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BEYOND QUANTUM GROUPS AND HOPF ALGEBRAS: SUPERSYMMETRY AND SYMMETRY BREAKING IN QUANTUM FIELD THEORY AND QUANTUM GRAVITY

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Abstract. An Algebraic Topology approach to Supersymmetry (SUSY) and Symmetry Breaking in Quantum Field and Quantum Gravity theories is presented with a view to developing a wide range of physical applications, such as: controlled nuclear fusion and other nuclear reaction studies in quantum chromodynamics, nonlinear physics at high energy densities, dynamic Jahn-Teller effects, superfluidity, high temperature superconductors, multiple scattering by molecular systems, molecular or atomic paracrystal structures, nanomaterials, ferromagnetism in glassy materials, spin glasses, quantum phase transitions and supergravity. This approach requires a unified conceptual framework that utilizes extended symmetries and quantum groupoid, algebroid and functorial representations of non–Abelian higher dimensional structures pertinent to quantized spacetime topology and state space geometry of quantum operator algebras. Fourier transforms, generalized Fourier–Stieltjes transforms, and duality relations link, respectively, the quantum groups and quantum groupoids with their dual algebraic structures; quantum double constructions are also discussed in this context in relation to quasitriangular, quasiHopf algebras, bialgebroids, Grassmann-Hopf algebras and Higher Dimensional Algebra. On the one hand, this quantum algebraic approach is known to provide solutions to the quantum Yang-Baxter equation. On the other hand, our novel approach to extended quantum symmetries and their associated representations is shown to be relevant to locally covariant General Relativity theories that are consistent with nonlocal quantum field theories.

1. Introduction

The theory of scattering by partially ordered, atomic or molecular, structures in terms of *paracrystals* and *lattice convolutions* was formulated in Hosemann and Bagchi (1962) using basic techniques of Fourier analysis and convolution products. A natural generalization of such molecular, partial symmetries and their corresponding analytical versions involves convolution algebras - a functional/distribution based theory that we will discuss in the context of a more general and original concept of a *convolution–algebroid of an extended symmetry groupoid of a paracrystal*, of

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any molecular or nuclear system, or indeed, any quantum system in general, including quantum fields and local quantum net configurations that are endowed with either partially disordered or 'completely' ordered structures. Further specific applications of the paracrystal theory to X-ray scattering, based on computer algorithms, programs and explicit numerical computations, were subsequently developed by the first author (Baianu, 1974) for one–dimensional paracrystals, or partially ordered lattices (Baianu, 1978) and other biological structures with partial structural disorder (Baianu, 1980). Such structures that are generally referred to as 'quasi-crystals' and paracrystals provide rather interesting physical examples of extended structural symmetries (cf. Hindeleh and Hosemann, 1988).

Further statistical analysis linked to structural symmetry and scattering theory considerations shows that a real paracrystal can be defined by a three dimensional convolution polynomial with a semi–empirically derived composition law, $*$, (Hosemann et al. 1981). As was shown in Baianu (1978)– supported with computed specific examples– several systems of convolution can be expressed analytically, thus allowing the numerical computation of X-ray, or neutron, scattering by partially disordered layer lattices via complex Fourier transforms of one-dimensional structural models using fast digital computers. The range of paracrystal theory applications is however much wider than the one-dimensional lattices with disorder, thus spanning very diverse non–crystalline systems, from metallic glasses and spin glasses to superfluids, high-temperature superconductors, and extremely hot anisotropic plasmas.

A salient, and well–fathomed concept from the mathematical perspective concerns that of a C^*-algebra of a (discrete) group (see e.g. Connes, 1994). The underlying vector space is that of complex valued functions with finite support, and the multiplication of the algebra is the fundamental *convolution product* which it is convenient for our purposes to write slightly differently from the common formula as

$$(f * g)(z) = \sum_{xy=z} f(x)g(y) \ , \tag{1.1}$$

and *-operation

$$f^*(x) = \overline{f(x^{-1})} \ . \tag{1.2}$$

The more usual expression of these formulas has a sum over the elements of the group. For topological groups, where the underlying vector space consists of continuous complex valued functions, this product requires the availability of some structure of measure and of measurable functions, with the sum replaced by an integral. Notice also that this algebra has an identity, the distribution function δ_1, which has value 1 on the identity 1 of the group, and has zero value elsewhere.

Given this convolution/distribution representation that combines crystalline ('perfect' or global–group, and/or group–like symmetries) with partial symmetries of paracrystals and glassy solids on the one hand, and also with non–commutative harmonic analysis (Mackey, 1992) on the other hand, we propose that several extended quantum symmetries can be represented algebraically in terms of certain structured *groupoids*, their *C^*–convolution quantum algebroids*, paragroup/ *quantized groups* and/or other more general mathematical structures that will be introduced in this report. It is already known that such extensions to groupoid and algebroid/coalgebroid symmetries require also a generalization of non–commutative harmonic analysis which involves certain Haar measures, generalized Fourier-Stieltjes transforms and certain categorical duality relationships representing very general mathematical symmetries as well. Proceeding from the abstract structures endowed with extended symmetries to numerical applications in quantum physics always involves representations through specification of concrete elements, objects and transformations. Thus, groupoid and functorial representations that generalize group representations in several, meaningful ways are key to linking abstract, quantum operator algebras and symmetry properties with actual numerical computations of quantum eigenvalues and their eigenstates, as well as a wide variety of numerical factors involved in computing quantum dynamics. The well-known connection between groupoid convolution representations and matrices (Weinstein, 1996) is only one of the several numerical computations made possibile via groupoid representations. A very promising approach to nonlinear (anharmonic) analysis of aperiodic quantum systems represented by rigged Hilbert space bundles may involve the computation of representation coefficients of Fourier-Stieltjes groupoid transforms that we will also discuss briefly in Section 7.

