | f(hx) = \rho_V(h) f(x)\} \forall x \in G, h \in H$$

and the action $g(f)(x) = f(xg) \forall g \in G$.

4.7.2

Let us check that this is indeed a representation:

$g(f)(hx) = f(hxg) = \rho_V(h) f(xg) = \rho_V(h) g(f)(x)$, and $g(g'(f))(x) = g'(f)(xg) = f(xgg') = (gg')(f)(x)$ for any $g, g', x \in G$ and $h \in H$.

Remark 4.29. In fact, $\mathrm{Ind}_H^G V$ is naturally equivalent to $\mathrm{Hom}_H(k[G], V)$.

Remark 4.30. Notice that if we choose a representative x_σ from every left H-coset σ of G, then any $f \in \mathrm{Ind}_H^G V$ is uniquely determined by $\{f(x_\sigma)\}$.

Because of this,

$$dim(\mathrm{Ind}_H^G V) = dim V \cdot \frac{|G|}{|H|}.$$

Problem 4.31. *Check that if $K \subset H \subset G$ are groups and V a representation of K then $Ind_H^G Ind_K^H V$ is isomorphic to $Ind_K^G V$.*

4.7.3 The character of induced representation

Let us now compute the character χ of $\mathrm{Ind}_H^G V$.

Theorem 4.32. *(The Mackey formula) One has*

$$\chi(g) = \frac{1}{|H|} \sum_{x \in G, xgx^{-1} \in H} \chi_V(xgx^{-1})$$

Proof. For a left H-coset of G, σ let us define

$$V_\sigma = \{f \in \mathrm{Ind}_H^G V | f(g) = 0 \ \forall g \ \notin \ \sigma\}.$$

Then one has

$$\mathrm{Ind}_H^G V = \bigoplus_\sigma V_\sigma,$$

and so

$$\chi(g) = \sum_\sigma \chi_\sigma(g),$$

where $\chi_\sigma(g)$ is the trace of the diagonal block of $\rho(g)$ corresponding to V_σ.

Since $g(\sigma) = \sigma g$ is a left H-coset for any left H-coset σ, $\chi_\sigma(g) = 0$ if $\sigma \neq \sigma g$.

Now assume that $\sigma = \sigma g$. Choose $x_\sigma \in \sigma$. Then $x_\sigma g = h x_\sigma$ where $h = x_\sigma g x_\sigma^{-1} \in H$. Consider the vector space homomorphism $\alpha : V_\sigma \to V$ with $\alpha(f) = f(x_\sigma)$. Since $f \in V_\sigma$ is uniquely determined by $f(x_\sigma)$, α is an isomorphism. We have

$$\alpha(gf) = g(f)(x_\sigma) = f(x_\sigma g) = f(h x_\sigma) = \rho_V(h) f(x_\sigma) = h\alpha(f),$$

and $gf = \alpha^{-1} h \alpha(f)$. This means that $\chi_\sigma(g) = \chi_V(h)$. Therefore

$$\chi(g) = \sum_{\sigma \in H \backslash G, \sigma g = \sigma} \chi_V(x_\sigma g x_\sigma^{-1}).$$

Since it does not matter which representative x_σ of σ we choose, this expression can be simplified to the statement of the theorem.

□

4.7.4 Frobenius reciprocity

A very important result about induced representations is the Frobenius Reciprocity Theorem which connects the operations Ind and Res.

Theorem 4.33. (Frobenius Reciprocity)

Let $H < G$ be groups, V be a representation of G and W a representation of H. Then $\mathrm{Hom}_G(V, \mathrm{Ind}_H^G W)$ is naturally isomorphic to $\mathrm{Hom}_H(\mathrm{Res}_H^G V, W)$.

Proof. Let $E = \mathrm{Hom}_G(V, \mathrm{Ind}_H^G W)$ and $E' = \mathrm{Hom}_H(\mathrm{Res}_H^G V, W)$. Define $F : E \to E'$ and $F^{-1} : E' \to E$ as follows: $F(\alpha)v = (\alpha v)(e)$ for any $\alpha \in E$ and $(F^{-1}(\beta)v)(x) = \beta(xv)$ for any $\beta \in E'$.

In order to check that F and F^{-1} are well defined and inverse to each other, we need to check the following five statements.

Let $\alpha \in E$, $\beta \in E'$, $v \in V$, and $x, g \in G$.

(a) $F(\alpha)$ is an H-homomorphism, i.e. $F(\alpha)hv = hF(\alpha)v$.

Indeed, $F(\alpha)hv = (\alpha hv)(e) = (h\alpha v)(e) = (\alpha v)(he) = (\alpha v)(eh) = h \cdot (\alpha v)(e) = hF(\alpha)v$.

(b) $F^{-1}(\beta)v \in \mathrm{Ind}_H^G W$, i.e. $(F^{-1}(\beta)v)(hx) = h(F^{-1}(\beta)v)(x)$.

Indeed, $(F^{-1}(\beta)v)(hx) = \beta(hxv) = h\beta(xv) = h(F^{-1}(\beta)v)(x)$.

(c) $F^{-1}(\beta)$ is a G-homomorphism, i.e. $F^{-1}(\beta)gv = g(F^{-1}(\beta)v)$.

Indeed, $(F^{-1}(\beta)gv)(x) = \beta(xgv) = (F^{-1}(\beta)v)(xg) = (g(F^{-1}(\beta)v))(x)$.

(d) $F \circ F^{-1} = Id_{E'}$.

This holds since $F(F^{-1}(\beta))v = (F^{-1}(\beta)v)(e) = \beta(v)$.

(e) $F^{-1} \circ F = Id_E$, i.e. $(F^{-1}(F(\alpha))v)(x) = (\alpha v)(x)$.

Indeed, $(F^{-1}(F(\alpha))v)(x) = F(\alpha xv) = (\alpha xv)(e) = (x\alpha v)(e) = (\alpha v)(x)$, and we are done. □

4.7.5 Examples

Here are some examples of induced representations (we use the notation for representations from the character tables).

1. Let $G = S_3$, $H = \mathbb{Z}_2$. Using the Frobenius reciprocity, we obtain: $\mathrm{Ind}_H^G \mathbb{C}_+ = \mathbb{C}^2 \oplus \mathbb{C}_+$, $\mathrm{Ind}_H^G \mathbb{C}_- = \mathbb{C}^2 \oplus \mathbb{C}_-$.
2. Let $G = S_3$, $H = \mathbb{Z}_3$. Then we obtain $\mathrm{Ind}_H^G \mathbb{C}_+ = \mathbb{C}_+ \oplus \mathbb{C}_-$, $\mathrm{Ind}_H^G \mathbb{C}_\epsilon = \mathrm{Ind}_H^G \mathbb{C}_{\epsilon^2} = \mathbb{C}^2$.
3. Let $G = S_4$, $H = S_3$. Then $\mathrm{Ind}_H^G \mathbb{C}_+ = \mathbb{C}_+ \oplus \mathbb{C}_-^3$, $\mathrm{Ind}_H^G \mathbb{C}_- = \mathbb{C}_- \oplus \mathbb{C}_+^3$, $\mathrm{Ind}_H^G \mathbb{C}^2 = \mathbb{C}^2 \oplus \mathbb{C}_-^3 \oplus \mathbb{C}_+^3$.

Problem 4.34. *Compute the decomposition into irreducibles of the representations of A_5 induced from*

(a) $\mathbb{Z}_2$

(b) $\mathbb{Z}_3$

(c) $\mathbb{Z}_5$

(d) A_4

(e) $\mathbb{Z}_2 \times \mathbb{Z}_2$

4.8 Representations of S_n

In this subsection we give a description of the representations of the symmetric group S_n for any n.

Definition 4.35. A **partition** λ of n is a representation of n in the form $n = \lambda_1 + \lambda_2 + ... + \lambda_p$, where λ_i are positive integers, and $\lambda_i \geq \lambda_{i+1}$.

To such λ we will attach a **Young diagram** Y_λ, which is the union of rectangles $-i \leq y \leq -i+1$, $0 \leq x \leq \lambda_i$ in the coordinate plane, for $i = 1, ..., p$. Clearly, Y_λ is a collection of n unit squares. A **Young tableau** corresponding to Y_λ is the result of filling the numbers $1, ..., n$ into the squares of Y_λ in some way (without repetitions). For example, we will consider the Young tableau T_λ obtained by filling in the numbers in the increasing order, left to right, top to bottom.

We can define two subgroups of S_n corresponding to Y_λ:

1. Row subgroup P_λ: the subgroup which maps every element of $\{1, ..., n\}$ into an element standing in the same row in T_λ.

2. Column subgroup Q_λ: the subgroup which maps every element of $\{1, ..., n\}$ into an element standing in the same column in T_λ.

Clearly, $P_\lambda \cap Q_\lambda = \{1\}$.

Define the *Young projectors:*

$$a_\lambda := \frac{1}{|P_\lambda|} \sum_{g \in P_\lambda} g,$$

$$b_\lambda := \frac{1}{|Q_\lambda|} \sum_{g \in Q_\lambda} (-1)^g g,$$

where $(-1)^g$ denotes the sign of the permutation g. Set $c_\lambda = a_\lambda b_\lambda$. Since $P_\lambda \cap Q_\lambda = \{1\}$, this element is nonzero.

The irreducible representations of S_n are described by the following theorem.

Theorem 4.36. *The subspace $V_\lambda := \mathbb{C}[S_n]c_\lambda$ of $\mathbb{C}[S_n]$ is an irreducible representation of S_n under left multiplication. Every irreducible representation of S_n is isomorphic to V_λ for a unique λ.*

The modules V_λ are called the **Specht modules**.

The proof of this theorem is given in the next subsection.

Example 4.37.

For the partition $\lambda = (n)$, $P_\lambda = S_n$, $Q_\lambda = \{1\}$, so c_λ is the symmetrizer, and hence V_λ is the trivial representation.

For the partition $\lambda = (1, ..., 1)$, $Q_\lambda = S_n$, $P_\lambda = \{1\}$, so c_λ is the antisymmetrizer, and hence V_λ is the sign representation.

$n = 3$. For $\lambda = (2, 1)$, $V_\lambda = \mathbb{C}^2$.

$n = 4$. For $\lambda = (2, 2)$, $V_\lambda = \mathbb{C}^2$; for $\lambda = (3, 1)$, $V_\lambda = \mathbb{C}^3_-$; for $\lambda = (2, 1, 1)$, $V_\lambda = \mathbb{C}^3_+$.

Corollary 4.38. *All irreducible representations of S_n can be given by matrices with rational entries.*

Problem 4.39. *Find the sum of dimensions of all irreducible representations of the symmetric group S_n.*

Hint. *Show that all irreducible representations of S_n are real, i.e. admit a nondegenerate invariant symmetric form. Then use the Frobenius-Schur theorem.*

4.9 Proof of Theorem 4.36

Lemma 4.40. *Let $x \in \mathbb{C}[S_n]$. Then $a_\lambda x b_\lambda = \ell_\lambda(x) c_\lambda$, where ℓ_λ is a certain linear function.*

Proof. If $g \in P_\lambda Q_\lambda$, then g has a unique representation as pq, $p \in P_\lambda, q \in Q_\lambda$, so $a_\lambda g b_\lambda = (-1)^q c_\lambda$. Thus, to prove the required statement, we need to show that if $g \notin P_\lambda Q_\lambda$ then $a_\lambda g b_\lambda = 0$.

To show this, it is sufficient to find a transposition t such that $t \in P_\lambda$ and $g^{-1}tg \in Q_\lambda$; then

$$a_\lambda g b_\lambda = a_\lambda t g b_\lambda = a_\lambda g (g^{-1}tg) b_\lambda = -a_\lambda g b_\lambda,$$

so $a_\lambda g b_\lambda = 0$. In other words, we have to find two elements i, j standing in the same row in the tableau $T = T_\lambda$, and in the same column in the tableau $T' = gT$. Thus, it suffices to show that if such a pair does not exist, then $g \in P_\lambda Q_\lambda$, i.e. there exists $p \in P_\lambda$, $q' \in Q'_\lambda := gQ_\lambda g^{-1}$ such that $pT = q'T'$ (so that $g = pq, q = g^{-1}q'g \in Q_\lambda$).

Any two elements in the first row of T must be in different columns of T', so there exist $q'_1 \in Q'_\lambda$ which moves all these elements to the first row. So there is $p_1 \in P_\lambda$ such that p_1T and q'_1T' have the same first row. Now do the same procedure with the second row, finding elements p_2, q'_2 such that p_2p_1T and $q'_2q'_1T'$ have the same first two rows. Continuing so, we will construct the desired elements p, q'. The lemma is proved. □

Let us introduce the **lexicographic ordering** on partitions: $\lambda > \mu$ if the first nonvanishing $\lambda_i - \mu_i$ is positive.

Lemma 4.41. *If $\lambda > \mu$ then $a_\lambda \mathbb{C}[S_n] b_\mu = 0$.*

Proof. Similarly to the previous lemma, it suffices to show that for any $g \in S_n$ there exists a transposition $t \in P_\lambda$ such that $g^{-1}tg \in Q_\mu$. Let $T = T_\lambda$ and $T' = gT_\mu$. We claim that there are two integers which are in the same row of T and the same column of T'. Indeed, if $\lambda_1 > \mu_1$, this is clear by the pigeonhole principle (already for the first row). Otherwise, if $\lambda_1 = \mu_1$, like in the proof of the previous lemma, we can find elements $p_1 \in P_\lambda, q'_1 \in gQ_\mu g^{-1}$ such that p_1T and q'_1T' have the same first row, and repeat the argument for the second row, and so on. Eventually, having done $i-1$ such steps, we'll have $\lambda_i > \mu_i$, which means that some two elements of the i-th row of the first tableau are in the same column of the second tableau, completing the proof. □

Lemma 4.42. *c_λ is proportional to an idempotent. Namely, $c_\lambda^2 = \frac{n!}{\dim V_\lambda} c_\lambda$.*

Proof. Lemma 4.40 implies that c_λ^2 is proportional to c_λ. Also, it is easy to see that the trace of c_λ in the regular representation is $n!$ (as the coefficient of the identity element in c_λ is 1). This implies the statement. □

Lemma 4.43. *Let A be an algebra and e be an idempotent in A. Then for any left A-module M, one has $Hom_A(Ae, M) = eM$ (acting by right multiplication).*

Proof. Note that $1-e$ is also idempotents in A. Thus the statement immediately follows from the fact that $\mathrm{Hom}_A(A, M) = M$ and the decomposition $A = Ae \oplus A(1-e)$. □

Now we are ready to prove Theorem 4.36. Let $\lambda \geq \mu$. Then by Lemmas 4.42, 4.43

$$\mathrm{Hom}_{S_n}(V_\lambda, V_\mu) = \mathrm{Hom}_{S_n}(\mathbb{C}[S_n]c_\lambda, \mathbb{C}[S_n]c_\mu) = c_\lambda \mathbb{C}[S_n]c_\mu.$$

The latter space is zero for $\lambda > \mu$ by Lemma 4.41, and 1-dimensional if $\lambda = \mu$ by Lemmas 4.40 and 4.42. Therefore, V_λ are irreducible, and V_λ is not isomorphic to V_μ if $\lambda \neq \mu$. Since the number of partitions equals the number of conjugacy classes in S_n, the representations V_λ exhaust all the irreductible representations of S_n. The theorem is proved.

4.10 Induced representations for S_n

Denote by U_λ the representation $\mathrm{Ind}_{P_\lambda}^{S_n}\mathbb{C}$. It is easy to see that U_λ can be alternatively defined as $U_\lambda = \mathbb{C}[S_n]a_\lambda$.

Proposition 4.44. *$Hom(U_\lambda, V_\mu) = 0$ for $\mu < \lambda$, and $\dim Hom(U_\lambda, V_\lambda) = 1$. Thus, $U_\lambda = \oplus_{\mu \geq \lambda} K_{\mu\lambda} V_\mu$, where $K_{\mu\lambda}$ are nonnegative integers and $K_{\lambda\lambda} = 1$.*

Definition 4.45. The integers $K_{\mu\lambda}$ are called the **Kostka numbers**.

Proof. By Lemmas 4.42 and 4.43,

$$\mathrm{Hom}(U_\lambda, V_\mu) = \mathrm{Hom}(\mathbb{C}[S_n]a_\lambda, \mathbb{C}[S_n]a_\mu b_\mu) = a_\lambda \mathbb{C}[S_n]a_\mu b_\mu,$$

and the result follows from Lemmas 4.40 and 4.41. □

Now let us compute the character of U_λ. Let $C_{\mathbf{i}}$ be the conjugacy class in S_n having i_l cycles of length l for all $l \geq 1$. Also let $x_1, ..., x_N$ be variables, and let

$$H_m(x) = \sum_i x_i^m$$

be the power sum polynomials.

Theorem 4.46. *Let $N \geq p$ (where p is the number of parts of λ). Then $\chi_{U_\lambda}(C_{\mathbf{i}})$ is the coefficient*[5] *of $x^\lambda := \prod x_j^{\lambda_j}$ in the polynomial*

$$\prod_{m \geq 1} H_m(x)^{i_m}.$$

Proof. The proof is obtained easily from the Mackey formula. Namely, $\chi_{U_\lambda}(C_{\mathbf{i}})$ is the number of elements $x \in S_n$ such that $xgx^{-1} \in P_\lambda$ (for a representative $g \in C_{\mathbf{i}}$), divided by $|P_\lambda|$. Thus,

$$\chi_{U_\lambda}(C_{\mathbf{i}}) = \frac{n!}{|C_{\mathbf{i}}| \prod_j \lambda_j!} |C_{\mathbf{i}} \cap P_\lambda|.$$

Now, it is easy to see that

$$\frac{n!}{|C_{\mathbf{i}}|} = \prod_m m^{i_m} i_m!$$

[5] If $j > p$, we define λ_j to be zero.