Currently, however, there are important aspects of quantum dynamics left out of the invariant, simplified picture provided by group symmetries and their corresponding representations of quantum operator algebras (Gilmore, 2005). Often physicists deal with such problems in terms of either spontaneous symmetry breaking or approximate symmetries that require underlying explanations or ad-hoc dynamic restrictions that are semi-empirical. A well-studied example of this kind is that of the dynamic Jahn-Teller effect and the corresponding 'theorem' (Ch. 21 on pp. 807–831, as well as p.735 of Abragam and Bleaney, 1970) which in its simplest form stipulates that a quantum state with electronic non–Kramers degeneracy may be unstable against small distortions of the surroundings, that would lower the symmetry of the crystal field and thus lift the degeneracy (i.e., cause observable splitting of the corresponding energy levels); this effect occurs in certain paramagnetic ion systems *via* dynamic distortions of the crystal field symmetries around paramagnetic or high-spin centers by moving ligands that are diamagnetic. The established physical explanation is that the Jahn-Teller coupling replaces a purely electronic degeneracy by a vibronic degeneracy (of *exactly the same* symmetry!). The dynamic, or spontaneous breaking of crystal field symmetry (for example, distortions of the octahedral or cubic symmetry) results in certain systems in the appearance of doublets of symmetry γ_3 or singlets of symmetry γ_1 or γ_2). Such dynamic systems could be locally expressed in terms of symmetry representations of a Lie algebroid, or globally in terms of a special Lie (or Lie–Weinstein) symmetry groupoid representations that can also take into account the spin exchange interactions between the Jahn-Teller centers exhibiting such quantum dynamic effects. Unlike the simple symmetries expressed by group representations, the latter can accommodate a much wider range of possible or approximate symmetries that are indeed characteristic of real, molecular systems with varying crystal field symmetry, as for example around certain transition ions dynamically bound to ligands in liquids where motional narrowing becomes very important. This well known example illustrates the importance of the interplay between symmetry and dynamics in quantum processes which is undoubtedly involved in many other instances including: *quantum chromodynamics, superfluidity, spontaneous symmetry breaking, quantum gravity and Universe dynamics.*

Therefore, the various interactions and interplay between the symmetries of quantum operator state space geometry and quantum dynamics at various levels leads to both algebraic and topological structures that are variable and complex, well beyond symmetry groups and well-studied group algebras (such as Lie algebras–see for example (Gilmore, 2005)). A unified treatment of quantum phenomena/dynamics and structures may thus become possible with the help of Algebraic Topology, non-Abelian treatments; such powerful mathematical tools are capable of revealing novel, fundamental aspects related to extended symmetries and quantum dynamics through a detailed analysis of the variable geometry of (quantum) operator algebra state spaces. At the center stage of non-Abelian Algebraic Topology are groupoid and algebroid structures with their internal and external symmetries (Weinstein, 1987) that allow one to treat physical spacetime structures and dynamics within an unified categorical, higher dimensional algebra framework (Brown, Glazebrook and Baianu, 2007). As already suggested in our previous report, the interplay between extended symmetries and dynamics generates higher dimensional structures of quantized spacetimes that exhibit novel properties not found in lower dimensional representations of groups, group algebras or Abelian groupoids.

It is also our intention here to explore, uncover, and then develop, new links between several important but seemingly distinct mathematical approaches to extended quantum symmetries that were not considered in previous reports.

2. Quantum Groups, Quantum Operator Algebras and Related Symmetries.

Quantum theories adopted a new lease of life post 1955 when von Neumann beautifully reformulated Quantum Mechanics (QM) in the mathematically rigorous context of Hilbert spaces and operator algebras. From a current physics perspective, von Neumann's approach to quantum mechanics has done however much more: it has not only paved the way to expanding the role of symmetry in physics, as for example with the Wigner-Eckhart theorem and its applications, but also revealed the fundamental importance in quantum physics of the state space geometry of (quantum) operator algebras.

The basic definition of von Neumann and Hopf algebras (see for example Majid, 1995), as well as the topological groupoid definition, are recalled in the Appendix to maintain a self-contained presentation. Subsequent developments of the quantum operator algebra were aimed at identifying more general quantum symmetries than those defined for example by symmetry groups, groups of unitary operators and Lie groups. Several fruitful quantum algebraic concepts were developed, such as: the Ocneanu *paragroups*-later found to be represented by Kac–Moody algebras, quantum 'groups' represented either as Hopf algebras or locally compact groups with Haar measure, 'quantum' groupoids represented as weak Hopf algebras, and so on. The Ocneanu paragroups case is particularly interesting as it can be considered as an extension through quantization of certain finite group symmetries to infinitely-dimensional von Neumann type II_1 algebras, and are, in effect, *'quantized groups'* that can be nicely constructed as Kac algebras; in fact, it was recently shown that a paragroup can be constructed from a crossed product by an outer action of a Kac algebra. This suggests a relation to categorical aspects of paragroups (rigid monoidal tensor categories) ([T-V], [Y2]). The strict symmetry of the group of (quantum) unitary operators is thus naturally extended through paragroups to the symmetry of the latter structure's unitary representations; furthermore, if a subfactor of the von Neumann algebra arises as a crossed product by a finite group action, the paragroup for this subfactor contains a very similar group structure to that of the original finite group, and also has a unitary representation theory similar to that of the original finite group. Last-but-not least, a paragroup yields a *complete invariant* for irreducible inclusions of AFD von Neumannn II_1 factors with finite index and finite depth (Theorem 2.6. of Sato, 2001). This can be considered as a kind of internal, 'hidden' quantum symmetry of von Neumann algebras.

On the other hand, unlike paragroups, quantum locally compact groups are not readily constructed as either Kac or Hopf C*-algebras. In recent years the techniques of Hopf symmetry and those of weak Hopf C*-algebras, sometimes called *quantum 'groupoids'* (cf Böhm et al.,1999), provide important tools–in addition to the paragroups– for studying the broader relationships of the Wigner fusion rules algebra, $6j$–symmetry (Rehren, 1997), as well as the study of the noncommutative symmetries of subfactors within the Jones tower constructed from finite index depth 2 inclusion of factors, also recently considered from the viewpoint of related Galois correspondences (Nikshych and Vainerman, 2000).