(it is the order of the centralizer Z_g of g), so we get

$$\chi_{U_\lambda}(C_{\mathbf{i}}) = \frac{\prod_m m^{i_m} i_m!}{\prod_j \lambda_j!} |C_{\mathbf{i}} \cap P_\lambda|.$$

Now, since $P_\lambda = \prod_j S_{\lambda_j}$, we have

$$|C_{\mathbf{i}} \cap P_\lambda| = \sum_r \prod_{j \geq 1} \frac{\lambda_j!}{\prod_{m \geq 1} m^{r_{jm}} r_{jm}!},$$

where $r = (r_{jm})$ runs over all collections of nonnegative integers such that

$$\sum_m m r_{jm} = \lambda_j, \sum_j r_{jm} = i_m.$$

Thus we get

$$\chi_{U_\lambda}(C_{\mathbf{i}}) = \sum_r \prod_m \frac{i_m!}{\prod_j r_{jm}!}$$

But this is exactly the coefficient of $\prod x^\lambda$ in

$$\prod_{m \geq 1} (x_1^m + ... + x_N^m)^{i_m}$$

(r_{jm} is the number of times we take x_j^m). □

4.11 The Frobenius character formula

Let $\Delta(x) = \prod_{1 \leq i < j \leq N} (x_i - x_j)$. Let $\rho = (N-1, N-2, ..., 0) \in \mathbb{C}^N$. The following theorem, due to Frobenius, gives a character formula for the Specht modules V_λ.

Theorem 4.47. *Let $N \geq p$. Then $\chi_{V_\lambda}(C_{\mathbf{i}})$ is the coefficient of $x^{\lambda+\rho} := \prod x_j^{\lambda_j + N - j}$ in the polynomial*

$$\Delta(x) \prod_{m \geq 1} H_m(x)^{i_m}.$$

Proof. Denote χ_{V_λ} shortly by χ_λ. Let us denote the class function defined in the theorem by θ_λ. It follows from Theorem 4.46 that this function has the property $\theta_\lambda = \sum_{\mu \geq \lambda} L_{\mu\lambda} \chi_\mu$, where $L_{\mu\lambda}$ are integers and $L_{\lambda\lambda} = 1$. Therefore, to show that $\theta_\lambda = \chi_\lambda$, it suffices to show that $(\theta_\lambda, \theta_\lambda) = 1$ (see Lemma 4.73).

We have

$$(\theta_\lambda, \theta_\lambda) = \frac{1}{n!} \sum_{\mathbf{i}} |C_{\mathbf{i}}| \theta_\lambda(C_{\mathbf{i}})^2,$$

which is the coefficient of $x^{\lambda+\rho} y^{\lambda+\rho}$ in the series $R(x,y) = \Delta(x)\Delta(y)S(x,y)$, where

$$S(x,y) = \sum_{\mathbf{i}} \prod_m \frac{(\sum_{j,k} x_j^m y_k^m / m)^{i_m}}{i_m!}.$$

Summing over $\mathbf{i}$ and m, we get

$$S(x,y) = \prod_m \exp(\sum_{j,k} x_j^m y_k^m / m) = \exp(-\sum_{j,k} \log(1 - x_j y_k)) = \prod_{j,k} (1 - x_j y_k)^{-1}$$

Thus,

$$R(x,y) = \frac{\prod_{i<j}(x_i - x_j)(y_i - y_j)}{\prod_{i,j}(1 - x_i y_j)}.$$

Now we need the following lemma.

Lemma 4.48.

$$\frac{\prod_{i<j}(z_j - z_i)(y_i - y_j)}{\prod_{i,j}(z_i - y_j)} = \det(\frac{1}{z_i - y_j}).$$

Proof. Multiply both sides by $\prod_{i,j}(z_i - y_j)$. Then the right hand side must vanish on the hyperplanes $x_i = x_j$ and $y_i = y_j$ (i.e. be divisible by $\Delta(x)\Delta(y)$), and is a homogeneous polynomial of degree $N(N-1)$. This implies that the right hand side and the left hand side are proportional. The proportionality coefficient (which is equal to 1) is found by induction by multiplying both sides by $x_N - y_N$ and then setting $x_N = y_N$. □

Now setting in the lemma $z_i = 1/x_i$, we get

Corollary 4.49. *(Cauchy identity)*

$$R(x,y) = \det(\frac{1}{1 - x_i y_j}).$$

Corollary 4.49 easily implies that the coefficient of $x^{\lambda+\rho}y^{\lambda+\rho}$ is 1. Indeed, if $\sigma \neq 1$ is a permutation in S_N, the coefficient of this monomial in $\frac{1}{\prod(1 - x_j y_{\sigma(j)})}$ is obviously zero. □

4.12 Problems

In the following problems, we do not make a distinction between Young diagrams and partitions.

Problem 4.50. *For a Young diagram μ, let $A(\mu)$ be the set of Young diagrams obtained by adding a square to μ, and $R(\mu)$ be the set of Young diagrams obtained by removing a square from μ.*

(a) Show that $Ind_{S_{n-1}}^{S_n} V_\mu = \oplus_{\lambda \in A(\mu)} V_\lambda$.

(b) Show that $\mathrm{Res}_{S_{n-1}}^{S_n} V_\mu = \oplus_{\lambda \in R(\mu)} V_\lambda$.

Problem 4.51. *The content $c(\lambda)$ of a Young diagram λ is the sum $\sum_j \sum_{i=1}^{\lambda_j}(i - j)$. Let $C = \sum_{i<j}(ij) \in \mathbb{C}[S_n]$ be the sum of all transpositions. Show that C acts on the Specht module V_λ by multiplication by $c(\lambda)$.*

Problem 4.52. *Show that the element $(12) + ... + (1n)$ acts on V_λ by a scalar if and only if λ is a rectangular Young diagram, and compute this scalar.*

4.13 The hook length formula

Let us use the Frobenius character formula to compute the dimension of V_λ. According to the character formula, $\dim V_\lambda$ is the coefficient of $x^{\lambda+\rho}$ in $\Delta(x)(x_1 + ... + x_N)^n$. Let $l_j = \lambda_j + N - j$. Then, we get

$$\dim V_\lambda = \sum_{s \in S_N : l_j \geq N - s(j)} (-1)^s \frac{n!}{\prod_j (l_j - N + s(j))!} = \frac{n!}{l_1!...l_N!} \sum_{s \in S_n} (-1)^s \prod_j l_j(l_j - 1)...(l_j - N + s(j) + 1) =$$

$$\frac{n!}{\prod_j l_j!}\det(l_j(l_j-1)...(l_j-N+i+1)).$$

Using column reduction and the Vandermonde determinant formula, we see from this expression that

$$\dim V_\lambda = \frac{n!}{\prod_j l_j!}\det(l_j^{N-i}) = \frac{n!}{\prod_j l_j!}\prod_{1\le i<j\le N}(l_i-l_j) \qquad (3)$$

(where $N \geq p$).

In this formula, there are many cancellations. After making some of these cancellations, we obtain the hook length formula. Namely, for a square (i,j) in a Young diagram λ ($i,j\geq 1, i\leq \lambda_j$), define the hook of (i,j) to be the set of all squares (i',j') in λ with $i'\geq i$, $j'=j$ or $i'=i$, $j'\geq j$. Let $h(i,j)$ be the length of the hook of i,j, i.e. the number of squares in it.

Theorem 4.53. *(The hook length formula) One has*

$$\dim V_\lambda = \frac{n!}{\prod_{i\le\lambda_j} h(i,j)}.$$

Proof. The formula follows from formula (3). Namely, note that

$$\frac{l_1!}{\prod_{1<j\le N}(l_1-l_j)} = \prod_{1\le k\le l_1, k\ne l_1-l_j} k.$$

It is easy to see that the factors in this product are exactly the hooklengths $h(i,1)$. Now delete the first row of the diagram and proceed by induction. □

4.14 Schur-Weyl duality

We start with a simple result which is called the Double Centralizer Theorem.

Theorem 4.54. *Let A, B be two subalgebras of the algebra $\mathrm{End}\, E$ of endomorphisms of a finite dimensional vector space E, such that A is semisimple, and $B = \mathrm{End}_A E$. Then:*

(i) $A = \mathrm{End}_B E$ (i.e., the centralizer of the centralizer of A is A);

(ii) B is semisimple;

(iii) as a representation of $A\otimes B$, E decomposes as $E = \oplus_{i\in I} V_i\otimes W_i$, where V_i are all the irreducible representations of A, and W_i are all the irreducible representations of B. In particular, we have a natural bijection between irreducible representations of A and B.

Proof. Since A is semisimple, we have a natural decomposition $E = \oplus_{i\in I} V_i\otimes W_i$, where $W_i := \mathrm{Hom}_A(V_i, E)$, and $A = \oplus_i \mathrm{End}\, V_i$. Therefore, by Schur's lemma, $B = \mathrm{End}_A(E)$ is naturally identified with $\oplus_i \mathrm{End}(W_i)$. This implies all the statements of the theorem. □

We will now apply it to the following situation: $E = V^{\otimes n}$, where V is a finite dimensional vector space over a field of characteristic zero, and A is the image of $\mathbb{C}[S_n]$ in $\mathrm{End}\, E$. Let us now characterize the algebra B. Let $gl(V)$ be $\mathrm{End}\, V$ regarded as a Lie algebra with operation $ab-ba$.

Theorem 4.55. *The algebra $B = \mathrm{End}_A E$ is the image of the universal enveloping algebra $U(gl(V))$ under its natural action on E. In other words, B is generated by elements of the form*

$$\Delta_n(b) := b\otimes 1\otimes ...\otimes 1 + 1\otimes b\otimes ...\otimes 1 + ... + 1\otimes 1\otimes ...\otimes b,$$

$b\in gl(V)$.

Proof. Clearly, the image of $U(gl(V))$ is contained in B, so we just need to show that any element of B is contained in the image of $U(gl(V))$. By definition, $B = S^n \operatorname{End} V$, so the result follows from part (ii) of the following lemma.

Lemma 4.56. *Let k be a field of characteristic zero.*

(i) For any finite dimensional vector space U over k, the space S^nU is spanned by elements of the form $u \otimes ... \otimes u$, $u \in U$.

(ii) For any algebra A over k, the algebra S^nA is generated by elements $\Delta_n(a)$, $a \in A$.

Proof. (i) The space S^nU is an irreducible representation of $GL(U)$ (Problem 3.19). The subspace spanned by $u \otimes ... \otimes u$ is a nonzero subrepresentation, so it must be everything.

(ii) By the fundamental theorem on symmetric functions, there exists a polynomial P with rational coefficients such that $P(H_1(x), ..., H_n(x)) = x_1...x_n$ (where $x = (x_1, ..., x_n)$). Then

$$P(\Delta_n(a), \Delta_n(a^2), ..., \Delta_n(a^n)) = a \otimes ... \otimes a.$$

The rest follows from (i). □

□

Now, the algebra A is semisimple by Maschke's theorem, so the double centralizer theorem applies, and we get the following result, which goes under the name "Schur-Weyl duality".

Theorem 4.57. *(i) The image A of $\mathbb{C}[S_n]$ and the image B of $U(gl(V))$ in $\operatorname{End}(V^{\otimes n})$ are centralizers of each other.*

(ii) Both A and B are semisimple. In particular, $V^{\otimes n}$ is a semisimple $gl(V)$-module.

(iii) We have a decomposition of $A \otimes B$-modules $V^{\otimes n} = \oplus_\lambda V_\lambda \otimes L_\lambda$, where the summation is taken over partitions of n, V_λ are Specht modules for S_n, and L_λ are some distinct irreducible representations of $gl(V)$ (or zero).

4.15 Schur-Weyl duality for the $GL(V)$

The Schur-Weyl duality for the Lie algebra $gl(V)$ implies a similar statement for the group $GL(V)$.

Proposition 4.58. *The image of $GL(V)$ in $\operatorname{End}(V^{\otimes n})$ spans B.*

Proof. Denote the span of $g^{\otimes n}$, $g \in GL(V)$, by B'. Let $b \in \operatorname{End} V$ be any element.

We claim that B' contains $b^{\otimes n}$. Indeed, for all values of t but finitely many, $t \cdot \mathrm{Id} + b$ is invertible, so $(t \cdot \mathrm{Id} + b)^{\otimes n}$ belongs to B'. This implies that this is true for all t, in particular for $t = 0$, since $(t \cdot \mathrm{Id} + b)^{\otimes n}$ is a polynomial in t.

The rest follows from Lemma 4.56. □

Corollary 4.59. *As a representation of $S_n \times GL(V)$, $V^{\otimes n}$ decomposes as $\oplus_\lambda V_\lambda \otimes L_\lambda$, where $L_\lambda = Hom_{S_n}(V_\lambda, V^{\otimes n})$ are distinct irreducible representations of $GL(V)$ or zero.*

Example 4.60. If $\lambda = (n)$ then $V_\lambda = S^nV$, and if $\lambda = (1^n)$ (n copies of 1) then $V_\lambda = \wedge^n V$. It was shown in Problem 3.19 that these representations are indeed irreducible (except that $\wedge^n V$ is zero if $n > \dim V$).

4.16 Schur polynomials

Let $\lambda = (\lambda_1, ..., \lambda_p)$ be a partition of n, and $N \geq p$. Let

$$D_\lambda(x) = \sum_{s \in S_N} (-1)^s \prod_{j=1}^{N} x_{s(j)}^{\lambda_j + N - j} = \det(x_i^{\lambda_j + N - j}).$$

Define the polynomials

$$S_\lambda(x) := \frac{D_\lambda(x)}{D_0(x)}$$

(clearly $D_0(x)$ is just $\Delta(x)$). It is easy to see that these are indeed polynomials, as D_λ is antisymmetric and therefore must be divisible by Δ. The polynomials S_λ are called the **Schur polynomials**.

Proposition 4.61.

$$\prod_m (x_1^m + ... + x_N^m)^{i_m} = \sum_{\lambda: p \leq N} \chi_\lambda(C_{\mathbf{i}}) S_\lambda(x).$$

Proof. The identity follows from the Frobenius character formula and the antisymmetry of $\Delta(x) \prod_m (x_1^m + ... + x_N^m)^{i_m}$. □

Certain special values of Schur polynomials are of importance. Namely, we have

Proposition 4.62.

$$S_\lambda(1, z, z^2, ..., z^{N-1}) = \prod_{1 \leq i < j \leq N} \frac{z^{\lambda_i + N - i} - z^{\lambda_j + N - j}}{z^{N-i} - z^{N-j}}$$

Therefore,

$$S_\lambda(1, ..., 1) = \prod_{1 \leq i < j \leq N} \frac{\lambda_i - \lambda_j + j - i}{j - i}$$

Proof. The first identity is obtained from the definition using the Vandermonde determinant. The second identity follows from the first one by setting $z = 1$. □

4.17 The characters of L_λ

Proposition 4.61 allows us to calculate the characters of the representations L_λ.

Namely, let $\dim V = N$, $g \in GL(V)$, and $x_1, ..., x_N$ be the eigenvalues of g on V. To compute the character $\chi_{L_\lambda}(g)$, let us calculate $\mathrm{Tr}_{V^{\otimes n}}(g^{\otimes n} s)$, where $s \in S_n$. If $s \in C_{\mathbf{i}}$, we easily get that this trace equals

$$\prod_m \mathrm{Tr}(g^m)^{i_m} = \prod_m H_m(x)^{i_m}.$$

On the other hand, by the Schur-Weyl duality

$$\mathrm{Tr}_{V^{\otimes n}}(g^{\otimes n} s) = \sum_\lambda \chi_\lambda(C_{\mathbf{i}}) \mathrm{Tr}_{L_\lambda}(g).$$

Comparing this to Proposition 4.61 and using linear independence of columns of the character table of S_n, we obtain

Theorem 4.63. *(Weyl character formula) The representation L_λ is zero if and only if $N < p$, where p is the number of parts of λ. If $N \geq p$, the character of L_λ is the Schur polynomial $S_\lambda(x)$. Therefore, the dimension of L_λ is given by the formula*

$$\dim L_\lambda = \prod_{i<j} \frac{\lambda_i - \lambda_j + j - i}{j - i}$$

This shows that irreducible representations of $GL(V)$ which occur in $V^{\otimes n}$ for some n are labeled by Young diagrams with any number of squares but at most $N = \dim V$ rows.

Proposition 4.64. *The representation $L_{\lambda+1^N}$ (where $1^N = (1, 1, ..., 1) \in \mathbb{Z}^N$) is isomorphic to $L_\lambda \otimes \wedge^N V$.*

Proof. Indeed, $L_\lambda \otimes \wedge^N V \subset V^{\otimes n} \otimes \wedge^N V \subset V^{n+N}$, and the only component of $V^{\otimes n+N}$ that has the same character as $L_\lambda \otimes \wedge^N V$ is $L_{\lambda+1^N}$. This implies the statement. □

4.18 Polynomial representations of $GL(V)$

Definition 4.65. We say that a finite dimensional representation Y of $GL(V)$ is **polynomial** if its matrix elements are polynomial functions of the entries of g, g^{-1}, $g \in GL(V)$ (i.e. belong to $k[g_{ij}][1/\det g]$).

For example, $V^{\otimes n}$ and hence all L_λ are polynomial. Also define $L_{\lambda - r \cdot 1^N} := L_\lambda \otimes (\wedge^N V^*)^{\otimes r}$ (this definition makes sense by Proposition 4.64). This is also a polynomial representation. Thus we have attrached a unique irreducible polynomial representation L_λ of $GL(V) = GL_N$ to any sequence $(\lambda_1, ..., \lambda_N)$ of integers (not necessarily positive) such that $\lambda_1 \geq ... \geq \lambda_N$. This sequence is called the **highest weight** of L_λ.

Theorem 4.66. *Every finite dimensional polynomial representation of $GL(V)$ is completely reducible, and decomposes into summands of the form L_λ (which are pairwise non-isomorphic).*

Proof. Let Y be a polynomial representation of $GL(V)$. Denoting the ring of polynomial functions on $GL(V)$ by R, we get an embedding $\xi : Y \to Y \otimes R$ given by $(u, \xi(v))(g) := u(gv)$. It is easy to see that ξ is a homomorphism of representations (where the action of $GL(V)$ on the first component of $Y \otimes R$ is trivial). Thus, it suffices to prove the theorem for a subrepresentation $Y \subset R^m$. Now, every element of R is a polynomial of g_{ij} times a nonpositive power of $\det(g)$. Thus, R is a quotient of a direct sum of representations of the form $S^r(V \otimes V^*) \otimes (\wedge^N V^*)^{\otimes s}$. So we may assume that Y is contained in a quotient of a (finite) direct sum of such representations. As $V^* = \wedge^{N-1} V \otimes \wedge^N V^*$, Y is contained in a direct sum of representations of the form $V^{\otimes n} \otimes (\wedge^N V^*)^{\otimes s}$, and we are done. □

Remark 4.67. Since the scalars in $GL(V)$ and $gl(V)$ act by scalars in the representations L_λ, the above results extend in a straightforward manner to representations of the Lie algebra $sl(V)$ of traceless operators on V and the group $SL(V)$ of operators with determinant 1. The only difference is that in this case the representations L_λ and $L_{\lambda+1^m}$ are isomorphic, so the representations are parametrized by integer sequences $\lambda_1 \geq ... \geq \lambda_N$ up to a simultaneous shift by a constant.

On can show that any finite dimensional representation of $sl(V)$ is completely reducible, and any irreducible one is of the form L_λ. In particular, for $\dim V = 2$ one recovers the representation theory of $sl(2)$ studied in Problem 1.55.

4.19 Problems

Problem 4.68. *(a) Show that the S_n-representation $V'_\lambda := \mathbb{C}[S_n]b_\lambda a_\lambda$ is isomorphic to V_λ.*

Hint. Calculate $Hom_{S_n}(V_\mu, V'_\lambda)$.

(b) Let $\phi : \mathbb{C}[S_n] \to \mathbb{C}[S_n]$ be the automorphism sending s to $(-1)^s s$ for any permutation s. Show that ϕ maps any representation V of S_n to $V \otimes \mathbb{C}_-$. Show also that $\phi(\mathbb{C}[S_n]a) = \mathbb{C}[S_n]\phi(a)$, for $a \in \mathbb{C}[S_n]$. Use (a) to deduce that $V_\lambda \otimes \mathbb{C}_- = V_{\lambda^*}$, where λ^* is the conjugate partition to λ, obtained by reflecting the Young diagram of λ.

Problem 4.69. *Let $R_{k,N}$ be the algebra of polynomials on the space of k-tuples of complex N by N matrices $X_1, ..., X_k$ invariant under simultaneous conjugation. An example of an element of $R_{k,N}$ is the function $T_w := \mathrm{Tr}(w(X_1, ..., X_k))$, where w is any finite word on a k-letter alphabet. Show that $R_{k,N}$ is generated by the elements T_w.*

Hint. Use Schur-Weyl duality.

4.20 Representations of $GL_2(\mathbb{F}_q)$

4.20.1 Conjugacy classes in $GL_2(\mathbb{F}_q)$

Let $\mathbb{F}_q$ be a finite field of size q of characteristic other than 2. Then

$$|GL_2(\mathbb{F}_q)| = (q^2 - 1)(q^2 - q),$$

since the first column of an invertible 2 by 2 matrix must be non-zero and the second column may not be a multiple of the first one. Factoring,

$$|GL_2(\mathbb{F}_q)| = q(q + 1)(q - 1)^2.$$

The goal of this section is to describe the irreducible representations of $GL_2(\mathbb{F}_q)$.
To begin, let us find the conjugacy classes in $GL_2(\mathbb{F}_q)$.

Representatives	Number of elements in a conjugacy class	Number of classes
Scalar $\begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}$	1 (this is a central element)	$q-1$ (one for every non-zero x)
Parabolic $\begin{pmatrix} x & 1 \\ 0 & x \end{pmatrix}$	$q^2 - 1$ (elements that commute with this one are of the form $\begin{pmatrix} t & u \\ 0 & t \end{pmatrix}$, $t \neq 0$)	$q-1$ (one for every non-zero x)
Hyperbolic $\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}$, $y \neq x$	$q^2 + q$ (elements that commute with this one are of the form $\begin{pmatrix} t & 0 \\ 0 & u \end{pmatrix}$, $t, u \neq 0$)	$\frac{1}{2}(q-1)(q-2)$ ($x, y \neq 0$ and $x \neq y$)
Elliptic $\begin{pmatrix} x & \epsilon y \\ y & x \end{pmatrix}$, $x \in \mathbb{F}_q$, $y \in \mathbb{F}_q^\times$, $\epsilon \in \mathbb{F}_q \setminus \mathbb{F}_q^2$ (characteristic polynomial over $\mathbb{F}_q$ is irreducible)	$q^2 - q$ (the reason will be described below)	$\frac{1}{2}q(q-1)$ (matrices with y and $-y$ are conjugate)

More on the conjugacy class of elliptic matrices: these are the matrices whose characteristic polynomial is irreducible over $\mathbb{F}_q$ and which therefore don't have eigenvalues in $\mathbb{F}_q$. Let A be such a matrix, and consider a quadratic extension of $\mathbb{F}_q$,

$$\mathbb{F}_q(\sqrt{\epsilon}), \epsilon \in \mathbb{F}_q \setminus \mathbb{F}_q^2.$$

Over this field, A will have eigenvalues

$$\alpha = \alpha_1 + \sqrt{\epsilon}\alpha_2$$

and

$$\overline{\alpha} = \alpha_1 - \sqrt{\epsilon}\alpha_2,$$

with corresponding eigenvectors

$$v, \ \overline{v} \quad (Av = \alpha v, \ A\overline{v} = \overline{\alpha v}).$$

Choose a basis

$$\{e_1 = v + \overline{v}, \ e_2 = \sqrt{\epsilon}(v - \overline{v})\}.$$

In this basis, the matrix A will have the form

$$\begin{pmatrix} \alpha_1 & \epsilon\alpha_2 \\ \alpha_2 & \alpha_1 \end{pmatrix},$$

justifying the description of representative elements of this conjugacy class.
In the basis $\{v, \overline{v}\}$, matrices that commute with A will have the form

$$\begin{pmatrix} \lambda & 0 \\ 0 & \overline{\lambda} \end{pmatrix},$$

for all

$$\lambda \in \mathbb{F}_{q^2}^{\times},$$

so the number of such matrices is $q^2 - 1$.

4.20.2 Representations of $GL_2(\mathbb{F}_q)$

In this section, G will denote the group $GL_2(\mathbb{F}_q)$.

4.20.3 1-dimensional representations

First, we describe the 1-dimensional representations of G.

Proposition 4.70. $[G, G] = SL_2(\mathbb{F}_q)$.

Proof. Clearly,

$$\det(xyx^{-1}y^{-1}) = 1,$$

so

$$[G, G] \subseteq SL_2(\mathbb{F}_q).$$

To show the converse, it suffices to show that the matrices

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

are commutators (as such matrices generate $SL_2(\mathbb{F}_q)$.) Clearly, by using transposition, it suffices to show that only the first two matrices are commutators. But it is easy to see that the matrix

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

is the commutator of the matrices

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \ B = \begin{pmatrix} 1 & 1/2 \\ 0 & 1 \end{pmatrix},$$

while the matrix

$$\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$$

is the commutator of the matrices

$$A = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}, \ B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

This completes the proof. □

Therefore,

$$G/[G,G] \cong \mathbb{F}_q^\times \text{ via } g \to \det(g).$$

The one-dimensional representations of G thus have the form

$$\rho(g) = \xi\big(\det(g)\big),$$

where ξ is a homomorphism

$$\xi : \mathbb{F}_q^\times \to \mathbb{C}^\times;$$

so there are $q-1$ such representations, denoted $\mathbb{C}_\xi$.

4.20.4 Principal series representations

Let

$$B \subset G, \ B = \{\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}\}$$

(the set of upper triangular matrices); then

$$|B| = (q-1)^2 q,$$

$$[B,B] = U = \{\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}\},$$

and

$$B/[B,B] \cong \mathbb{F}_q^\times \times \mathbb{F}_q^\times$$

(the isomorphism maps an element of $B/[B,B]$ to its two eigenvalues).
Let

$$\lambda : B \to \mathbb{C}^\times$$

be a homomorphism defined by

$$\lambda \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \lambda_1(a)\lambda_2(c), \text{for some pair of homomorphisms } \lambda_1, \lambda_2 : \mathbb{F}_q^\times \to \mathbb{C}^\times.$$

Define

$$V_{\lambda_1,\lambda_2} = Ind_B^G \mathbb{C}_\lambda,$$

where $\mathbb{C}_\lambda$ is the 1-dimensional representation of B in which B acts by λ. We have

$$\dim(V_{\lambda_1,\lambda_2}) = \frac{|G|}{|B|} = q + 1.$$

Theorem 4.71. *1. $\lambda_1 \neq \lambda_2 \Rightarrow V_{\lambda_1,\lambda_2}$ is irreducible.*

2. $\lambda_1 = \lambda_2 = \mu \Rightarrow V_{\lambda_1,\lambda_2} = \mathbb{C}_\mu \oplus W_\mu$, where W_μ is a q-dimensional irreducible representation of G.

3. $W_\mu \cong W_\nu$ iff $\mu = \nu$; $V_{\lambda_1,\lambda_2} \cong V_{\lambda_1',\lambda_2'}$ iff $\{\lambda_1, \lambda_2\} = \{\lambda_1', \lambda_2'\}$ (in the second case, $\lambda_1 \neq \lambda_2, \lambda_1' \neq \lambda_2'$).

Proof. From the Mackey formula, we have

$$tr_{V_{\lambda_1,\lambda_2}}(g) = \frac{1}{|B|} \sum_{a \in G,\, aga^{-1} \in B} \lambda(aga^{-1}).$$

If

$$g = \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix},$$

the expression on the right evaluates to

$$\lambda_1\lambda_2(x)\frac{|G|}{|B|} = \lambda_1(x)\lambda_2(x)\big(q+1\big).$$

If

$$g = \begin{pmatrix} x & 1 \\ 0 & x \end{pmatrix},$$

the expression evaluates to

$$\lambda_1\lambda_2(x) \cdot 1,$$

since here

$$aga^{-1} \in B \Rightarrow a \in B.$$

If

$$g = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix},$$

the expression evaluates to

$$\big(\lambda_1(x)\lambda_2(y) + \lambda_1(y)\lambda_2(x)\big) \cdot 1,$$

since here

$$aga^{-1} \in B \Rightarrow a \in B \text{ or } a \text{ is an element of } B \text{ multiplied by the transposition matrix.}$$

If

$$g = \begin{pmatrix} x & \varepsilon y \\ y & x \end{pmatrix},$$

the expression on the right evaluates to 0 because matrices of this type don't have eigenvalues over $\mathbb{F}_q$.
From the definition, $\lambda_i(x)(i = 1, 2)$ is a root of unity, so

$$|G|\langle \chi_{V_{\lambda_1,\lambda_2}}, \chi_{V_{\lambda_1,\lambda_2}}\rangle = (q+1)^2(q-1) + (q^2-1)(q-1) + 2(q^2+q)\frac{(q-1)(q-2)}{2} + (q^2+q)\sum_{x\neq y}\lambda_1(x)\lambda_2(y)\overline{\lambda_1(y)\lambda_2(x)}.$$

The last two summands come from the expansion

$$|a+b|^2 = |a|^2 + |b|^2 + a\overline{b} + \overline{a}b.$$

If

$$\lambda_1 = \lambda_2 = \mu,$$

the last term is equal to

$$(q^2+q)(q-2)(q-1),$$

and the total in this case is

$$(q+1)(q-1)[(q+1)+(q-1)-2q(q-2)] = (q+1)(q-1)2q(q-1) = 2|G|,$$

so

$$\langle \chi_{V_{\lambda_1,\lambda_2}}, \chi_{V_{\lambda_1,\lambda_2}}\rangle = 2.$$

Clearly,

$$\mathbb{C}_\mu \subseteq \mathrm{Ind}_B^G \mathbb{C}_{\mu,\mu},$$

since

$$\mathrm{Hom}_G(\mathbb{C}_\mu, \mathrm{Ind}_B^G\mathbb{C}_{\mu,\mu}) = \mathrm{Hom}_B(\mathbb{C}_\mu, \mathbb{C}_\mu) = \mathbb{C} \text{ (Theorem 4.33)}.$$

$\therefore \mathrm{Ind}_B^G\mathbb{C}_{\mu,\mu} = \mathbb{C}_\mu \oplus W_\mu$; W_μ is irreducible; and the character of W_μ is different for distinct values of μ, proving that W_μ are distinct.

If $\lambda_1 \neq \lambda_2$, let $z = xy^{-1}$, then the last term of the summation is

$$(q^2+q)\sum_{x\neq y}\lambda_1(z)\overline{\lambda_2(z)} = (q^2+q)\sum_{x;z\neq 1}\frac{\lambda_1}{\lambda_2}(z) = (q^2+q)(q-1)\sum_{z\neq 1}\frac{\lambda_1}{\lambda_2}(z).$$

Since

$$\sum_{z\in\mathbb{F}_q^\times}\frac{\lambda_1}{\lambda_2}(z) = 0,$$

because the sum of all roots of unity of a given order $m > 1$ is zero, the last term becomes

$$-(q^2+q)(q-1)\sum_{z\neq 1}\frac{\lambda_1}{\lambda_2}(1) = -(q^2+q)(q-1).$$

The difference between this case and the case of $\lambda_1 = \lambda_2$ is equal to

$$-(q^2+q)[(q-2)(q-1)+(q-1)] = |G|,$$

so this is an irreducible representation.

To prove the third assertion of the theorem, we look at the characters on hyperbolic elements and note that the function

$$\lambda_1(x)\lambda_2(y) + \lambda_1(y)\lambda_2(x)$$

determines λ_1, λ_2 up to permutation.

□

4.20.5 Complimentary series representations

Let $\mathbb{F}_{q^2} \supset \mathbb{F}_q$ be a quadratic extension $\mathbb{F}_q(\sqrt{\varepsilon}), \varepsilon \in \mathbb{F}_q \setminus \mathbb{F}_q^2$. We regard this as a 2-dimensional vector space over $\mathbb{F}_q$; then $GL_2(\mathbb{F}_q)$ is the group of linear transformations of $\mathbb{F}_{q^2}$ over $\mathbb{F}_q$. Let $K \subset GL_2(\mathbb{F}_q)$ be the cyclic group of multiplications by elements of $\mathbb{F}_q^\times$,

$$K = \{\begin{pmatrix} x & \varepsilon y \\ y & x \end{pmatrix}\},\ |K| = q^2 - 1.$$

For $\nu : K \to \mathbb{C}^\times$ a homomorphism, let

$$Y_\nu = \mathrm{Ind}_K^G \nu.$$

This representation, of course, is very reducible. Let us compute its character, using the Mackey formula. We get

$$\chi \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix} = q(q-1)\nu(x);$$

$$\chi(A) = 0 \text{ for } A \text{ parabolic or hyperbolic;}$$

$$\chi \begin{pmatrix} x & \varepsilon y \\ y & x \end{pmatrix} = \nu \begin{pmatrix} x & \varepsilon y \\ y & x \end{pmatrix} + \nu \begin{pmatrix} x & \varepsilon y \\ y & x \end{pmatrix}^q.$$

The last assertion is because if we regard the matrix as an element of $\mathbb{F}_{q^2}$, conjugation is an automorphism of $\mathbb{F}_{q^2}$ over $\mathbb{F}_q$, but the only nontrivial automorphism of $\mathbb{F}_{q^2}$ over $\mathbb{F}_q$ is the q^{th} power map.

We thus have

$$\mathrm{Ind}_K^G \nu^q = \mathrm{Ind}_K^G \nu$$

because they have the same character. Therefore, for $\nu^q \neq \nu$ we get $\frac{1}{2}q(q-1)$ representations.

Next, we look at the following tensor product:

$$W_\varepsilon \otimes V_{\alpha,\varepsilon},$$

where ε is the trivial character and W_ε is defined as in the previous section. The character of this representation is

$$\chi \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix} = q(q+1)\alpha(x);$$

$$\chi(A) = 0 \text{ for } A \text{ parabolic or elliptic;}$$

$$\chi \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} = \alpha(x) + \alpha(y).$$

Thus the "virtual representation"

$$W_\varepsilon \otimes V_{\alpha,\varepsilon}, - V_{\alpha,\varepsilon} - \mathrm{Ind}_K^G \nu$$

where α is the restriction of ν to scalars has character

$$\chi \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix} = (q-1)\alpha(x);$$

$$\chi \begin{pmatrix} x & 1 \\ 0 & x \end{pmatrix} = -\alpha(x);$$

$$\chi \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} = 0;$$

$$\chi \begin{pmatrix} x & \epsilon y \\ y & x \end{pmatrix} = -\nu \begin{pmatrix} x & \epsilon y \\ y & x \end{pmatrix} - \nu^q \begin{pmatrix} x & \epsilon y \\ y & x \end{pmatrix}.$$

In all that follows, we will have $\nu^q \neq \nu$.

The following two lemmas will establish that the inner product of this character with itself is equal to 1, that its value at 1 is positive, and that the above conditions imply that it is the character of an irreducible representation of G.

Lemma 4.72. *Let χ be the character of the "virtual representation" defined above. Then*

$$\langle \chi, \chi \rangle = 1$$

and

$$\chi(1) > 0.$$

Proof.

$$\chi(1) = q(q+1) - (q+1) - q(q-1) = q - 1 > 0.$$

We now compute the inner product $\langle \chi, \chi \rangle$. Since α is a root of unity, this will be equal to

$$\frac{1}{(q-1)^2 q(q+1)} \left[(q-1)\cdot(q-1)^2\cdot 1 + (q-1)\cdot 1\cdot(q^2-1) + \frac{q(q-1)}{2}\cdot \sum_{\zeta \text{ elliptic}} (\nu(\zeta)+\nu^q(\zeta))\overline{(\nu(\zeta)+\nu^q(\zeta))}\right]$$

Because ν is also a root of unity, the last term of the expression evaluates to

$$2 + \sum_{\zeta \text{ elliptic}} \nu^{q-1}(\zeta) + \nu^{1-q}(\zeta).$$

Let's evaluate the last summand.