We shall proceed at first by pursuing the relationships between these mainly algebraic concepts and their extended quantum symmetries, also including relevant computation examples; then we shall consider several further extensions of symmetry and algebraic topology in the context of local quantum physics/algebraic quantum field theory, symmetry breaking, quantum chromodynamics and the development of novel supersymmetry theories of quantum gravity. In this respect one can also take spacetime 'inhomogeneity' as a criterion for the comparisons between physical, partial or local, symmetries: on the one hand, the example of paracrystals reveals thermodynamic disorder (entropy) within its own spacetime framework, whereas in spacetime itself, whatever the selected model, the inhomogeneity arises through (super) gravitational effects. More specifically, in the former case one has the technique of the generalized Fourier–Stieltjes transform (along with convolution and Haar measure), and in view of the latter, we may compare the resulting 'broken'/paracrystal–type symmetry with that of the supersymmetry predictions for weak gravitational fields (e.g., 'ghost' particles) along with the broken supersymmetry in the presence of intense gravitational fields. Another significant extension of quantum symmetries may result from the superoperator algebra/algebroids of Prigogine's quantum *superoperators* which are defined only for irreversible, infinite-dimensional systems (Prigogine, 1980).

2.1. **Solving Quantum Problems by Algebraic Methods: Applications to Molecular Structure, Quantum Chemistry and Quantum Theories.**

As already discussed in the Introduction, one often deals with continuity and continuous transformations in natural systems, be they physical, chemical or self-organizing. Such continuous 'symmetries' often have a special type of underlying continuous group, called a *Lie group*. Briefly, a *Lie group* G is generally considered having a (smooth) C^∞ manifold structure, and acts upon itself smoothly. Such a *globally smooth* structure is surprisingly simple in two ways: it always admits an *Abelian fundamental group*, and seemingly also related to this global property, it admits an associated, *unique*– as well as *finite*–*Lie algebra* that completely specifies *locally* the properties of the Lie group everywhere.

2.1.1. *The Finite Lie Algebra of Quantum Commutators and their Unique (*continuous*) Lie Groups.*

Lie algebras can greatly simplify quantum computations and the initial problem of defining the form and symmetry of the quantum Hamiltonian subject to boundary and initial conditions in the quantum system under consideration. However, unlike most regular abstract algebras, a Lie algebra is *not associative*, and it is in fact a *vector space* (Heynman and Lifschitz, 1958). It is also perhaps this feature that makes the Lie algebras somewhat compatible, or 'consistent', with quantum logics that are also thought to have *non–associative, non–distributive and non–commutative lattice* structures. Nevertheless, the need for 'quantizing' Lie algebras in the sense of a certain *non-commutative* 'deformation' apparently remains for a quantum system, especially if one starts with a 'classical' *Poisson* algebra (Landsman and Ramazan, 2001). This requirement remains apparently even for the generalized version of a Lie algebra, called a *Lie algebroid* (see its definition and related remarks in Sections 4 and 5).

The presence of Lie groups in many classical physics problems, in view of its essential *continuity* property and its *Abelian* fundamental group, is not surprising. What is surprising, however, at first sight, is the appearance of *Lie groups and Lie algebras* in the context of commutators of observable operators even in quantum systems *with no classical analogue observables such as the spin*, as– for example– the SU(2) and its corresponding, unique $\mathfrak{su}(2)$– algebra

As a result of quantization, one would expect to deal with an algebra such as the Hopf (quantum group) which is *associative*. On the other hand, the application of the correspondence principle to the simple, classical harmonic oscillator system leads to a quantized harmonic oscillator and remarkably simple *commutator* algebraic expressions, which correspond precisely to the definition of a Lie algebra. Furthermore, this (*Lie*) algebraic procedure of assembling the quantum Hamiltonian from simple observable operator commutators is readily extended to *coupled, quantum harmonic oscillators*, as shown in great detail in Fernandez and Castro (1996).

2.2. **Some basic examples.**

Example 2.1. *The Lie Algebra of a Quantum Harmonic Oscillator.* One wishes to solve the time-independent Schrödinger equations of motion in order to determine the stationary states of the quantum harmonic oscillator which has a quantum Hamiltonian of the form:

$$\mathbf{H} = (\frac{1}{2m}) \cdot P^2 + \frac{k}{2} \cdot X^2 \ , \tag{2.1}$$

where X and P are, respectively, the coordinate and conjugate momentum operators. X and P satisfy the Heisenberg commutation/'uncertainty' relations $[X, P] = i\hbar I$, where the identity operator I is employed to simplify notation. A simpler, equivalent form of the above Hamiltonian is obtained by defining physically dimensionless coordinate and momentum:

$$\mathbf{x} = (\frac{X}{\alpha}) \ , \ \mathbf{p} = (\frac{\alpha P}{\hbar}) \text{ and } \alpha = \sqrt{\frac{\hbar}{mk}} \ . \tag{2.2}$$

With these new dimensionless operators, $\mathbf{x}$ and $\mathbf{p}$, the quantum Hamiltonian takes the form:

$$\mathbf{H} = (\frac{\hbar\omega}{2}) \cdot (\mathbf{p}^2 + \mathbf{x}^2) \ , \tag{2.3}$$

which in units of $\hbar \cdot \omega$ is simply:

$$\mathbf{H}' = (\frac{1}{2}) \cdot (\mathbf{p}^2 + \mathbf{x}^2) \ . \tag{2.4}$$

The commutator of $\mathbf{x}$ with its conjugate operator $\mathbf{p}$ is simply $[\mathbf{x}, \mathbf{p}] = i$.