Since $\mathbb{F}_{q^2}^\times$ is cyclic and $\nu^q \neq \nu$,

$$\sum_{\zeta \in \mathbb{F}_{q^2}^\times} \nu^{q-1}(\zeta) = \sum_{\zeta \in \mathbb{F}_{q^2}^\times} \nu^{1-q}(\zeta) = 0.$$

Therefore,

$$\sum_{\zeta \text{ elliptic}} \nu^{q-1}(\zeta) + \nu^{1-q}(\zeta) = 0 - \sum_{\zeta \in \mathbb{F}_q^\times} \nu^{q-1}(\zeta) + \nu^{1-q}(\zeta) = 0 - 2(q-1) = -2(q-1)$$

since $\mathbb{F}_q^\times$ is cyclic of order $q-1$. Therefore,

$$\langle \chi, \chi \rangle = \frac{1}{(q-1)^2 q(q+1)} \left((q-1)\cdot(q-1)^2\cdot 1 + (q-1)\cdot 1\cdot(q^2-1) + \frac{q(q-1)}{2}\cdot(2(q^2-q) - 2(q-1))\right) = 1.$$

$\square$

Lemma 4.73. *Let*

$$V_1, V_2, \dots, V_m$$

be (possibly reducible) representations of a finite group G over $\mathbb{C}$, and let

$$\chi_1, \chi_2, \dots, \chi_m$$

be the respective characters. Let

$$p_1, p_2, \dots p_m \in \mathbb{Z}$$

(not necessarily nonnegative!), and let

$$\chi = p_1\chi_1 + p_2 + \cdots + p_m\chi_m.$$

If $\langle \chi, \chi \rangle = 1$ *and* $\chi(1) > 0$, *then* χ *is a character of an irreducible representation of* G.

Proof. Let

$$W_1, W_2, \dots, W_s$$

be the irreducible representations of G, and let

$$V_i = \bigoplus_j a_{ij} W_j \ (a_{ij} \in \mathbb{Z}^+ \cup \{0\}).$$

If ξ_j are characters of W_j, then

$$\chi_i = \sum_j a_{ij}\xi_j \Rightarrow \chi = \sum_{i,j} p_i a_{ij} \xi_j = \sum q_j \xi_j,$$

where

$$q_j = \sum_i p_i a_{ij} \in \mathbb{Z}.$$

Since

$$\langle \chi, \chi \rangle = \sum_j q_j^2 = 1,$$

we must have $q_j = 0$ except for one value of $j = j_0$, with $q_{j_0} = \pm 1$. Thus,

$$\chi = \pm \xi_{j_0},$$

and since $\chi(1) > 0$ we have

$$\chi = \xi_{j_0}.$$

□

We have now shown that for any ν with $\nu^q \neq \nu$ the representation Y_ν with the same character as

$$W_\varepsilon \otimes V_{\alpha,\varepsilon,} - V_{\alpha,\varepsilon} - \mathrm{Ind}_K^G \nu$$

exists and is irreducible. These characters are distinct for distinct pairs $(\alpha,\ \nu)$ (up to switch $\nu \to \nu^q$), so there are $\frac{q(q-1)}{2}$ such representations, each of dimension $q - 1$.

We have thus found $q - 1$ 1-dimensional representations of G, $\frac{q(q-1)}{2}$ principal series representations, and $\frac{q(q-1)}{2}$ complimentary series representations, for a total of $q^2 - 1$ representations, i.e. the number of conjugacy classes in G. We can also check the sum of squares formula:

$$(q-1)\cdot 1^2 + (q-1)\cdot q^2 + \frac{(q-1)(q-2)}{2}\cdot(q+1)^2 + \frac{q(q-1)}{2}\cdot(q-1)^2 = (q-1)^2 q(q+1) = |G|.$$

This implies that we have in fact found all irreducible representations of $GL_2(\mathbb{F}_q)$.

4.21 Artin's theorem

Theorem 4.74. *Let X be a conjugation-invariant system of subgroups of a finite group G. Then two conditions are equivalent:*

(i) Any element of G belongs to a subgroup $H \in X$.

(ii) The character of any irreducible representation of G belongs to the $\mathbb{Q}$-span of characters of induced representations $\mathrm{Ind}_H^G V$, where $H \in X$ and V is an irreducible representation of H.

Proof. Proof that (ii) implies (i). Assume that $g \in G$ does not belong to any of the subgroups $H \in X$. Then, since X is conjugation invariant, it cannot be conjugated into such a subgroup. Hence by the Mackey formula, $\chi_{\mathrm{Ind}_H^G(V)}(g) = 0$ for all $H \in X$ and V. So by (ii), for any irreducible representation W of G, $\chi_W(g) = 0$. But irreducible characters span the space of class functions, so any class function vanishes on g, which is a contradiction.

Proof that (i) implies (ii). Let U be a virtual representation of G (i.e. a linear combination of irreducible representations with nonzero integer coefficients) such that $(\chi_U, \chi_{\mathrm{Ind}_H^G V}) = 0$ for all H, V. So by Frobenius reciprocity, $(\chi_{U|_H}, \chi_V) = 0$. This means that χ_U vanishes on H for any $H \in X$. Hence by (i), χ_U is identically zero. This implies (ii). □

Corollary 4.75. *Any irreducible character of a finite group is a rational linear combination of induced characters from its cyclic subgroups.*

4.22 Representations of semidirect products

Let G, A be finite groups and $\phi : G \to \mathrm{Aut}(A)$ be a homomorphism. For $a \in A$, denote $\phi(g)a$ by $g(a)$. The semidirect product $G \ltimes A$ is defined to be the product $A \times G$ with multiplication law

$$(a_1, g_1)(a_2, g_2) = (a_1 g_1(a_2), g_1 g_2).$$

Clearly, G and A are subgroups of $G \ltimes A$ in a natural way.

We would like to study irreducible complex representations of $G \ltimes A$. For simplicity, let us do it when A is abelian.

In this case, irreducible representations of A are 1-dimensional and form the character group $A^\vee$, which carries an action of G. Let O be an orbit of this action, $x \in O$ a chosen element, and G_x the stabilizer of x in G. Let U be an irreducible representation of G_x. Then we define a representation $V_{(O,U)}$ of $G \ltimes A$ as follows.

As a representation of G, we set

$$V_{(O,x,U)} = \mathrm{Ind}_{G_x}^G U = \{f : G \to U | f(hg) = hf(g), h \in G_x\}.$$

Next, we introduce an additional action of A on this space by $(a \circ f)(g) = (x, g(a))f(g)$. Then it's easy to check that these two actions combine into an action of $G \ltimes A$. Also, it is clear that this representation does not really depend on the choice of x, in the following sense. Let $x, y \in O$, and $g \in G$ be such that $gxg^{-1} = y$, and let $g(U)$ be the representation of G_y obtained from the representation U of G_x by the action of g. Then $V_{(O,x,U)}$ is (naturally) isomorphic to $V_{(O,y,U')}$. Thus we will denote $V_{(O,x,U)}$ by $V_{(O,U)}$.

Theorem 4.76. *(i) The representations $V_{(O,U)}$ is irreducible.*

(ii) They are pairwise nonisomorphic.

(iii) They form a complete set of irreducible representations of $G \ltimes A$.

(iv) The character of $V = V_{(U,O)}$ *is given by the Mackey-type formula*

$$\chi_V(a,g) = \frac{1}{|G_x|} \sum_{h \in G: hgh^{-1} \in G_X} x(h(a)) \chi_U(hgh^{-1}).$$

Proof. (i) Let us decompose $V = V_{(O,U)}$ as an A-module. Then we get

$$V = \oplus_{y \in O} V_y,$$

where $V_y = \{v \in V_{(O,U)} | av = (y,a)v, a \in A\}$. So if $W \subset V$ is a subrepresentation, then $W = \oplus_{y \in O} W_y$, where $W_y \subset V_y$. Now, V_y is a representation of G_y, which goes to U under any isomorphism $G_y \to G_x$ determined by $g \in G$ conjugating x to y. Hence, V_y is irreducible over G_y, so $W_y = 0$ or $W_y = V_y$ for each y. Also, if $hyh^{-1} = z$ then $hW_y = W_z$, so either $W_y = 0$ for all y or $W_y = V_y$ for all y, as desired.

(ii) The orbit O is determined by the A-module structure of V, and the representation U by the structure of V_x as a G_x-module.

(iii) We have

$$\sum_{U,O} \dim V_{(U,O)}^2 = \sum_{U,O} |O|^2 (\dim U)^2 =$$

$$\sum_O |O|^2 |G_x| = \sum_O |O||G/G_x||G_x| = |G| \sum_O |O| = |G||A^\vee| = |G \ltimes A|.$$

□

(iv) The proof is essentially the same as that of the Mackey formula.

5 Quiver Representations

5.1 Problems

Problem 5.1. Field embeddings. *Recall that* $k(y_1, ..., y_m)$ *denotes the field of rational functions of* $y_1, ..., y_m$ *over a field* k. *Let* $f : k[x_1, ..., x_n] \to k(y_1, ..., y_m)$ *be an injective homomorphism. Show that* $m \geq n$. *(Look at the growth of dimensions of the spaces* W_N *of polynomials of degree* N *in* x_i *and their images under* f *as* $N \to \infty$*). Deduce that if* $f : k(x_1, ..., x_n) \to k(y_1, ..., y_m)$ *is a field embedding, then* $m \geq n$.

Problem 5.2. Some algebraic geometry.

Let k *be an algebraically closed field, and* $G = GL_n(k)$. *Let* V *be an polynomial representation of* G. *Show that if* G *has finitely many orbits on* V *then* $\dim(V) \leq n^2$. *Namely:*

(a) Let $x_1, ..., x_N$ *be linear coordinates on* V. *Let us say that a subset* X *of* V *is Zariski dense if any polynomial* $f(x_1, ..., x_N)$ *which vanishes on* X *is zero (coefficientwise). Show that if* G *has finitely many orbits on* V *then* G *has at least one dense orbit on* V.

(b) Use (a) to construct a field embedding $k(x_1, ..., x_N) \to k(g_{pq})$, *then use Problem 5.1.*

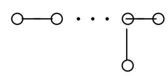
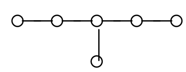
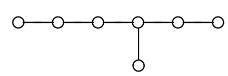
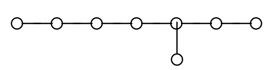
(c) generalize the result of this problem to the case when $G = GL_{n_1}(k) \times ... \times GL_{n_m}(k)$.

Problem 5.3. Dynkin diagrams.

Let Γ be a graph, i.e. a finite set of points (vertices) connected with a certain number of edges (we allow multiple edges). We assume that Γ is connected (any vertex can be connected to any other by a path of edges) and has no self-loops (edges from a vertex to itself). Suppose the vertices of Γ are labeled by integers $1, ..., N$. Then one can assign to Γ an $N \times N$ matrix $R_\Gamma = (r_{ij})$, where r_{ij} is the number of edges connecting vertices i and j. This matrix is obviously symmetric, and is called the adjacency matrix. Define the matrix $A_\Gamma = 2I - R_\Gamma$, where I is the identity matrix.

Main definition: Γ *is said to be a Dynkin diagram if the quadratic from on $\mathbb{R}^N$ with matrix A_Γ is positive definite. Dynkin diagrams appear in many areas of mathematics (singularity theory, Lie algebras, representation theory, algebraic geometry, mathematical physics, etc.) In this problem you will get a complete classification of Dynkin diagrams. Namely, you will prove*

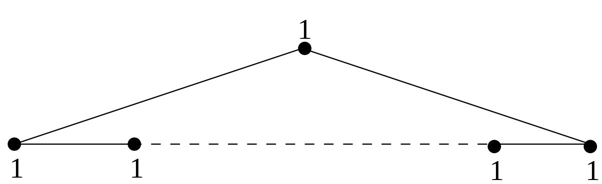
Theorem. Γ *is a Dynkin diagram if and only if it is one on the following graphs:*

- A_n :
- D_n: 
- E_6 : 
- E_7 : 
- E_8 : 

(a) Compute the determinant of A_Γ where $\Gamma = A_N, D_N$. (Use the row decomposition rule, and write down a recusive equation for it). Deduce by Sylvester criterion that A_N, D_N are Dynkin diagrams[6]

(b) Compute the determinants of A_Γ for E_6, E_7, E_8 (use row decomposition and reduce to (a)). Show they are Dynkin diagrams.

(c) Show that if Γ is a Dynkin diagram, it cannot have cycles. For this, show that $det(A_\Gamma) = 0$ for a graph Γ below [7]

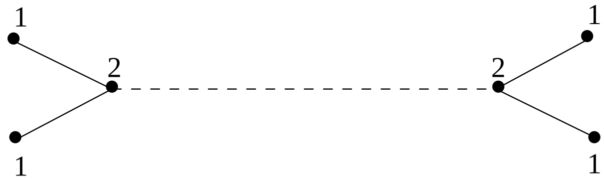


[6] The Sylvester criterion says that a symmetric bilinear form $(,)$ on $\mathbb{R}^N$ is positive definite iff for any $k \le N$, $\det_{1 \le i,j \le k}(e_i, e_j) > 0$.

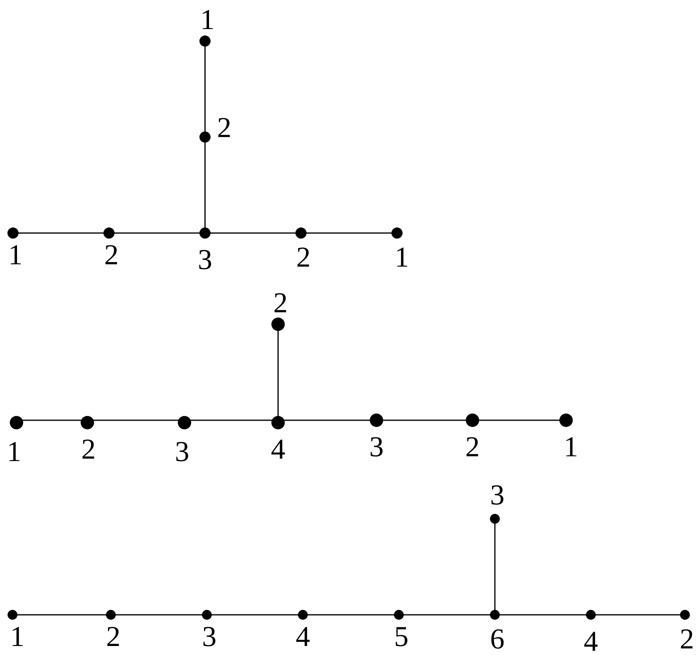
[7] Please ignore the numerical labels.

(show that the sum of rows is 0). Thus Γ has to be a tree.

(d) Show that if Γ is a Dynkin diagram, it cannot have vertices with 4 or more incoming edges, and that Γ can have no more than one vertex with 3 incoming edges. For this, show that $det(A_\Gamma) = 0$ for a graph Γ below:



(e) Show that $det(A_\Gamma) = 0$ for a graphs Γ below:



(f) Deduce from (a)-(e) the classification theorem for Dynkin diagrams.

(g) A (simply laced) affine Dynkin diagram is a connected graph without self-loops such that the quadratic form defined by A_Γ is positive semidefinite. Classify affine Dynkin diagrams. (Show that they are exactly the forbidden diagrams from (c)-(e)).

Problem 5.4. *Let Q be a quiver with set of vertices D. We say that Q is of finite type if it has finitely many indecomposable representations. Let b_{ij} be the number of edges from i to j in Q ($i, j \in D$).*

There is the following remarkable theorem, proved by P. Gabriel in the 1970-s.

Theorem. *A connected quiver Q is of finite type if and only if the corresponding unoriented graph (i.e. with directions of arrows forgotten) is a Dynkin diagram.*

In this problem you will prove the only if direction of this theorem (i.e. why other quivers are NOT of finite type).

(a) Show that if Q is of finite type then for any numbers $x_i \geq 0$ which are not simultaneously zero, one has $q(x_1, ..., x_r) > 0$, where

$$q(x_1, ..., x_r) := \sum_{i \in D} x_i^2 - \frac{1}{2} \sum_{i,j \in D} b_{ij} x_i x_j.$$

Hint. It suffices to check the result for integers: $x_i = n_i$. For this, consider the space W of representations V of Q such that $\dim V_i = n_i$. Show that the group $\times_i GL_{n_i}(k)$ acts with finitely many orbits on $W \oplus k$, and use Problem 5.2 to derive the inequality.

(b) Deduce that q is a positive definite quadratic form.

(c) Show that a quiver of finite type can have no self-loops. Then, using Problem 5.3, deduce the theorem.

Problem 5.5. *Let $G \neq 1$ be a finite subgroup of $SU(2)$, and V be the 2-dimensional representation of G coming from its embedding into $SU(2)$. Let V_i, $i \in I$, be all the irreducible representations of G. Let r_{ij} be the multiplicity of V_i in $V \otimes V_j$.*

(a) Show that $r_{ij} = r_{ji}$.

(b) The McKay graph of G, $M(G)$, is the graph whose vertices are labeled by $i \in I$, and i is connected to j by r_{ij} edges. Show that $M(G)$ is connected. (Use Problem 3.26)

(c) Show that $M(G)$ is an affine Dynkin graph (one of the "forbidden" graphs in Problem 5.3). For this, show that the matrix $a_{ij} = 2\delta_{ij} - r_{ij}$ is positive semidefinite but not definite, and use Problem 5.3.

Hint. Let $f = \sum x_i \chi_{V_i}$, where χ_{V_i} be the characters of V_i. Show directly that $((2-\chi_V)f, f) \geq 0$. When is it $= 0$? Next, show that $M(G)$ has no self-loops, by using that if G is not cyclic then G contains the central element $-Id \in SU(2)$.

(d) Which groups from Problem 3.24 correspond to which diagrams?

(e) Using the McKay graph, find the dimensions of irreducible representations of all finite $G \subset SU(2)$ (namely, show that they are the numbers labeling the vertices of the affine Dynkin diagrams on our pictures). Compare with the results on subgroups of $SO(3)$ we obtained earlier.

5.2 Indecomposable representations of the quivers A_1, A_2, A_3

One central question when looking at representations of quivers is whether a certain quiver has only finitely many indecomposable represenions. We already proved that only those quives whose underlying undirected graph is a Dynkin diagram may have this property. To see if they actually do have this property, we first explicitly decompose representations of certain easy quivers.

Remark 5.6. By an object of the type $1 \longrightarrow 0$ we mean a map from a one-dimensional vector space to the zero space. Similarly, an object of the type $0 \longrightarrow 1$ is a map from the zero space into an one-dimensional space. The object $1 \longrightarrow 1$ means an isomorphism from a one-dimensional to another one-dimensional space. Similarly, numbers in such diagrams always mean the dimension of the attached spaces and the maps are the canonical maps (unless specified otherwise)

Example 5.7 (A_1)**.** The quiver A_1 consists of a single vertex and has no edges. Since a representation of this quiver is just a single vector space, the only indecomposable representation is the ground field itself. Therefore the quiver A_1 has only one indecomposable representation, namely the field of complex numbers.

Example 5.8 (A_2)**.** The quiver A_2 consists of two vertices connected by a single edge.

$$\bullet \longrightarrow \bullet$$

A representation of this quiver consists of two vector spaces V, W and an operator $A : V \to W$.

$$\underset{V}{\bullet} \xrightarrow{A} \underset{W}{\bullet}$$

To decompose this representation, we first let V' be a complement to the kernel of A in V and let W' be a complement to the image of A in W. Then we can decompose the representation as follows

$$\underset{V}{\bullet} \xrightarrow{A} \underset{W}{\bullet} = \underset{\ker V}{\bullet} \xrightarrow{0} \underset{0}{\bullet} \oplus \underset{V'}{\bullet} \overset{A}{\xrightarrow{\sim}} \underset{\mathrm{Im} A}{\bullet} \oplus \underset{0}{\bullet} \xrightarrow{0} \underset{W'}{\bullet}$$

The first summand is a direct sum of objects of the type $1 \longrightarrow 0$, the second a multiple of $1 \longrightarrow 1$, the third of $0 \longrightarrow 1$. We see that the quiver A_2 has three indecomposable representations, namely

$$1 \longrightarrow 0, \quad 1 \longrightarrow 1 \quad \text{and} \quad 0 \longrightarrow 1 .$$

Example 5.9 (A_3)**.** The quiver A_3 consists of three vertices and two connections between them. So we have to choose between two possible orientations.

$$\bullet \longrightarrow \bullet \longrightarrow \bullet \quad \text{or} \quad \bullet \longrightarrow \bullet \longleftarrow \bullet$$

1. We first look at the orientation

   $$\bullet \longrightarrow \bullet \longrightarrow \bullet .$$

   Then a representation of this quiver looks like

   $$\underset{V}{\bullet} \xrightarrow{A} \underset{W}{\bullet} \xrightarrow{B} \underset{Y}{\bullet} .$$

   Like in 5.8 we first split away

   $$\underset{\ker A}{\bullet} \xrightarrow{0} \underset{0}{\bullet} \xrightarrow{0} \underset{0}{\bullet} .$$

   This object is a multiple of $1 \longrightarrow 0 \longrightarrow 0$. Next, let Y' be a complement of $\mathrm{Im} B$. Then we can also split away

   $$\underset{0}{\bullet} \xrightarrow{0} \underset{0}{\bullet} \xrightarrow{0} \underset{Y'}{\bullet}$$

   which is a multiple of the object $0 \longrightarrow 0 \longrightarrow 1$. This results in a situation where the map A is injective and the map B is surjective (we rename the spaces to simplify notation):

   $$\underset{V}{\bullet} \overset{A}{\hookrightarrow} \underset{W}{\bullet} \overset{B}{\twoheadrightarrow} \underset{Y}{\bullet} .$$

   Next, let $X = \ker(B \circ A)$ and let X' be a complement of X in V. Let W' be a complement of $A(X)$ in W such that $A(X') \subset W'$. Then we get

   $$\underset{V}{\bullet} \overset{A}{\hookrightarrow} \underset{W}{\bullet} \overset{B}{\twoheadrightarrow} \underset{Y}{\bullet} = \underset{X}{\bullet} \xrightarrow{A} \underset{A(X)}{\bullet} \xrightarrow{B} \underset{0}{\bullet} \oplus \underset{X'}{\bullet} \overset{A}{\hookrightarrow} \underset{W'}{\bullet} \overset{B}{\twoheadrightarrow} \underset{Y}{\bullet}$$

   The first of these summands is a multiple of $1 \xrightarrow{\sim} 1 \longrightarrow 0$. Looking at the second summand, we now have a situation where A is injective, B is surjective and furthermore $\ker(B \circ A) = 0$. To simplify notation, we redefine

   $$V = X', \ W = W'.$$

Next we let $X = \operatorname{Im}(B \circ A)$ and let X' be a complement of X in Y. Furthermore, let $W' = B^{-1}(X')$. Then W' is a complement of $A(V)$ in W. This yields the decomposition

$$\underset{V}{\bullet} \xhookrightarrow{A} \underset{W}{\bullet} \overset{B}{\twoheadrightarrow} \underset{Y}{\bullet} \;=\; \underset{V}{\bullet} \xrightarrow[\sim]{A} \underset{A(V)}{\bullet} \xrightarrow[\sim]{B} \underset{X}{\bullet} \;\oplus\; \underset{0}{\bullet} \longrightarrow \underset{W'}{\bullet} \overset{B}{\twoheadrightarrow} \underset{X'}{\bullet}$$

Here, the first summand is a multiple of $1 \xrightarrow{\sim} 1 \xrightarrow{\sim} 1$. By splitting away the kernel of B, the second summand can be decomposed into multiples of $0 \longrightarrow 1 \xrightarrow{\sim} 1$ and $0 \longrightarrow 1 \longrightarrow 0$. So, on the whole, this quiver has six indecomposable representations:

$$1 \longrightarrow 0 \longrightarrow 0\,, \quad 0 \longrightarrow 0 \longrightarrow 1\,, \quad 1 \xrightarrow{\sim} 1 \longrightarrow 0\,,$$

$$1 \xrightarrow{\sim} 1 \xrightarrow{\sim} 1\,, \quad 0 \longrightarrow 1 \xrightarrow{\sim} 1\,, \quad 0 \longrightarrow 1 \longrightarrow 0$$

2. Now we look at the orientation

$$\bullet \longrightarrow \bullet \longleftarrow \bullet\,.$$

Very similarly to the other orientation, we can split away objects of the type

$$1 \longrightarrow 0 \longleftarrow 0\,, \quad 0 \longrightarrow 0 \longleftarrow 1$$

which results in a situation where both A and B are injective:

$$\underset{V}{\bullet} \xhookrightarrow{A} \underset{W}{\bullet} \xhookleftarrow{B} \underset{Y}{\bullet}\,.$$

By identifying V and Y as subspaces of W, this leads to the problem of classifying pairs of subspaces of a given space W up to isomorphism (the **pair of subspaces problem**). To do so, we first choose a complement W' of $V \cap Y$ in W, and set $V' = W' \cap V$, $Y' = W' \cap Y$. Then we can decompose the representation as follows:

$$\underset{V}{\bullet} \hookrightarrow \underset{W}{\bullet} \hookleftarrow \underset{Y}{\bullet} \;=\; \underset{V'}{\bullet} \hookrightarrow \underset{W'}{\bullet} \hookleftarrow \underset{Y'}{\bullet} \;\oplus\; \underset{V\cap Y}{\bullet} \xrightarrow{\sim} \underset{V\cap Y}{\bullet} \xleftarrow{\sim} \underset{V\cap Y}{\bullet}.$$

The second summand is a multiple of the object $1 \xrightarrow{\sim} 1 \xleftarrow{\sim} 1$. We go on decomposing the first summand. Again, to simplify notation, we let

$$V = V',\ W = W',\ Y = Y'$$

We can now assume that $V \cap Y = 0$. Next, let W' be a complement of $V \oplus Y$ in W. Then we get

$$\underset{V}{\bullet} \hookrightarrow \underset{W}{\bullet} \hookleftarrow \underset{Y}{\bullet} \;=\; \underset{V}{\bullet} \hookrightarrow \underset{V\oplus Y}{\bullet} \hookleftarrow \underset{Y}{\bullet} \;\oplus\; \underset{0}{\bullet} \longrightarrow \underset{W'}{\bullet} \longleftarrow \underset{0}{\bullet}$$

The second of there summands is a multiple of the indecomposable object $0 \longrightarrow 1 \longleftarrow 0$. The first summand can be further decomposed as follows:

$$\underset{V}{\bullet} \hookrightarrow \underset{V\oplus Y}{\bullet} \hookleftarrow \underset{Y}{\bullet} \;=\; \underset{V}{\bullet} \xrightarrow{\sim} \underset{V}{\bullet} \longleftarrow \underset{0}{\bullet} \;\oplus\; \underset{0}{\bullet} \longrightarrow \underset{Y}{\bullet} \xleftarrow{\sim} \underset{Y}{\bullet}$$

These summands are multiples of

$$1 \longrightarrow 1 \longleftarrow 0\,, \quad 0 \longrightarrow 1 \longleftarrow 1$$

So - like in the other orientation - we get 6 indecomposable representations of A_3:

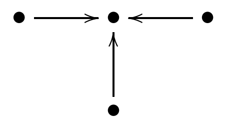
$$1 \longrightarrow 0 \longleftarrow 0\,, \quad 0 \longrightarrow 0 \longleftarrow 1\,, \quad 1 \xrightarrow{\sim} 1 \xleftarrow{\sim} 1\,,$$

$$0 \longrightarrow 1 \longleftarrow 0\,, \quad 1 \longrightarrow 1 \longleftarrow 0\,, \quad 0 \longrightarrow 1 \longleftarrow 1$$

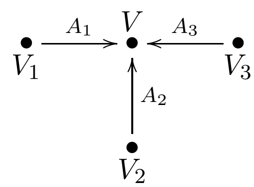
5.3 Indecomposable representations of the quiver D_4

As a last - slightly more complicated - example we consider the quiver D_4.

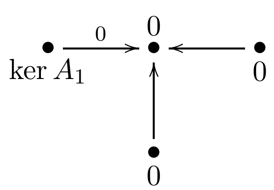
Example 5.10 (D_4)**.** We restrict ourselves to the orientation

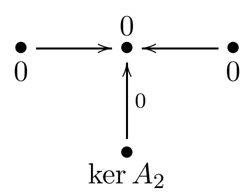


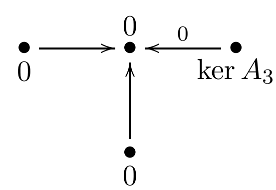
So a representation of this quiver looks like



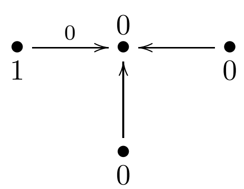
The first thing we can do is - as usual - split away the kernels of the maps A_1, A_2, A_3. More precisely, we split away the representations

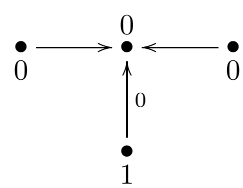


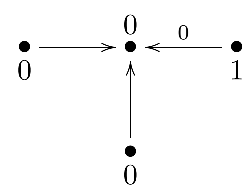




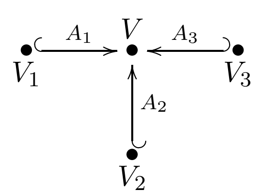
These representations are multiples of the indecomposable objects





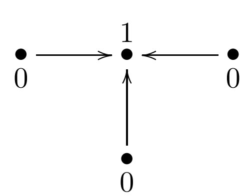


So we get to a situation where all of the maps A_1, A_2, A_3 are injective.



As in 2, we can then identify the spaces V_1, V_2, V_3 with subspaces of V. So we get to the **triple of subspaces problem** of classifying a triple of subspaces of a given space V.

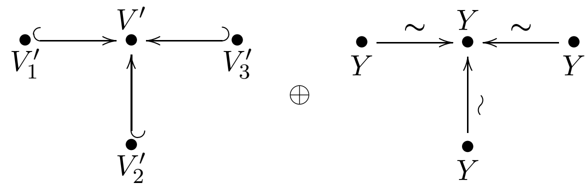
The next step is to split away a multiple of



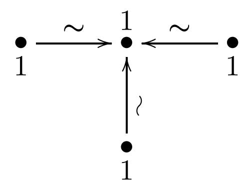
to reach a situation where

$$V_1 + V_2 + V_3 = V.$$

By letting $Y = V_1 \cap V_2 \cap V_3$, choosing a complement V' of Y in V, and setting $V_i' = V' \cap V_i$, $i = 1, 2, 3$, we can decompose this representation into



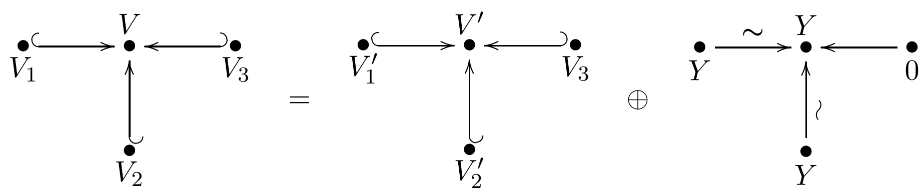
The last summand is a multiple of the indecomposable representation



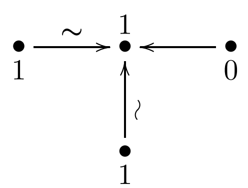
So - considering the first summand and renaming the spaces to simplify notation - we are in a situation where

$$V = V_1 + V_2 + V_3, \quad V_1 \cap V_2 \cap V_3 = 0$$

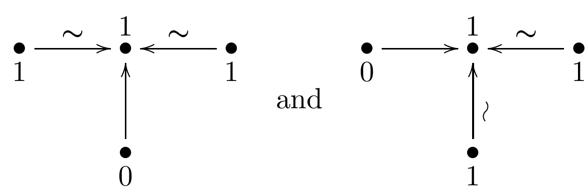
As a next step, we let $Y = V_1 \cap V_2$ and we choose a complement V' of Y in V such that $V_3 \subset V'$, and set $V_1' = V' \cap V_1, V_2' = V' \cap V_2$ of Y. This yields the decomposition



The second summand is a multiple of the indecomposable object



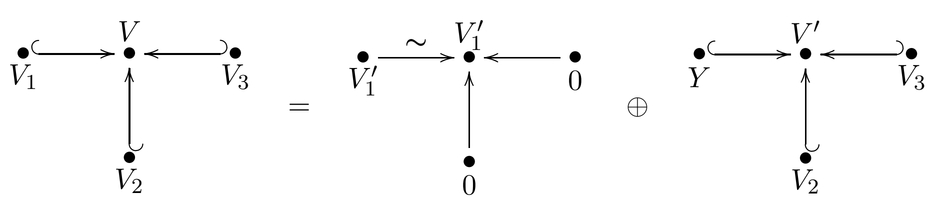
In the resulting situation we have $V_1 \cap V_2 = 0$. Similarly we can split away multiples of



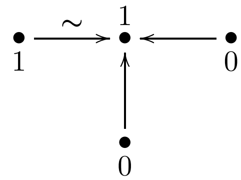
to reach a situation where the spaces V_1, V_2, V_3 do not intersect pairwise

$$V_1 \cap V_2 \ = \ V_1 \cap V_3 \ = \ V_2 \cap V_3 \ = \ 0$$

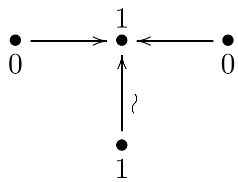
If $V_1 \not\subseteq V_2 \oplus V_3$ we let $Y = V_1 \cap (V_2 \oplus V_3)$. We let V_1' be a complement of Y in V_1. Since then $V_1' \cap (V_2 \oplus V_3) = 0$, we can select a complement V' of V_1' in V which contains $V_2 \oplus V_3$. This gives us the decomposition



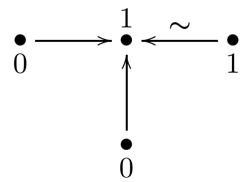
The first of these summands is a multiple of



By splitting these away we get to a situation where $V_1 \subseteq V_2 \oplus V_3$. Similarly, we can split away objects of the type



and



to reach a situation in which the following conditions hold

1. $V_1 + V_2 + V_3 = V$
2. $V_1 \cap V_2 = 0, \quad V_1 \cap V_3 = 0, \quad V_2 \cap V_3 = 0$
3. $V_1 \subseteq V_2 \oplus V_3, \quad V_2 \subseteq V_1 \oplus V_3, \quad V_3 \subseteq V_1 \oplus V_2$

But this implies that

$$V_1 \oplus V_2 = V_1 \oplus V_3 = V_2 \oplus V_3 = V.$$

So we get

$$\dim V_1 = \dim V_2 = \dim V_3 = n$$

and

$$\dim V = 2n$$

Since $V_3 \subseteq V_1 \oplus V_2$ we can write every element of V_3 in the form

$$x \in V_3, \quad x = (x_1, x_2),\ x_1 \in V_1,\ x_2 \in V_2$$

We then can define the projections

$$B_1 : V_3 \to V_1, \quad (x_1, x_2) \mapsto x_1$$

$$B_2 : V_3 \to V_2, \quad (x_1, x_2) \mapsto x_2$$

Since $V_3 \nsubseteq V_1, V_3 \nsubseteq V_2$, these maps have to be injective and therefore are isomorphisms. We then define the isomorphism

$$A = B_2 \circ B_1^{-1} : V_1 \to V_2$$

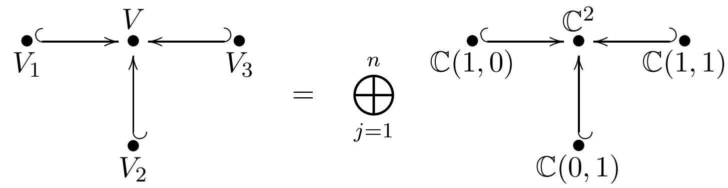
Let $e_1, \ldots, e_n$ be a basis for V_1. Then we get

$$V_1 = \mathbb{C}\, e_1 \oplus \mathbb{C}\, e_2 \oplus \cdots \oplus \mathbb{C}\, e_n$$

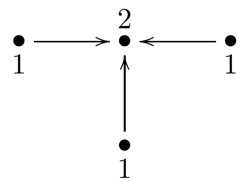
$$V_2 = \mathbb{C}\, Ae_1 \oplus \mathbb{C}\, Ae_2 \oplus \cdots \oplus \mathbb{C}\, Ae_n$$

$$V_3 = \mathbb{C}\, (e_1 + Ae_1) \oplus \mathbb{C}\, (e_2 + Ae_2) \oplus \cdots \oplus \mathbb{C}\, (e_n + Ae_n)$$

So we can think of V_3 as the graph of an isomorphism $A : V_1 \to V_2$. From this we obtain the decomposition



These correspond to the indecomposable object



Thus the quiver D_4 with the selected orientation has 12 indecomposable objects. If one were to explicitly decompose representations for the other possible orientations, one would also find 12 indecomposable objects.