Next one defines the superoperators $S_{Hx} = [H, x] = -i \cdot p$, and $S_{Hp} = [H, p] = i \cdot \mathbf{x}$ that will lead to new operators that act as generators of a Lie algebra for this quantum harmonic oscillator. The eigenvectors Z of these superoperators are obtained by solving the equation $S_H \cdot Z = \zeta Z$, where ζ are the eigenvalues, and Z can be written as $(c_1 \cdot x + c_2 \cdot p)$. The solutions are

$$\zeta = \pm 1 \text{ ,and } c_2 = \mp i \cdot c_1 \ . \tag{2.5}$$

Therefore, the two eigenvectors of S_H can be written as:

$$a^\dagger = c_1 * (x - ip) \text{ ,and } a = c_1(x + ip) \ , \tag{2.6}$$

respectively for $\zeta = \pm 1$. For $c_1 = \sqrt{2}$ one obtains normalized operators H, a and $a\dagger$ that generate a 4–dimensional Lie algebra with commutators:

$$[H, a] = -a \ , \ [H, a^\dagger] = a^\dagger \ , \text{ and } [a, a^\dagger] = I \ . \tag{2.7}$$

The term **a** is called the *annihilation* operator and the term $a\dagger$ is called the *creation* operator. This Lie algebra is solvable and generates after repeated application of $a\dagger$ all the eigenvectors of the quantum harmonic oscillator:

$$\Phi_n = (\frac{(a\dagger)^n}{\surd(n!)}) \cdot \Phi_0 \ . \tag{2.8}$$

The corresponding, possible eigenvalues for the energy, derived then as solutions of the Schrödinger equations for the quantum harmonic oscillator are:

$$E_n = \hbar \cdot \omega(n + \frac{1}{2}) \ , \ \text{where } n = 0, 1, \ldots, N \ . \tag{2.9}$$

The position and momentum eigenvector coordinates can be then also computed by iteration from *(finite)* matrix representations of the *(finite)* Lie algebra, using, for example, a simple computer programme to calculate linear expressions of the annihilation and creation operators. For example, one can show analytically that:

$$[a, x^k] = (\frac{k}{\surd 2}) \cdot (x_{k-1}) \ . \tag{2.10}$$

One can also show by introducing a *coordinate* representation that the eigenvectors of the harmonic oscillator can be expressed as *Hermite polynomials* in terms of the coordinates. In the coordinate representation the quantum *Hamiltonian* and *bosonic* operators have, respectively, the simple expressions:

$$\begin{aligned} H &= (\frac{1}{2}) \cdot [-\frac{d^2}{dx^2}) + (x^2)] \ , \\ a &= (\frac{1}{\surd 2}) \cdot (x + \frac{d}{dx}) \ , \\ a\dagger &= (\frac{1}{\surd 2}) \cdot (x - \frac{d}{dx}) \ . \end{aligned} \tag{2.11}$$

The ground state eigenfunction normalized to unity is obtained from solving the simple first-order differential equation $a\Phi_0(x) = 0$ and which leads to the expression:

$$\Phi_0(x) = (\pi^{-\frac{1}{4}}) \cdot \exp(-\frac{x^2}{2}) \ . \tag{2.12}$$

By repeated application of the creation operator written as

$$a\dagger = (-\frac{1}{\surd 2}) \cdot (\exp(\frac{x^2}{2})) \cdot (\frac{d}{dx^2}) \cdot \exp(-\frac{x^2}{2}) \ , \tag{2.13}$$

one obtains the n-th level eigenfunction:

$$\Phi_n(x) = (\frac{1}{(\surd\pi)2^n n!)}) \cdot (\mathbf{He}_n(x)) \ , \tag{2.14}$$

where $\mathbf{He}_n(x)$ is *the Hermite polynomial* of order n . With the special generating function of the Hermite polynomials

$$F(t, x) = (\pi^{-\frac{1}{4}}) \cdot (\exp((-\frac{x^2}{2}) + tx - (\frac{t^2}{4})) \ , \tag{2.15}$$

one obtains explicit analytical relations between the eigenfunctions of the quantum harmonic oscillator and the above special generating function:

$$F(t, x) = \sum_{n=0} (\frac{t^n}{\surd(2^n \cdot n!)}) \cdot \Phi_n(x) \ . \tag{2.16}$$

Such applications of the Lie algebra, and the related algebra of the *bosonic* operators as defined above are quite numerous in theoretical physics, and especially for various quantum field carriers in QFT that are all *bosons*. (Please note also additional examples of special 'Lie' superalgebras for gravitational and other fields in Section 6, such as gravitons and Goldstone quanta that are all *bosons* of different spin values and *'Penrose homogeneity'*).

In the interesting case of a *two-mode* bosonic quantum system formed by the tensor (direct) product of *one-mode* bosonic states: $\mid m, n >:=\mid m > \otimes \mid n >$, one can generate a 3–dimensional Lie algebra in terms of *Casimir* operators. *Finite*– dimensional Lie algebras are far more tractable, or easier to compute, than those with an infinite basis set. For example, such a Lie algebra as the 3–dimensional one considered above for the two-mode, bosonic states is quite useful for numerical computations of vibrational (IR, Raman, etc.) spectra of two–mode, *diatomic* molecules, as well as the computation of scattering states. Other perturbative calculations for more complex quantum systems, as well as calculations of exact solutions by means of Lie algebras have also been developed (see for example Fernandez and Castro,1996).

Example 2.2. *The* SU(2) *Quantum Group.* Let us consider the structure of the ubiquitous quantum SU(2) group (Woronowicz 1987, Chaician and Demichev 1996). Here A is taken to be a C*–algebra generated by elements α and β subject to the relations:

$$\begin{aligned} \alpha\alpha^* + \mu^2\beta\beta^* = 1 \ , \ \alpha^*\alpha + \beta^*\beta = 1 \ , \\ \beta\beta^* = \beta^*\beta \ , \ \alpha\beta = \mu\beta\alpha \ , \ \alpha\beta^* = \mu\beta^*\alpha \ , \\ \alpha^*\beta = \mu^{-1}\beta\alpha^* \ , \ \alpha^*\beta^* = \mu^{-1}\beta^*\alpha^* \ , \end{aligned} \tag{2.17}$$

where $\mu \in [-1, 1]\backslash\{0\}$. In terms of the matrix

$$u = \begin{bmatrix} \alpha & -\mu\beta^* \\ \beta & \alpha^* \end{bmatrix} \tag{2.18}$$

the coproduct Δ is then given via

$$\Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj} \ . \tag{2.19}$$

Example 2.3. *The $SL_q(2)$ Hopf algebra.* The Hopf algebra $SL_q(2)$ is defined by the generators a, b, c, d and the following relations:

$$ba = qab \ , \ db = qbd \ , \ ca = qac \ , \ dc = qcd \ , bc = cb \ , \tag{2.20}$$

together with

$$adda = (q^{-1} - q)bc \ , \ adq^{-1}bc = 1 \ , \tag{2.21}$$

and

$$\Delta \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \otimes \begin{bmatrix} a & b \\ c & d \end{bmatrix} \ , \ \epsilon \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \ , \ S \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} d & -qb \\ -q^{-1}c & a \end{bmatrix} \ . \tag{2.22}$$

2.3. **Hopf algebras.** Firstly, a unital associative algebra consists of a linear space A together with two linear maps

$$\begin{aligned} m &: A \otimes A \longrightarrow A \ , \ (\textit{multiplication}) \\ \eta &: \mathbb{C} \longrightarrow A \ , \ (\textit{unity}) \end{aligned} \tag{2.23}$$

satisfying the conditions

$$\begin{aligned} m(m \otimes \mathbf{1}) &= m(\mathbf{1} \otimes m) \\ m(\mathbf{1} \otimes \eta) &= m(\eta \otimes \mathbf{1}) = \mathrm{id} \ . \end{aligned} \tag{2.24}$$

This first condition can be seen in terms of a commuting diagram :

$$\begin{array}{ccc} A \otimes A \otimes A & \xrightarrow{m \otimes \mathrm{id}} & A \otimes A \\ {\scriptstyle \mathrm{id} \otimes m}\downarrow & & \downarrow{\scriptstyle m} \\ A \otimes A & \xrightarrow{m} & A \end{array} \tag{2.25}$$

Next suppose we consider 'reversing the arrows', and take an algebra A equipped with a linear homorphisms $\Delta : A \longrightarrow A \otimes A$, satisfying, for $a, b \in A$:

$$\begin{aligned} \Delta(ab) &= \Delta(a)\Delta(b) \\ (\Delta \otimes \mathrm{id})\Delta &= (\mathrm{id} \otimes \Delta)\Delta \ . \end{aligned} \tag{2.26}$$

We call Δ a *comultiplication*, which is said to be *coasociative* in so far that the following diagram commutes

$$\begin{array}{ccc} A \otimes A \otimes A & \xleftarrow{\Delta \otimes \mathrm{id}} & A \otimes A \\ {\scriptstyle \mathrm{id} \otimes \Delta}\uparrow & & \uparrow{\scriptstyle \Delta} \\ A \otimes A & \xleftarrow{\Delta} & A \end{array} \tag{2.27}$$

There is also a counterpart to η, the *counity* map $\varepsilon : A \longrightarrow \mathbb{C}$ satisfying

$$(\mathrm{id} \otimes \varepsilon) \circ \Delta = (\varepsilon \otimes \mathrm{id}) \circ \Delta = \mathrm{id} \ . \tag{2.28}$$

A *bialgebra* $(A, m, \Delta, \eta, \varepsilon)$ is a linear space A with maps $m, \Delta, \eta, \varepsilon$ satisfying the above properties.

Now to recover anything resembling a group structure, we must append such a bialgebra with an antihomomorphism $S : A \longrightarrow A$, satisfying $S(ab) = S(b)S(a)$, for $a, b \in A$. This map is defined implicitly via the property :

$$m(S \otimes \mathrm{id}) \circ \Delta = m(\mathrm{id} \otimes S) \circ \Delta = \eta \circ \varepsilon \ . \tag{2.29}$$

We call S the *antipode map*. A *Hopf algebra* is then a bialgebra $(A, m, \eta, \Delta, \varepsilon)$ equipped with an antipode map S .

Commutative and noncommutative Hopf algebras form the backbone of quantum 'groups' and are essential to the generalizations of symmetry. Indeed, in most respects a quantum 'group' is identifiable with a Hopf algebra. When such algebras are actually associated with proper groups of matrices there is considerable scope for their representations on both finite and infinite dimensional Hilbert spaces.

2.4. **Quasi-Hopf algebra.** A quasi-Hopf algebra is an extension of a Hopf algebra. Thus, a quasi-Hopf algebra is a *quasi-bialgebra* $\mathcal{B}_{\mathcal{H}} = (\mathcal{H}, \Delta, \varepsilon, \Phi)$ for which there exist $\alpha, \beta \in \mathcal{H}$ and a bijective antihomomorphism S (the 'antipode') of $\mathcal{H}$ such that

$$\sum_i S(b_i)\alpha c_i = \varepsilon(a)\alpha, \tag{2.30}$$

$$\sum_i b_i \beta S(c_i) = \varepsilon(a)\beta \tag{2.31}$$

for all $a \in \mathcal{H}$ and with

$$\Delta(a) = \sum_i b_i \otimes c_i, \tag{2.32}$$

and

$$\sum_i X_i \beta S(Y_i) \alpha Z_i = \mathbf{I}, \tag{2.33}$$

$$\sum_j S(P_j)\alpha Q_j \beta S(R_j) = \mathbf{I}, \tag{2.34}$$

where the expansions for the quantities Φ and Φ^{-1} are given by

$$\Phi = \sum_i X_i \otimes Y_i \otimes Z_i, \tag{2.35}$$

and

$$\Phi^{-1} = \sum_j P_j \otimes Q_j \otimes R_j. \tag{2.36}$$

As in the general case of a quasi-bialgebra, the property of being quasi-Hopf is unchanged by "twisting". Thus, "twisting" the comultiplication of a coalgebra