It appears as if the number of indecomposable representations does not depend on the orientation of the edges, and indeed - Gabriel's theorem will generalize this observation.

5.4 Roots

From now on, let Γ be a fixed graph of type A_n, D_n, E_6, E_7, E_8. We denote the adjecency matrix of Γ by R_Γ.

Definition 5.11 (Cartan Matrix)**.** We define the Cartan matrix as

$$A_\Gamma = 2\mathrm{Id} - R_\Gamma$$

On the lattice $\mathbb{Z}^n$ (or the space $\mathbb{R}^n$) we then define an inner product

$$B(x, y) = x^T A_\Gamma y$$

corresponding to the graph Γ.

Lemma 5.12. *1. B is positive definite*

2. $B(x, x)$ takes on only even values for $x \in \mathbb{Z}^n$.

Proof. 1. This follows by definition, since Γ is a Dynkin diagram.

2. By the definition of the Cartan matrix we get

$$B(x, x) = x^T Ay = \sum_{i,j} x_i\, a_{ij}\, x_j = 2 \sum_i x_i^2 + \sum_{i,j,\, i \neq j} x_i\, a_{ij}\, x_j$$

But since A is symmetric, we obtain

$$B(x, x) = 2 \sum_i x_i^2 + \sum_{i,j,\, i \neq j} x_i\, a_{ij}\, x_j = 2 \sum_i x_i^2 + 2 \cdot \sum_{i<j} a_{ij}\, x_i x_j$$

which is even.

□

Definition 5.13 (Root)**.** A root with respect to a certain positive inner product is a shortest (with respect to this inner product), nonzero vector in $\mathbb{Z}^n$.

So for the inner product B, a root is a nonzero vector $x \in \mathbb{Z}^n$ such that

$$B(x, x) = 2$$

Remark 5.14. There can be only finitely many roots, since all of them have to lie in a ball of some radius.

Definition 5.15. We call vectors of the form

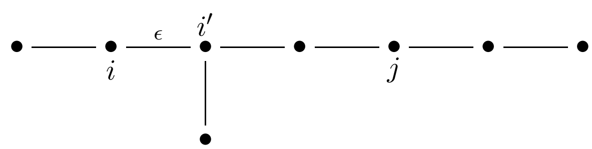
$$\alpha_i = (0, \dots, \overbrace{1}^{i-\text{th}}, \dots, 0)$$

simple roots.

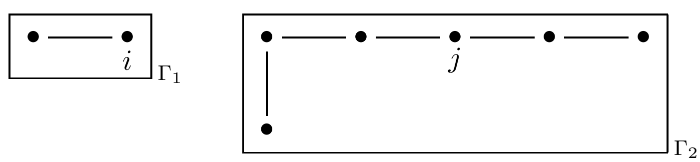
The α_i naturally form a basis of the lattice $\mathbb{Z}^n$.

Lemma 5.16. *Let α be a root, $\alpha = \sum_{i=1}^{n} k_i \alpha_i$. Then either $k_i \geq 0$ for all i or $k_i \leq 0$ for all i.*

Proof. Assume the contrary, i.e. $k_i > 0$, $k_j < 0$. Without loss of generality, we can also assume that $k_s = 0$ for all s between i and j. We can identify the indices i, j with vertices of the graph Γ.



Next, let ϵ be the edge connecting i with the next vertex towards j and i' be the vertex on the other end of ϵ. We then let Γ_1, Γ_2 be the graphs obtained from Γ by removing ϵ. Since Γ is supposed to be a Dynkin diagram - and therefore has no cycles or loops - both Γ_1 and Γ_2 will be connected graphs, which are not connected to each other.



Then we have $i \in \Gamma_1$, $j \in \Gamma_2$. We define

$$\beta = \sum_{m \in \Gamma_1} k_m \alpha_m, \quad \gamma = \sum_{m \in \Gamma_2} k_m \alpha_m$$

With this choice we get

$$\alpha = \beta + \gamma.$$

Since $k_i > 0, k_j < 0$ we know that $\beta \neq 0, \gamma \neq 0$ and therefore

$$B(\beta, \beta) \geq 2, \quad B(\gamma, \gamma) \geq 2.$$

Furthermore,

$$B(\beta, \gamma) = -k_i k_{i'}$$

since Γ_1, Γ_2 are only connected at ϵ. But this has to be a nonnegative number, since $k_i > 0$ and $k_{i'} \leq 0$. This yields

$$B(\alpha, \alpha) = B(\beta + \gamma, \beta + \gamma) = \underbrace{B(\beta, \beta)}_{\geq 2} + 2 \underbrace{B(\beta, \gamma)}_{\geq 0} + \underbrace{B(\gamma, \gamma)}_{\geq 2} \geq 4$$

But this is a contradiction, since α was assumed to be a root. $\square$

Definition 5.17 (positive and negative roots)**.** We call a root $\alpha = \sum_i k_i \alpha_i$ a positive root, if all $k_i \geq 0$. A root for which $k_i \leq 0$ for all i is called a negative root.

Remark 5.18. The Lemma states that every root is either positive or negative.

Example 5.19.

1. Let Γ be of the type A_{n-1}. Then the lattice $L = \mathbb{Z}^{n-1}$ can be realized as a subgroup of the lattice $\mathbb{Z}^n$ by letting $L \subseteq \mathbb{Z}^n$ be the subgroup of all vectors $(x_1, \ldots, x_n)$ such that

$$\sum_i x_i = 0.$$

The vectors

$$\begin{aligned} \alpha_1 &= (1, -1, 0, \ldots, 0) \\ \alpha_2 &= (0, 1, -1, 0, \ldots, 0) \\ &\vdots \\ \alpha_{n-1} &= (0, \ldots, 0, 1, -1) \end{aligned}$$

naturally form a basis of L. Furthermore, the standard inner product

$$(x, y) = \sum x_i y_i$$

on $\mathbb{Z}^n$ restricts to the inner product B given by Γ on L, since it takes the same values on the basis vectors:

$$(\alpha_i, \alpha_i) = 2$$

$$(\alpha_i, \alpha_j) = \begin{cases} -1 & i, j \text{ adjacent} \\ 0 & \text{otherwise} \end{cases}$$

This means that vectors of the form

$$(0, \ldots, 0, 1, 0, \ldots, 0, -1, 0, \ldots, 0) = \alpha_i + \alpha_{i+1} + \cdots + \alpha_{j-1}$$

and

$$(0, \ldots, 0, -1, 0, \ldots, 0, 1, 0, \ldots, 0) = -(\alpha_i + \alpha_{i+1} + \cdots + \alpha_{j-1})$$

are the roots of L. Therefore the number of positive roots in L equals

$$\frac{n(n-1)}{2}$$

2. As a fact we also state the number of positive roots in the other Dynkin diagrams:

D_n	$n(n-1)$
E_6	36 roots
E_7	63 roots
E_8	120 roots

Definition 5.20 (Root reflection)**.** Let $\alpha \in \mathbb{Z}^n$ be a positive root. The reflection s_α is defined by the formula

$$s_\alpha(v) = v - B(v, \alpha)\alpha$$

We denote s_{α_i} by s_i and call these **simple reflections**.

Remark 5.21. s_α fixes B, since

$$B(s_\alpha(v), s_\alpha(w)) = B(v - B(v,\alpha)\alpha, w - B(w,\alpha)\alpha) =$$

$$B(v,w) - B(v, B(w,\alpha)\alpha) - B(B(v,\alpha)\alpha, w) + B(B(v,\alpha)\alpha, B(w,\alpha)\alpha) = B(v,w)$$

Remark 5.22. As a linear operator of $\mathbb{R}^n$, s_α fixes any vector orthogonal to α and

$$s_\alpha(\alpha) = -\alpha$$

Therefore s_α is the reflection at the hyperplane orthogonal to α. The s_i generate a subgroup $W \subseteq O(\mathbb{R}^n)$, which is called *the Weyl group* of Γ. Since for every $w \in W$, $w(\alpha_i)$ is a root, and since there are only finitely many roots, W has to be finite.

5.5 Gabriel's theorem

Definition 5.23. Let Q be a quiver with any labeling $1, \dots, n$ of the vertices. Let $V = (V_1, \dots, V_n)$ be a representation of Q. We then call

$$d(V) = (\dim V_1, \dots, \dim V_n)$$

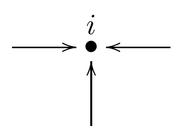
the dimension vector of this representation.

We are now able to formulate Gabriel's theorem using roots.

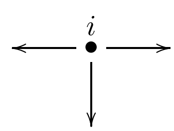
Theorem 5.24 (Gabriel's theorem)**.** *Let Q be a quiver of type A_n, D_n, E_6, E_7, E_8. Then Q has finitely many indecomposable representations. Namely, the dimension vector of any indecomposable representation is a positive root (with respect to B_Γ) and for any positive root α there is exactly one indecomposable representation with dimension vector α.*

5.6 Reflection Functors

Definition 5.25. Let Q be any quiver. We call a vertex $i \in Q$ a sink, if all edges connected to i point towards i.



We call a vertex $i \in Q$ a source, if all edges connected to i point away from i.



Definition 5.26. Let Q be any quiver and $i \in Q$ be a sink (a source). Then we let $\overline{Q_i}$ be the quiver obtained from Q by reversing all arrows pointing into (pointing out of) i.

We are now able to define the reflection functors (also called *Coxeter functors*).

Definition 5.27. Let Q be a quiver, $i \in Q$ be a sink. Let V be a representation of Q. Then we define the reflection functor

$$F_i^+ : \text{Rep}Q \to \text{Rep}\overline{Q_i}$$

by the rule

$$F_i^+(V)_k = V_k \quad \text{if } k \neq i$$

$$F_i^+(V)_i = \ker \left(\bigoplus_{j \to i} V_j \to V_i \right)$$

Also, all maps stay the same but those now pointing out of i; these are replaced by the obvious projections.

Definition 5.28. Let Q be a quiver, $i \in Q$ be a source. Let V be a representation of Q. Let ψ be the canonical map

$$\psi : V_i \to \bigoplus_{i \to j} V_j$$

Then we define the reflection functor

$$F_i^- : \text{Rep}Q \to \text{Rep}\overline{Q_i}$$

by the rule

$$F_i^-(V)_k = V_k \quad \text{if } k \neq i$$

$$F_i^-(V)_i = \text{Coker}\,(\psi) = \left(\bigoplus_{i \to j} V_j \right) / (\text{Im}\phi)$$

Again, all maps stay the same but those now pointing into i; these are replaced by the obvious projections.

Proposition 5.29. *Let Q be a quiver, V an indecomposable representation of Q.*

1. *Let $i \in Q$ be a sink. Then either* $\dim V_i = 1$, $\dim V_j = 0$ *for $j \neq i$* ***or***

$$\varphi : \bigoplus_{j \to i} V_j \to V_i$$

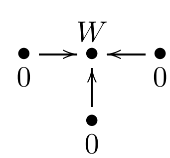
   is surjective.

2. *Let $i \in Q$ be a source. Then either* $\dim V_i = 1$, $\dim V_j = 0$ *for $j \neq i$* ***or***

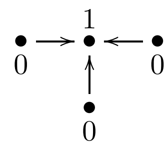
$$\psi : V_i \to \bigoplus_{i \to j} V_j$$

   is injective.

Proof. 1. Choose a complement W of Imφ. Then we get

$V \quad =$  $\oplus \quad V'$

Since V is indecomposable, one of these summands has to be zero. If the first summand is zero, then φ has to be surjective. If the second summand is zero, then the first has to be of the desired form, because else we could write it as a direct sum of several objects of the type



which is impossible, since V was supposed to be indecomposable.

2. Follows similarly by splitting away the kernel of ψ.

□

Proposition 5.30. *Let Q be a quiver, V be a representation of Q.*

1. If

$$\varphi : \bigoplus_{j \to i} V_j \to V_i$$

is surjective, then

$$F_i^- F_i^+ V = V$$

2. If

$$\psi : V_i \to \bigoplus_{i \to j} V_j$$

is injective, then

$$F_i^+ F_i^- V = V$$

Proof. In the following proof, we will always mean by $i \to j$ that i points into j in the original quiver Q. We only show the first statement and we also restrict ourselves to showing that the spaces of V and $F_i^- F_i^+ V$ are the same. It is enough to do so for the i-th space. Let

$$\varphi : \bigoplus_{j \to i} V_j \to V_i$$

be surjective and let

$$K = \ker \varphi.$$

When applying F_i^+, the space V_i gets exchanged by K. Furthermore, let

$$\psi : K \to \bigoplus_{j \to i} V_j$$

After applying F_i^-, K gets replaced by

$$K' = \left(\bigoplus_{j \to i} V_j \right) / (\mathrm{Im}\psi)$$

But

$$\mathrm{Im}\psi = K$$

and therefore

$$K' = \left(\bigoplus_{j\to i} V_j\right) / \left(\ker \bigoplus_{j\to i} V_j \to V_i\right) = \operatorname{Im} \bigoplus_{j\to i} V_j \to V_i$$

by homomorphism theorem. Since φ was assumed to be surjective, we get

$$K' = V_i$$

□

Proposition 5.31. *Let Q be a quiver, V be an indecomposable representation. Then F_i^+V and F_i^-V (whenever defined) are either indecomposable or 0.*

Proof. We prove the proposition for F_i^+V - the case F_i^-V follows similarly. By Proposition 5.29 it follows that either

$$\varphi : \bigoplus_{j\to i} V_j \to V_i$$

is surjective or $\dim V_i = 1, \dim V_j = 0, \quad j \neq i$. In the last case

$$F_i^+V = 0$$

So we can assume that φ is surjective. In this case, assume that F_i^+V is decomposable as

$$F_i^+V = X \oplus Y$$

with $X, Y \neq 0$. But F_i^+V is injective at i, since the maps are canonical projections, whose direct sum is the tautological embedding. Therefore X and Y also have to be injective at i and hence (by 5.30)

$$F_i^+F_i^-X = X, \quad F_i^+F_i^-Y = Y$$

In particular

$$F_i^-X \neq 0, \quad F_i^-Y \neq 0.$$

Therefore

$$V = F_i^-F_i^+V = F_i^-X \oplus F_i^-Y$$

which is a contradiction, since V was assumed to be indecomposable. So we can infer that

$$F_i^+V$$

is indecomposable. □

Proposition 5.32. *Let Q be a quiver and V a representation of Q.*

1. *Let $i \in Q$ be a sink and let V be surjective at i. Then*

$$d(F_i^+V) = s_i(d(V)).$$

2. *Let $i \in Q$ be a source and let V be injective at i. Then*

$$d(F_i^-V) = s_i(d(V)).$$

Proof. We only prove the first statement, the second one follows similarly. Let $i \in Q$ be a sink and let

$$\varphi : \bigoplus_{j \to i} V_j \to V_i$$

be surjective. Let $K = \ker \varphi$. Then

$$\dim K = \sum_{j \to i} \dim V_j - \dim V_i$$

Therefore we get

$$\left(d(F_i^+ V) - d(V)\right)_i = \sum_{j \to i} \dim V_j - 2 \dim V_i = -B\left(d(V), \alpha_i\right)$$

and

$$\left(d(F_i^+ V) - d(V)\right)_j = 0, \quad j \neq i.$$

This implies

$$\begin{aligned} & d(F_i^+ V) - d(V) = -B\left(d(V), \alpha_i\right) \alpha_i \\ \Leftrightarrow \quad & d(F_i^+ V) \;=\; d(V) - B\left(d(V), \alpha_i\right) \alpha_i \;=\; s_i\left(d(V)\right) \end{aligned}$$

□

5.7 Coxeter elements

Definition 5.33. Let Q be a quiver and let Γ be the underlying graph. Fix any labeling $1, \ldots, r$ of the vertices of Γ. Then the Coxeter element c of Q corresponding to this labeling is defined as

$$c = s_1 s_2 \ldots s_r$$

Lemma 5.34. *Let*

$$\beta = \sum_i k_i \alpha_i$$

with $k_i \geq 0$ for all i but not all $k_i = 0$. Then there is $N \in \mathbb{N}$, such that

$$c^N \beta$$

has at least one strictly negative coefficient.