$$\mathcal{C} = (C, \Delta, \epsilon)$$

over a field k produces another coalgebra $\mathcal{C}^{cop}$; because the latter is considered as a vector space over the field k, the new comultiplication of $\mathcal{C}^{cop}$ (obtained by "twisting") is defined by

$$\Delta^{cop}(c) = \sum c_{(2)} \otimes c_{(1)},$$

with $c \in \mathcal{C}$ and

$$\Delta(c) = \sum c_{(1)} \otimes c_{(2)}.$$

Note also that the linear dual $\mathcal{C}^*$ of $\mathcal{C}$ is an algebra with unit ϵ and the multiplication being defined by

$$\langle c^* {*} d^*, c \rangle = \sum \left\langle c^*, c_{(1)} \right\rangle \left\langle d^*, c_{(2)} \right\rangle,$$

for $c^*, d^* \in \mathcal{C}^*$ and $c \in \mathcal{C}$ ([105]).

Quasi-Hopf algebras emerged from studies of Drinfel'd twists and also from F-matrices associated with finite-dimensional irreducible representations of quantum affine algebra. Thus, F-matrices were employed to factorize the corresponding R-matrix. In turn, this leads to applications in Statistical Quantum Mechanics, in the form of quantum affine algebras; their representations give rise to solutions of the Yang-Baxter equation. This provides solvability conditions for various quantum statistics models, allowing characteristics of such models to be derived from their corresponding quantum affine algebras. The study of F-matrices has been applied to models such as the so-called Heisenberg 'XXZ model', in the framework of the algebraic 'Bethe ansatz'. Thus F-matrices and quantum groups together with quantum affine algebras provide an effective framework for solving two-dimensional integrable models by using the Quantum Inverse Scattering method as suggested by Drinfel'd and other authors.

"Heisenberg XXZ model and quantum Galilei group." F Bonechi et al 1992 J. Phys. A: Math. Gen. 25 L939-L943 doi: 10.1088/0305-4470/25/15/007

F Bonechi, E Celeghini, R Giachetti, E Sorace and M Tarlini Dipartimento di Fisica, Firenze Univ., Italy Abstract. The 1D Heisenberg spin model with an **anisotropy of the XXZ type** is analysed in terms of the symmetry given by the quantum Galilei group $Gamma_q(1)$. For a chain with an infinite number of sites the authors show that the magnon excitations and S=1/2, n-magnon bound states are determined by the algebra. In this case the $Gamma_q(1)$ symmetry provides a description naturally compatible with the Bethe ansatz. The recurrence relations determined by $Gamma_q(1)$ permit one to express the energy of the n-magnon bound states in a closed form in terms of Tchebischeff polynomials.

Print publication: Issue 15 (7 August 1992) The Quantum Galilei Group Authors: S. Giller, C. Gonera, P. Kosinski, P. Maslanka (Submitted on 5 May 1995) Abstract: The quantum Galilei group $G_\varkappa$ is defined. The bicrossproduct structure of $G_\varkappa$ and the corresponding Lie algebra is revealed. The projective representations for the two-dimensional quantum Galilei group are constructed. Comments: AMSTeX Subjects: Quantum Algebra (math.QA) Cite as: **arXiv:q-alg/9505007v1**

II. The deformed Galilean algebra

The deformed Galilean algebra g_χ can be obtained by a contraction procedure from the k-Poincaré algebra. We make the rescaling $P_0 \to P_0 c$, $Li \to Lic^{-1}$ and let $c \to \infty$ keeping $kc = \chi = const.$ The resulting structure reads :

$$[M_i, P_j] = i_{\epsilon ijk} P_k,$$
$$[M_i, P_0] = 0,$$
$$[Mi, Mj] = i\epsilon_{ijk} M_k,$$
$$[P_\mu, P_\nu] = 0,$$
$$[M_i, L_j] = i_{\epsilon ijk} L_k,$$
$$[L_i, P_0] = iP_i, [L_i, P_k] = 0,$$
$$[L_i, L_j] = 1/(4\chi^2)\epsilon_{ijk} P_k (P\dot{M})$$

,

$$\Delta M_i = M_i \otimes I + I \otimes M_i,$$
$$\Delta P_0 = P_0 \otimes I + I \otimes P0,$$
$$\Delta P_i = P_i \otimes e^{-P_0/2\chi} + e^{P_0/2\chi} \otimes P_0$$

...

$$S(P_\mu) = -P_\mu, S(M_i) = -M_i,$$
$$S(L_i) = -L_i - 3i/(2\chi) P_i$$

.

Note that this algebra is obtained by **contraction** in the strong deformation limit $k \to 0$. It seems that there exists no nonrelativistic limit

$(c \to \infty)$ with k being kept fixed [5]. The **Casimir operators** can be also obtained by contraction. They read :

$$C_1 = P^2,$$

$$C_2 = P^2/(4\chi^2)(P \times M)^2 + (P \times L)^2.$$

Obviously, in the limit $\chi \to \infty$ we recover standard Galilean structure. As in the case of k-deformed Poincaré algebra [4] one can show that our algebra has a **bicrossproduct structure**. Thus, we obtain $g_\chi = T \triangleright\!\triangleleft U(M,L)$, where $U(M,L)$ is the universal covering of the Lie algebra $e(3)$

$$[M_i, M_j] = i\epsilon_{ijk}M_k, [M_i, L_j] = i\epsilon_{ijk}L_k, [L_i, L_j] = 0,$$
$$\Delta M_i = M_i \otimes I + I \otimes M_i,$$
$$\Delta L_i = L_i \otimes I + I \otimes L_i,$$

(5) whereas T is defined by

$$[P_\mu, P_\nu] = 0,$$
$$\Delta P_0 = P_0 \otimes I + I \otimes P_0, \Delta P_i = P_i e^{-P_0/(\chi)} + I \otimes P_i.$$

2.5. Quasitriangular Hopf algebra.

Definition 2.1. A Hopf algebra, H, is called *quasitriangular* if there is an invertible element, R, of $H \otimes H$ such that:

(1) $R\,\Delta(x) = (T \circ \Delta)(x)\,R$ for all $x \in H$, where Δ is the coproduct on H, and the linear map $T : H \otimes H \to H \otimes H$ is given by

$$T(x \otimes y) = y \otimes x, \qquad (2.37)$$

(2) $(\Delta \otimes 1)(R) = R_{13}\,R_{23}$,
(3) $(\mathbf{1} \otimes \Delta)(R) = R_{13}\,R_{12}$, where $R_{12} = \phi_{12}(R)$,
(4) $R_{13} = \phi_{13}(R)$, and $R_{23} = \phi_{23}(R)$, where $\phi_{12} : H \otimes H \to H \otimes H \otimes H$,
(5) $\phi_{13} : H \otimes H \to H \otimes H \otimes H$, and $\phi_{23} : H \otimes H \to H \otimes H \otimes H$, are algebra morphisms determined by

$$\begin{aligned} \phi_{12}(a \otimes b) &= a \otimes b \otimes 1, \\ \phi_{13}(a \otimes b) &= a \otimes 1 \otimes b, \\ \phi_{23}(a \otimes b) &= 1 \otimes a \otimes b. \end{aligned} \qquad (2.38)$$

R is called the *R-matrix*.

An important part of the above algebra can be summarized in the following commutative diagrams involving the algebra morphisms, the coproduct on H and the identity map id :

$$\begin{array}{ccc} H \otimes H \otimes H & \xleftarrow{\phi_{12},\phi_{13}} & H \otimes H \\ {\scriptstyle \mathrm{id}\otimes\mathrm{id}\otimes\Delta}\uparrow & & \uparrow{\scriptstyle \Delta} \\ H \otimes H \otimes H & \xleftarrow{\phi_{23},\mathrm{id}\otimes\Delta} & H \otimes H \end{array} \qquad (2.39)$$

and

$$\begin{array}{ccc} H \otimes H \otimes H & \xleftarrow{\Delta\otimes \mathrm{id}} & H \otimes H \\ {\scriptstyle \mathrm{id}\otimes\Delta}\uparrow & & \uparrow{\scriptstyle \Delta} \\ H \otimes H & \xleftarrow{\Delta} & H \end{array} \tag{2.40}$$

Because of this property of quasitriangularity, the R-matrix, R, becomes a solution of the Yang-Baxter equation; thus, a module M of H can be used to determine quasi-invariants of links, braids, knots and higher dimensional structures with similar quantum symmetries. Furthermore, as a consequence of the property of quasitriangularity, one has that:

$$(\epsilon \otimes 1)R = (1 \otimes \epsilon)R = 1 \in H. \tag{2.41}$$

Finally, one also has:

$$R^{-1} = (S \otimes 1)(R), \tag{2.42}$$

$$R = (1 \otimes S)(R^{-1}), \tag{2.43}$$

and

$$(S \otimes S)(R) = R. \tag{2.44}$$

One can also prove that the antipode S is e a linear isomorphism, and therefore S^2 is an automorphism: S^2 is obtained by conjugating by an invertible element, $S(x) = uxu^{-1}$, with

$$u = m(S \otimes 1)R^{21}. \tag{2.45}$$

By employing Drinfel'd's quantum double construction one can assemble a quasitriangular Hopf algebra from a Hopf algebra and its dual.

2.5.1. *Twisting a quasi-triangular Hopf algebra.* The property of being a quasi-triangular Hopf algebra is left unchanged by 'twisting *via* an invertible element' $F = \sum_i f^i \otimes f_i \in \mathcal{A} \otimes \mathcal{A}$ such that $(\varepsilon \otimes id)F = (id \otimes \varepsilon)F = 1$, and also such that the following cocycle condition is satisfied:

$(F \otimes 1) \circ (\Delta \otimes id)F = (1 \otimes F) \circ (id \otimes \Delta)F$. Moreover, $u = \sum_i f^i S(f_i)$ is invertible and the twisted antipode is given by $S'(a) = uS(a)u^{-1}$, with the twisted comultiplication, R-matrix and co-unit change according to those defined for the quasi-triangular Quasi-Hopf algebra. Such a twist is known as an admissible (or Drinfel'd) twist.

2.6. **Quasi-triangular Quasi-Hopf algebra–QTQH.** A quasi-triangular quasi-Hopf algebra is an extended form of a quasi-Hopf algebra (defined by Drinfel'd in 1989), and also of a quasi-triangular Hopf algebra. Thus, a quasi-triangular quasi-Hopf algebra is defined as a *quintuple* $\mathcal{B}_{\mathcal{H}} = (\mathcal{H}, R, \Delta, \varepsilon, \Phi)$, with

$$\mathcal{B}_{\mathcal{H}} = (\mathcal{H}, \Delta, \varepsilon, \Phi), \tag{2.46}$$

being a quasi-Hopf algebra, and $R \in \mathcal{H} \otimes \mathcal{H}$ being known as the R-matrix (as defined above), which is an invertible element such that:

$$\begin{gathered} R\Delta(a) = \sigma \circ \Delta(a)R, a \in \mathcal{H} \\ \sigma : \mathcal{H} \otimes \mathcal{H} \to \mathcal{H} \otimes \mathcal{H} \quad , \\ x \otimes y \to y \otimes x, \end{gathered} \tag{2.47}$$

so that σ is the switch map and

$$(2.48)\quad \begin{aligned} (\Delta \otimes id)R &= \Phi_{321} R_{13} \Phi_{132}^{-1} R_{23} \Phi_{123} \\ (id \otimes \Delta)R &= \Phi_{231}^{-1} R_{13} \Phi_{213} R_{12} \Phi_{123}^{-1} \\ \text{where } \Phi_{abc} &= x_a \otimes x_b \otimes x_c \\ \text{and } \Phi_{123} &= \Phi = x_1 \otimes x_2 \otimes x_3 \in \mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H}. \end{aligned}$$

The quasi-Hopf algebra becomes triangular if in addition we have $R_{21}R_{12} = 1$.