Proof. c belongs to a finite group W. So there is $M \in \mathbb{N}$, such that

$$c^M = 1$$

We claim that

$$1 + c + c^2 + \cdots + c^{M-1} = 0$$

as operators on $\mathbb{R}^n$. This implies what we need, since β has at least one strictly positive coefficient, so one of the elements

$$c\beta, c^2\beta, \ldots, c^{M-1}\beta$$

must have at least one strictly negative one. Furthermore, it is enough to show that 1 is not an eigenvalue for c, since

$$(1 + c + c^2 + \cdots + c^{M-1})v = w \neq 0$$

$$\Rightarrow \quad cw = c\left(1 + c + c^2 + \cdots + c^{M-1}\right)v = (c + c^2 + c^3 + \cdots + c^{M-1} + 1)v = w$$

Assume the contrary, i.e. 1 is a eigenvalue of c and let v be a corresponding eigenvector.

$$cv = v \quad \Rightarrow \quad s_1 \dots s_r v = v$$

$$\Leftrightarrow \quad s_2 \dots s_r v = s_1 v$$

But since s_i only changes the i-th coordinate of v, we get

$$s_1 v = v \quad \text{and} \quad s_2 \dots s_r v = v$$

Repeating the same procedure, we get

$$s_i v = v$$

for all i. But this means

$$B(v, \alpha_i) = 0$$

for all i and since B is nondegenerate, we get $v = 0$. But this is a contradiction, since v is an eigenvector. □

5.8 Proof of Gabriel's theorem

Let V be an indecomposable representation of Q. We introduce a fixed labeling $1, \dots r$ on Q, such that $i < j$ if one can reach j from i. This is possible, since we can assign the highest label to any sink, remove this sink from the quiver, assign the next highest label to a sink of the remaining quiver and so on. This way we create a labeling of the desired kind.

We now consider the sequence

$$V^{(0)} = V,\ V^{(1)} = F_r^+ V,\ V^{(2)} = F_{r-1}^+ F_r^+ V, \dots$$

This sequence is well defined because of the selected labeling: r has to be a sink of Q, $r-1$ has to be a sink of $\overline{Q_r}$ and so on. Furthermore we note that $V^{(r)}$ is a representation of Q again, since every arrow has been reversed twice (since we applied a reflection functor to every vertex). This implies that we can define

$$V^{(r+1)} = F_r^+ V^{(r)}, \dots$$

and continue the sequence to infinity.

Theorem 5.35. *There is $m \in \mathbb{N}$, such that*

$$d\left(V^{(m)}\right) = \alpha_p$$

for some p.

Proof. If $V^{(i)}$ is surjective at the appropriate vertex k, then

$$d\left(V^{(i+1)}\right) = d\left(F_k^+ V^{(i)}\right) = s_k d\left(V^{(i)}\right)$$

This implies, that if $V^{(0)}, \dots, V^{(i-1)}$ are surjective at the appropriate vertices, then

$$d\left(V^{(i)}\right) = \dots s_{r-1} s_r d(V)$$

By Lemma 5.34 this cannot continue indefinitely - since $d\left(V^{(i)}\right)$ may not have any negative entries. Let i be smallest number such that $V^{(i)}$ is not surjective at the appropriate vertex. By Proposition 5.31 it is indecomposable. So, by Proposition 5.29, we get

$$d(V^{(i)}) = \alpha_p$$

for some p. □

We are now able to prove Gabriel's theorem as corollaries of this theorem.

Corollary 5.36. *Let Q be a quiver, V be any indecomposable representation. Then $d(V)$ is a positive root.*

Proof. By 5.35

$$s_{i_1} \dots s_{i_n} (d(V)) = \alpha_p.$$

Since the s_i preserve B, we get

$$B(d(V), d(V)) = B(\alpha_p, \alpha_p) = 2.$$

□

Corollary 5.37. *Let V, V' be indecomposable representations of Q such that $d(V) = d(V')$. Then V and V' are isomorphic.*

Proof. Let i be such that

$$d\left(V^{(i)}\right) = \alpha_p.$$

Then we also get $d\left(V'^{(i)}\right) = \alpha_p$. So

$$V'^{(i)} = V^{(i)} =: V^i.$$

Furthermore we have

$$V^{(i)} = F_k^+ \dots F_{r-1}^+ F_r^+ V^{(0)}$$
$$V'^{(i)} = F_k^+ \dots F_{r-1}^+ F_r^+ V'^{(0)}$$

But both $V^{(i-1)}, \dots, V^{(0)}$ and $V'^{(i-1)}, \dots, V'^{(0)}$ have to be surjective at the appropriate vertices. This implies

$$F_r^- F_{r-1}^- \dots F_k^- V^i = \begin{cases} F_r^- F_{r-1}^- \dots F_k^- F_k^+ \dots F_{r-1}^+ F_r^+ V^{(0)} & = V^{(0)} \quad = V \\ F_r^- F_{r-1}^- \dots F_k^- F_k^+ \dots F_{r-1}^+ F_r^+ V'^{(0)} & = V'^{(0)} \quad = V' \end{cases}$$

□

These two corollaries show that there are only finitely many indecomposable representations (since there are only finitely many roots) and that the dimension vector of each of them is a positive root. The last statement of Gabriel's theorem follows from

Corollary 5.38. *For every positive root α, there is an indecomposable representation V with*

$$d(V) = \alpha$$

Proof. Consider the sequence

$$s_r\alpha, s_{r-1}s_r\alpha, \dots$$

Consider the first element of this sequence which is a negative root (this has to happen by 5.34) and look at one step before that, call this element β. So β is a positive root and $s_i\beta$ is a negative root for some i. But since the s_i only change one coordinate, we get

$$\beta = \alpha_i$$

and

$$(s_q \dots s_{r-1}s_r)\alpha = \alpha_i.$$

We let $\mathbb{C}_{(i)}$ be the representation having dimension vector α_i. Then we define

$$V = F_r^- F_{r-1}^- \dots F_q^- \mathbb{C}_{(i)}$$

This is an indecomposable representation and

$$d(V) = \alpha.$$

□

Example. Let us demonstrate by example how reflection functors work. Consider the quiver D_4 with the orientation of all arrows towards the node (which is labeled by 4). Start with the 1-dimensional representation V_{α_4} sitting at the 4-th vertex. Apply to V_{α_4} the functor $F_3^- F_2^- F_1^-$. This yields

$$F_1^- F_2^- F_3^- V_{\alpha_4} = V_{\alpha_1+\alpha_2+\alpha_3+\alpha_4}.$$

Now applying F_4^- we get

$$F_4^- F_1^- F_2^- F_3^- V_{\alpha_4} = V_{\alpha_1+\alpha_2+\alpha_3+2\alpha_4}.$$

Note that this is exactly the inclusion of 3 lines into the plane, which is the most complicated indecomposable representation of the D_4 quiver.

5.9 Problems

Problem 5.39. *Let Q_n be the cyclic quiver of length n, i.e. n vertices connected by n oriented edges forming a cycle. Obviously, the classification of indecomposable representations of Q_1 is given by the Jordan normal form theorem. Obtain a similar classification of indecomposable representations of Q_2. In other words, classify pairs of linear operators $A : V \to W$ and $B : W \to V$ up to isomorphism. Namely:*

(a) Consider the following pairs (for $n \geq 1$):

1) $E_{n,\lambda}$: $V = W = \mathbb{C}^n$, A is the Jordan block of size n with eigenvalue λ, $B = 1$ ($\lambda \in \mathbb{C}$).

2) $E_{n,\infty}$: is obtained from $E_{n,0}$ by exchanging V with W and A with B.

3) H_n: $V = \mathbb{C}^n$ with basis v_i, $W = \mathbb{C}^{n-1}$ with basis w_i, $Av_i = w_i$, $Bw_i = v_{i+1}$ for $i < n$, and $Av_n = 0$.

4) K_n is obtained from H_n by exchanging V with W and A with B.

Show that these are indecomposable and pairwise nonisomorphic.

(b) Show that if E is a representation of Q_2 such that AB is not nilpotent, then $E = E' \oplus E''$, where $E'' = E_{n,\lambda}$ for some $\lambda \neq 0$.

(c) Consider the case when AB is nilpotent, and consider the operator X on $V \oplus W$ *given by* $X(v, w) = (Bw, Av)$. *Show that X is nilpotent, and admits a basis consisting of chains (i.e. sequences* $u, Xu, X^2u, ...X^{l-1}u$ *where* $X^l u = 0$*) which are compatible with the direct sum decomposition (i.e. for every chain* $u \in V$ *or* $u \in W$*). Deduce that (1)-(4) are the only indecomposable representations of* Q_2.

(d) generalize this classification to the Kronecker quiver, which has two vertices 1 and 2 and two edges both going from 1 to 2.

(e)(optional) can you generalize this classification to Q_n, $n > 2$*?*

Problem 5.40. *Let* $L \subset \frac{1}{2}\mathbb{Z}^8$ *be the lattice of vectors where the coordinates are either all integers or all half-integers (but not integers), and the sum of all coordinates is an even integer.*

(a) Let $\alpha_i = e_i - e_{i+1}$, $i = 1, ..., 6$, $\alpha_7 = e_6 + e_7$, $\alpha_8 = 1/2 \sum_{i=1}^{8} e_i$. *Show that* α_i *are a basis of L (over* $\mathbb{Z}$*).*

(b) Show that roots in L (under the usual inner product) form a root system of type E_8 *(compute the inner products of* α_i*).*

(c) Show that the E_7 *and* E_6 *lattices can be obtained as the sets of vectors in the* E_8 *lattice L where the first two, respectively three, coordinates (in the basis* e_i*) are equal.*

(d) Show that E_6, E_7, E_8 *have 72,126,240 roots, respectively (enumerate types of roots in terms of the presentations in the basis* e_i*, and count the roots of each type).*

Problem 5.41. *Let* V_α *be the indecomposable representation of a Dynkin quiver Q which corresponds to a positive root* α. *For instance, if* α_i *is a simple root, then* V_{α_i} *has a 1-dimensional space at i and 0 everywhere else.*

(a) Show that if i is a source then $\mathrm{Ext}^1(V, V_{\alpha_i}) = 0$ *for any representation V of Q, and if i is a sink, then* $\mathrm{Ext}^1(V_{\alpha_i}, V) = 0$.

(b) Given an orientation of the quiver, find a Jordan-Holder series of V_α *for that orientation.*

6 Introduction to categories

6.1 The definition of a category

We have now seen many examples of representation theories and of operations with representations (direct sum, tensor product, induction, restriction, reflection functors, etc.) A context in which one can systematically talk about this is provided by **Category Theory**.

Category theory was founded by Saunders MacLane and Samuel Eilenberg in early 1940-s. It is a fairly abstract theory which seemingly has no content, for which reason it was christened "abstract nonsense". Nevertheless, it is a very flexible and powerful language, which has become totally indispensible in many areas of mathematics, such as algebraic geometry, topology, representation theory, and many others.

Definition 6.1. A category C is the following data:

(i) a class of objects $Ob(C)$;

(ii) for every objects $X, Y \in Ob(C)$, the class $\mathrm{Hom}_C(X, Y) = \mathrm{Hom}(X, Y)$ of morphisms (or arrows) from X, Y (for $f \in \mathrm{Hom}(X, Y)$, one may write $a : X \to Y$);

(iii) For any objects $X, Y, Z \in Ob(C)$, a composition map $\mathrm{Hom}(Y, Z) \times \mathrm{Hom}(X, Y) \to \mathrm{Hom}(X, Z)$, $(f, g) \to f \circ g$,

which satisfy the following axioms:

1. The composition is associative, i.e. $(f \circ g) \circ h = f \circ (g \circ h)$;

2. For each $X \in Ob(C)$, there is a morphism $1_X \in Hom(X, X)$, called the unit morphism, such that $1_X \circ f = f$ and $g \circ 1_X = g$ for any f, g for which compositions make sense.

Remark. We will sometimes write $X \in C$ instead of $X \in Ob(C)$.

Example 6.2. 1. The category **Sets** of sets (morphisms are arbitrary maps).

2. The categories **Groups**, **Rings** (morphisms are homomorphisms).

3. The category $\mathbf{Vect}_k$ of vector spaces over a field k (morphisms are linear maps).

4. The category $\mathrm{Rep}(A)$ of representations of an algebra A (morphisms are homomorphisms of representations).

5. The category of topological spaces (morphisms are continuous maps).

6. The homotopy category of topological spaces (morphisms are homotopy classes of continuous maps).

Important remark. Unfortunately, one cannot simplify this definition by replacing the word "class" by the much more familiar word "set". Indeed, this would rule out the important Example 1, as it is well known that there is no set of all sets, and working with such a set leads to contadictions. The precise definition of a class and the precise distinction between a class and a set is the subject of set theory, and cannot be discussed here. Luckily, for many practical purposes this distinction is not essential.

We also mention that in many examples, including examples 1-4, the word "class" in (ii) can be replaced by "set". Categories with this property (that $\mathrm{Hom}(X, Y)$ is a set for any X, Y) are called locally small; many categories that we encounter are of this kind.

Definition 6.3. A full subcategory of a category C is a category C' whose objects are a subclass of objects of C, and $\mathrm{Hom}_{C'}(X, Y) = \mathrm{Hom}_C(X, Y)$.

6.2 Functors

We would like to define arrows between categories. Such arrows are called **functors**.

Definition 6.4. A functor $F : C \to D$ between categories C and D is

(i) a map $F : Ob(C) \to Ob(D)$;

(ii) for each $X, Y \in C$, a map (also denoted by F) $Hom(X, Y) \to \mathrm{Hom}(F(X), F(Y))$ which preserves compositions and identity morphisms.

Note that functors can be composed in an obvious way. Also, any category has the identity functor.

Example 6.5. 1. A category C with one object X (such that $\mathrm{Hom}(X, X)$ is a set) is the same thing as a monoid. A functor between such categories is a homomorphism of monoids.

2. Forgetful functors **Groups** $\to$ **Sets**, **Rings** $\to$ **Groups**.

3. The opposite category of a given category is the same category with the order of arrows and compositions reversed. Then $V \to V^*$ is a functor $\mathbf{Vect}_k \to \mathbf{Vect}_k^{op}$.

4. The Hom functors: If C is a locally small category then we have the functor $C \to$ **Sets** given by $Y \mapsto \mathrm{Hom}(X, Y)$ and $C^{op} \to$ **Sets** given by $Y \to \mathrm{Hom}(Y, X)$.

5. The assignment $X \to \mathrm{Fun}(X, \mathbb{Z})$ is a functor **Sets** $\to$ $\mathbf{Rings}^{op}$.

6. Let Q be a quiver. Consider the category $C(Q)$ whose objects are the vertices and morphisms are oriented paths between them. Then functors from $C(Q)$ to $\mathbf{Vect}_k$ are representations of Q over k.

7. Let $K \subset G$ be groups. Then we have the induction functor $\mathrm{Ind}_K^G : \mathrm{Rep}(K) \to \mathrm{Rep}(G)$, and $\mathrm{Res}_K^G : \mathrm{Rep}(G) \to \mathrm{Rep}(K)$.

8. We have an obvious notion of the Cartesian product of categories (obtained by taking the Cartesian products of the classes of objects and morphisms of the factors). The functors of direct sum and tensor product are then functors $\mathbf{Vect}_k \times \mathbf{Vect}_k \to \mathbf{Vect}_k$. Also the operations $V \mapsto V^{\otimes n}$, $V \mapsto S^n V$, $V \mapsto \wedge^n V$ are functors on $\mathbf{Vect}_k$. More generally, if π is a representation of S_n, we have functors $V \to \mathrm{Hom}_{S_n}(\pi, V^{\otimes n})$. Such functors (for irreducible π) are called the Schur functors. They are labeled by Young diagrams.

9. The reflection functors $F_i^{\pm} : \mathrm{Rep}(Q) \to \mathrm{Rep}(\bar{Q}_i)$ are functors between representation categories of quivers.

6.3 Morphisms of functors

One of the important features of functors between categories which distinguishes them from usual maps or functions is that the functors between two given categories themselves form a category, i.e. one can define a nontrivial notion of a morphism between two functors.

Definition 6.6. Let C, D be categories and $F, G : C \to D$ be functors between them. A morphism $a : F \to G$ (also called a natural transformation or a functorial morphism) is a collection of morphisms $a_X : F(X) \to G(X)$ labeled by the objects X of C, which is *functorial* in X, i.e., for any morphism $f : X \to Y$ (for $X, Y \in C$) one has $a_X \circ F(f) = G(f) \circ a_X$.

A morphism $a : F \to G$ is an isomorphism if there is another morphism $a^{-1} : G \to F$ such that $a \circ a^{-1}$ and $a^{-1} \circ a$ are the identities. The set of morphisms from F to G is denoted by $\mathrm{Hom}(F, G)$.

Example 6.7. 1. Let $\mathbf{FVect}_k$ be the category of finite dimensional vector spaces over k. Then the functors id and $**$ on this category are isomorphic. The isomorphism is defined by the standard maps $a_V : V \to V^{**}$ given by $a_V(u)(f) = f(u)$, $u \in V$, $f \in V^*$. But these two functors are not isomorphic on the category of all vector spaces $\mathbf{Vect}_k$, since for an infinite dimensional vector space V, V is not isomorphic to V^{**}.

2. Let $\mathbf{FVect}_k'$ be the category of finite dimensional k-vector spaces, where the morphisms are the isomorphisms. We have a functor F from this category to itself sending any space V to V^* and any morphism a to $(a^*)^{-1}$. This functor satisfies the property that V is isomorphic to $F(V)$ for any V, but it is not isomorphic to the identity functor. This is because the isomorphism $V \to F(V) = V^*$ cannot be chosen to be compatible with $\mathrm{Hom}(V, V) = GL(V)$, as V is not isomorphic to V^* as a representation of $GL(V)$.

3. Let A be an algebra over a field k, and $F : A - \mathbf{mod} \to \mathbf{Vect}_k$ be the forgetful functor. Then as follows from Problem 1.22, $\mathrm{Hom}(F, F) = A$.

4. The set of endomorphisms of the identity functor on the category $A - \mathbf{mod}$ is the center of A (check it!).

6.4 Equivalence of categories

When two algebraic or geometric objects are isomorphic, it is usually not a good idea to say that they are equal (i.e. literally the same). The reason is that such objects are usually equal in many different ways, i.e. there are many ways to pick an isomorphism, but by saying that the objects are equal we are misleading the reader or listener into thinking that we are providing a certain choice of the identification, which we actually do not do. A vivid example of this is a finite dimensional vector space V and its dual space V^*.

For this reason in category theory, one most of the time tries to avoid saying that two objects or two functors are equal. In particular, this applies to the definition of isomorphism of categories.

Namely, the naive notion of isomorphism of categories is defined in the obvious way: a functor $F : C \to D$ is an isomorphism if there exists $F^{-1} : D \to C$ such that $F \circ F^{-1}$ and $F^{-1} \circ F$ are equal to the identity functors. But this definition is not very useful. We might suspect so since we have used the word "equal" which we are not supposed to use. And in fact here is an example of two categories which are the same for all practical purposes but are not isomorphic; it demonstrates the definiency of our definition.

Namely, let C_1 be the simplest possible category: $Ob(C)$ consists of one object X, with $\mathrm{Hom}(X, X) = \{1_X\}$. Also, let C_2 have two objects X, Y and 4 morphisms: $1_X, 1_Y, a : X \to Y$ and $b : Y \to X$. So we must have $a \circ b = 1_Y$, $b \circ a = 1_X$.

It is easy to check that for any category D, there is a natural bijection between the collections of isomorphism classes of functors $C_1 \to D$ and $C_2 \to D$ (both are identified with the collection of isomorphism classes of objects of D). This is what we mean by saying that C_1 and C_2 are the same for all practical purposes. Nevertheless they are not isomorphic, since C_1 has one object, and C_2 has two objects (even though these two objects are isomorphic).

This shows that we should adopt a more flexible and less restrictive notion of isomorphism of categories. This is accomplished by the definition of an **equivalence of categories.**

Definition 6.8. A functor $F : C \to D$ is an equivalence of categories if there exists $F' : D \to C$ such that $F \circ F'$ and $F^{-1} \circ F$ are **isomorphic** to the identity functors.

In this situation, F' is said to be *a quasi-inverse* to F.

In particular, the above categories C_1 and C_2 are equivalent (check it!).

Also, the category **FSet** of finite sets is equivalent to the category whose objects are nonnegative integers, and morphisms are given by $\mathrm{Hom}(m, n) = \mathrm{Maps}(\{1, ..., m\}, \{1, ..., n\})$. Are these categories isomorphic? The answer to this question depends on whether you believe that there is only one finite set with a given number of elements, or that there are many of those. It seems better to think that there are many (without asking "how many"), so that isomorphic sets need not be literally equal, but this is really a matter of choice. In any case, the answer to this question is irrelevant for any practical purpose, and thinking about it will give you a headache.

6.5 Representable functors

A fundamental notion in category theory is that of a **representable functor**. Namely, let C be a (locally small) category, and $F : C \to \mathbf{Sets}$ be a functor. We say that F is **representable** if there exists an object $X \in C$ such that F is isomorphic to the functor $\mathrm{Hom}(X, ?)$.

In a similar way, one can talk about representable functors from C^{op} to Sets. Namely, one calls such a functor representable if it is of the form $\mathrm{Hom}(?, X)$ for some object $X \in C$.

Not every functor is representable, but if a representing object X exists, then it is unique. Namely, we have the following lemma.

Lemma 6.9. *(Yoneda Lemma) If X exists, then it is unique up to a unique isomorphism. I.e., if X, Y are two objects in C, then any isomorphism of functors $\phi : Hom(X, ?) \to Hom(Y, ?)$ gives rise to an isomorphism $a_\phi : X \to Y$.*

Proof. (Sketch) One sets $a_\phi = \phi_Y^{-1}(1_Y)$, and shows that it is invertible by constructing the inverse, which is $a_\phi^{-1} = \phi_X(1_X)$. It remains to show that the composition both ways is the identity, which we will omit here. □

Example 6.10. Let A be an algebra. Then the forgetful functor on A-modules is representable, and the representing object is the free rank 1 module (=the regular representation) $M = A$. But if A is infinite dimensional, and we restrict attention to the category of finite dimensional modules, then the forgetful functor, in general, is not representable (this is so, for example, if A is the algebra of complex functions on $\mathbb{Z}$ with finitely many nonzero values).

6.6 Adjoint functors

Another fundamental notion in category theory is the notion of **adjoint functors**.

Definition 6.11. Functors $F : C \to D$ and $G : D \to C$ are said to be a pair of adjoint functors if for any $X \in C$, $Y \in D$ we are given an isomorphism $\xi_{XY} : \mathrm{Hom}_C(F(X), Y) \to \mathrm{Hom}_D(X, G(Y))$ which is functorial in X and Y; in other words, if we are given an isomorphism of functors $\mathrm{Hom}(F(?), ?) \to \mathrm{Hom}(?, G(?))$ $(C \times D \to \mathbf{Sets})$. In this situation, we say that F is left adjoint to G and G is right adjoint to F.

Remark 6.12. This terminology is motivated by the analogy between categories and inner product spaces. Namely, the "inner product" on a category is the assignment $X, Y \to \mathrm{Hom}(X, Y)$ (so it takes values in the category **Sets**). Yoneda's lemma may be interpreted as nondegeneracy of the inner product (and the representability property is analogous to the property of a linear functional to be the inner product with a vector, which is not always the case in an infinite dimensional space, even if the inner product is nondegenerate). With this analogy in mind, the above definition is parallel to the definition of adjoint operators: $FV \to W$ and $G : W \to V$ are a pair of adjoint operators if $(Fv, w) = (v, Gw)$ for all v, w. Note that if the inner products are not symmetric, then the left and right adjoint of an operator don't necessarily coincide; the same applies to adjoint functors.

Not every functor has a left or right adjoint, but if it does, it is unique and can be constructed canonically (i.e. if we somehow found two such functors, then there is a canonical isomorphism between them). This follows easily from the Yoneda lemma, as if F, G are a pair of adjoint functors then $F(X)$ represents the functor $Y \to \mathrm{Hom}(X, G(Y))$, and $G(Y)$ represents the functor $X \to \mathrm{Hom}(F(X), Y)$.

Example 6.13. 1. Let V be a finite dimensional representation on a group G. Then the left and right adjoint to the functor $V\otimes$ on the category of representations of G is the functor $V^*\otimes$.

2. The functor Res_K^G is left adjoint to Ind_K^G. This is nothing but the statement of Frobenius reciprocity.

3. Let $\mathbf{Assoc}_k$ be the category of associative unital algebras, and $\mathbf{Lie}_k$ the category of Lie algebras over some field k. We have a functor $L : \mathbf{Assoc}_k \to \mathbf{Lie}_k$, which attaches to an associative algebra the same space regarded as a Lie algebra, with bracket $[a, b] = ab - ba$. Then the functor L has a left adjoint, which is the functor U of taking the universal enveloping algebra of a Lie algebra.

4. We have the functor $GL_1 : \mathbf{Assoc}_k \to \mathbf{Groups}$, given by $A \mapsto GL_1(A) = A^\times$. This functor has a left adjoint, which is the functor $G \mapsto k[G]$, the group algebra of G.

5. The left adjoint to the forgetful functor $\mathbf{Assoc}_k \to \mathbf{Vect}_k$ is the functor of tensor algebra: $V \mapsto TV$. Also, if we denote by $\mathbf{Comm}_k$ the category of commutative algebras, then the left adjoint to the forgetful functor $\mathbf{Comm}_k \to \mathbf{Vect}_k$ is the functor of the symmetric algebra: $V \mapsto SV$.

One can give many more examples, spanning many fields. These examples show that adjoint functors are ubiquitous in mathematics.

6.7 Abelian categories

The type of categories that most often appears in representation theory is **abelian categories**. The standard definition of an abelian category is rather long, so we will not give it here; rather, we will use as the definition what is really the statement of the Freyd-Mitchell theorem:

Definition 6.14. An abelian category is a full subcategory C of the category of A-mod of left modules over a ring A, which is closed under taking finite direct sums, submodules, and quotient modules.

Example 6.15. The category of modules over an algebra A and the category of finite dimensional modules over A are abelian categories.

We see from this definition that in an abelian category, $\mathrm{Hom}(X, Y)$ is an abelian group for each X, Y, compositions are group homomorphisms with respect to each argument, there is the zero object, the notion of an injective morphism (monomorphism) and surjective morphism (epimorphism), and every morphism has a kernel and a cokernel.

Remark 6.16. The good thing about Definition 6.14 is that it allows us to visualize objects, morphisms, kernels, and cokernels in terms of classical algebra. But the definition also has a big drawback, which is that the ring A is not determined by C. In particular, two different rings can have equivalent categories of modules. This is why people prefer to use the standard definition, which is free from this drawback, even though it is more abstract.

We say that an abelian category C is k-linear if the groups $\mathrm{Hom}_C(X, Y)$ are equipped with a structure of a vector space over k, and composition maps are k-linear in each argument. In particular, the categories in Example 6.15 are k-linear.

6.8 Exact functors

Definition 6.17. A sequence of objects and morphisms

$$X_0 \to X_1 \to ... \to X_{n+1}$$

in an abelian category is said to be **a complex** if the composition of any two consecutive arrows is zero. The **cohomology** of this complex is $H^i = \mathrm{Ker}\,(d_i)/\mathrm{Im}(d_{i-1})$, where $d_i : X_i \to X_{i+1}$ (thus the cohomology is defined for $1 \le i \le n$). The complex is said to be **exact in the i-th term** if $H^i = 0$, and is said to be **an exact sequence** if it is exact in all terms. A **short exact sequence** is an exact sequence of the form

$$0 \to X \to Y \to Z \to 0.$$

Clearly, $0 \to X \to Y \to Z \to 0$ is a short exact sequence iff $X \to Y$ is injective, $Y \to Z$ is surjective, and the induced map $Y/X \to Z$ is an isomorphism.

Definition 6.18. A functor F between two abelian categories is **additive** if it induces homomorphisms on Hom groups. Also, for k-linear categories one says that F is k-linear if it induces k-linear maps between Hom spaces.

It is easy to show that if F is an additive functor, then $F(X \oplus Y)$ is canonically isomorphic to $F(X) \oplus F(Y)$.

Example 6.19. The functors Ind_K^G, Res_K^G, $\mathrm{Hom}_G(V, ?)$ in the theory of group representations over a field k are additive and k-linear.

Definition 6.20. An additive functor $F : C \to D$ between abelian categories is **left exact** if for any exact sequence

$$0 \to X \to Y \to Z,$$

the sequence

$$0 \to F(X) \to F(Y) \to F(Z)$$

is exact. F is **right exact** for any exact sequence

$$X \to Y \to Z \to 0,$$

the sequence

$$F(X) \to F(Y) \to F(Z) \to 0$$

is exact. F is **exact** if it is both left and right exact.

Definition 6.21. An abelian category C is **semisimple** if any short exact sequence in this category splits, i.e. is isomorphic to a sequence

$$0 \to X \to X \oplus Y \to Y \to 0$$

(where the maps are obvious).

Example 6.22. The category of representations of a finite group G over a field of characteristic not dividing $|G|$ (or 0) is semisimple.

Note that in a semisimple category, any additive functor is automatically exact on both sides.

Example 6.23. (i) The functors Ind_K^G, Res_K^G are exact.

(ii) The functor $\mathrm{Hom}(X, ?)$ is left exact, but not necessarily right exact. To see that it need not be right exact, it suffices to consider the exact sequence

$$0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0,$$

and apply the functor $\mathrm{Hom}(\mathbb{Z}/2\mathbb{Z}, ?)$.

(iii) The functor $X \otimes_A$ for a right A-module X (on the category of left A-modules) is right exact, but not necessarily left exact. To see this, it suffices to tensor multiply the above exact sequence by $\mathbb{Z}/2\mathbb{Z}$.

7 Structure of finite dimensional algebras

In this section we return to studying the structure of finite dimensional algebras. Throughout the section, we work over an algebraically closed field k (of any characteristic).

7.1 Projective modules

Let A be an algebra, and P be a left A-module.

Theorem 7.1. *The following properties of P are equivalent:*

(i) If $\alpha : M \to N$ is a surjective morphism, and $\nu : P \to N$ any morphism, then there exists $\mu : P \to M$ such that $\alpha \circ \mu = \nu$.

(ii) Any surjective morphism $\alpha : M \to P$ splits, i.e. there exists $\mu : P \to M$ such that $\alpha \circ \mu = \mathrm{id}$.

(iii) There exists another A-module Q such that $P \oplus Q$ is a free A-module, i.e. a direct sum of copies of A.

(iv) The functor $Hom_A(P, ?)$ on the category of A-modules is exact.

Proof. To prove that (i) implies (ii), take $N = P$. To prove that (ii) implies (iii), take M to be free (this can always be done since any module is a quotient of a free module). To prove that (iii) implies (iv), note that the functor $\mathrm{Hom}_A(P, ?)$ is exact if A is free (as $\mathrm{Hom}_A(A, N) = N$), so the statement follows, as if the direct sum of two exact sequences is exact, then each of them is exact. To prove that (iv) implies (i), let K be the kernel of the map α, and apply the exact functor $\mathrm{Hom}_A(P, ?)$ to the exact sequence

$$0 \to K \to M \to N \to 0.$$

□

Definition 7.2. A module satisfying any of the conditions (i)-(iv) of Theorem 7.1 is said to be **projective**.

7.2 Lifting of idempotents

Let A be a ring, and $I \subset A$ a nilpotent ideal.

Proposition 7.3. *Let $e_0 \in A/I$ be an idempotent, i.e. $e_0^2 = e_0$. There exists an idempotent $e \in A$ which is a lift of e_0 (i.e. it projects to e_0 under the reduction modulo I). This idempotent is unique up to conjugation by an element of $1 + I$.*

Proof. Let us first establish the statement in the case when $I^2 = 0$. In this case, let e_* be any lift of e_0 to A. Then $e_*^2 - e_* = a \in I$, and $ea = ae$. We look for e in the form $e = e_* + b$, $b \in I$. The equation for b is $e_0 b + b e_0 - b = a$.

Set $b = (2e_0 - 1)a$. Then

$$e_0 b + b e_0 - b = 2e_0 a - (2e_0 - 1)a = a,$$

so e is an idempotent. To classify other solutiuons, set $e' = e + c$. For e' to be an idempotent, we must have $ec + ce - c = 0$. This is equivalent to saying that $ece = 0$ and $(1 - e)c(1 - e) = 0$, so $c = ec(1 - e) + (1 - e)ce = [e, [e, c]]$. Hence $e' = (1 + [c, e])e(1 + [c, e])^{-1}$.

Now, in the general case, we prove by induction in k that there exists a lift e_k of e_0 to A/I^{k+1}, and it is unique up to conjugation by an element of $1 + I^k$ (this is sufficient as I is nilpotent). Assume it is true for $k = m - 1$, and let us prove it for $k = m$. So we have an idempotent $e_{m-1} \in A/I^m$, and we have to lift it to A/I^{m+1}. But $(I^m)^2 = 0$ in A/I^{m+1}, so we are done. □

Corollary 7.4. *Let $e_{01}, ..., e_{0m}$ be a system of orthogonal idempotents in A/I, i.e. $e_{0i}e_{0j} = \delta_{ij}e_{0i}$, and $\sum e_{0i} = 1$. Then there exists a system of orthogonal idempotents $e_1, ..., e_m$ ($e_ie_j = \delta_{ij}e_i$, $\sum e_i = 1$) in A which lifts $e_{01}, ..., e_{0m}$.*

Proof. The proof is by induction in m. For $m = 2$ this follows from Lemma 7.3. For $m > 2$, we lift e_{01} to e_1 using Lemma 7.3, and then apply the induction assumption to the algebra $(1 - e_1)A(1 - e_1)$. □

7.3 Projective covers

Obviously, every projective module over an algebra A is a direct sum of indecomposable projective modules, so to understand projective modules over A, it suffices to classify indecomposable projective modules.

Let A be a finite dimensional algebra, with simple modules $M_1, ..., M_n$.

Theorem 7.5. *(i) For each $i = 1, ..., n$ there exists a unique indecomposable projective module P_i such that $\dim Hom(P_i, M_j) = \delta_{ij}$.*

(ii) $A = \oplus_{i=1}^n (\dim M_i)P_i$.

(iii) any indecomposable projective module over A is isomorphic to P_i for some i.

Proof. Recall that $A/\mathrm{Rad}(A) = \oplus_{i=1}^n \mathrm{End}(M_i)$, and $\mathrm{Rad}(A)$ is a nilpotent ideal. Pick a basis of M_i, and let $e_{ij}^0 = E_{jj}^i$, the rank 1 projectors projecting to the basis vectors of this basis ($j = 1, ..., \dim M_i$). Then e_{ij}^0 are orthogonal idempotents in $A/\mathrm{Rad}(A)$. So by Corollary 7.4 we can lift them to orthogonal idempotents e_{ij} in A. Now define $P_{ij} = Ae_{ij}$. Then $A = \oplus_i \oplus_{j=1}^{\dim M_i} P_{ij}$, so P_{ij} are projective. Also, we have $\mathrm{Hom}(P_{ij}, M_k) = e_{ij}M_k$, so $\dim \mathrm{Hom}(P_{ij}, M_k) = \delta_{ik}$. Finally, P_{ij} is independent of j up to an isomorphism, as e_{ij} for fixed i are conjugate under $A^\times$ by Proposition 7.3; thus we will denote P_{ij} by P_i.

We claim that P_i is indecomposable. Indeed, if $P_i = Q_1 \oplus Q_2$, then $\mathrm{Hom}(Q_l, M_j) = 0$ for all j either for $l = 1$ or for $l = 2$, so either $Q_1 = 0$ or $Q_2 = 0$.

Also, there can be no other indecomposable projective modules, since any indecomposable projective module has to occur in the decomposition of A. The theorem is proved. □