The twisting of $\mathcal{B}_{\mathcal{H}}$ by $F \in \mathcal{H} \otimes \mathcal{H}$ is the same as for a quasiHopf algebra, with the additional definition of the twisted R-matrix. A quasitriangular, quasiHopf algebra with $\Phi = 1$ is a quasitriangular Hopf algebra because the last two conditions in the definition above reduce to the quasitriangularity condition for a Hopf algebra. Therefore, just as in the case of the twisting of a quasiHopf algebra, the property of being quasi-triangular of a quasiHopf algebra is preserved by twisting.

2.7. **Yang–Baxter equations.**

2.7.1. *Parameter-dependent Yang–Baxter equation.* Consider A to be an unital associative algebra.Then, the parameter-dependent Yang–Baxter equation is an equation for $R(u)$, the parameter–dependent invertible element of the tensor product $A \otimes A$ (here, u is the parameter, which usually ranges over all real numbers in the case of an additive parameter, or over all positive real numbers in the case of a multiplicative parameter). The Yang–Baxter equation is usually stated as:

$$R_{12}(u)\ R_{13}(u+v)\ R_{23}(v) = R_{23}(v)\ R_{13}(u+v)\ R_{12}(u), \tag{2.49}$$

for all values of u and v, in the case of an additive parameter, and

$$R_{12}(u)\ R_{13}(uv)\ R_{23}(v) = R_{23}(v)\ R_{13}(uv)\ R_{12}(u), \tag{2.50}$$

for all values of u and v, in the case of a multiplicative parameter, where

$$\begin{aligned} R_{12}(w) &= \phi_{12}(R(w)) \\ R_{13}(w) &= \phi_{13}(R(w)) \\ R_{23}(w) &= \phi_{23}(R(w)) \end{aligned} \tag{2.51}$$

for all values of the parameter w, and

$$\begin{aligned} \phi_{12} &: H \otimes H \to H \otimes H \otimes H \\ \phi_{13} &: H \otimes H \to H \otimes H \otimes H \\ \phi_{23} &: H \otimes H \to H \otimes H \otimes H \end{aligned} \tag{2.52}$$

are algebra morphisms determined by the following (strict) conditions:

$$\begin{aligned} \phi_{12}(a \otimes b) &= a \otimes b \otimes 1 \\ \phi_{13}(a \otimes b) &= a \otimes 1 \otimes b \\ \phi_{23}(a \otimes b) &= 1 \otimes a \otimes b. \end{aligned} \tag{2.53}$$

2.7.2. *The Parameter–independent Yang–Baxter equation.* Let A be a unital associative algebra. The parameter–independent Yang–Baxter equation is an equation for R, an invertible element of the tensor product $A \otimes A$. The Yang–Baxter equation is

$$\begin{aligned} R_{12}\ R_{13}\ R_{23} &= R_{23}\ R_{13}\ R_{12} \\ \text{where } R_{12} &= \phi_{12}(R) \\ R_{13} &= \phi_{13}(R) \\ \text{and } R_{23} &= \phi_{23}(R). \end{aligned} \tag{2.54}$$

Let V be a module over A. Let $T : V \otimes V \to V \otimes V$ be the linear map satisfying $T(x \otimes y) = y \otimes x$ for all $x, y \in V$, then a representation of the braid group, B_n, can be constructed on $V^{\otimes n}$ by $\sigma_i = 1^{\otimes i-1} \otimes \check{R} \otimes 1^{\otimes n-i-1}$ for $i = 1, \ldots, n-1$, where $\check{R} = T \circ R$ on $V \otimes V$. This representation may thus be used to determine quasi–invariants of braids, knots and links.

2.8. **Generalization of the Quantum Yang-Baxter Equation.** The quantum Yang-Baxter equation was generalized by Lambe and Redford (1997) in ref. [105] to:

$$R = qb(\sum_{i=1}^{n} e_{ii} \otimes e_{ii}) + b(\sum_{i>j} e_{ii} \otimes e_{jj}) + c(\sum_{i<j} e_{ii} \otimes e_{jj}) + (qb - q^{-1}c)(\sum_{i>j} e_{ij} \otimes e_{ji}),$$

for $b, c \neq 0$. A solution of the quantum Yang-Baxter has the form $R : M \otimes M \to M \otimes M$, with M being a finite dimensional vector space over a field k. Most of the solutions are stated for a mathematical field but in many cases a commutative ring with unity may be sufficient instead.

2.9. $\mathrm{SU}(3), \mathrm{SU}(5), \mathrm{SU}(10)$ **and** E_6 **Representations in Quantum Chromodynamics and Unified Theories involving Spontaneous Symmetry Breaking.**

There have been several attempts to take into consideration extended quantum symmetries that would include, or 'embed', the SU(2) and SU(3) symmetries in larger symmetry groups such as $\mathrm{SU}(5), \mathrm{SU}(10)$ and the exceptional Lie group E_6, but so far with only limited success as their representations make several predictions that are so far unsupported by high energy physics experiments (Gilmore, 2003). To remove unobserved particles from such predictions one has invariably to resort to ad–hoc spontaneous symmetry breaking assumptions that would require still further explanations, and so on. So far the only thing that is certain is the fact that the $\mathrm{U}(1) \times \mathrm{SU}(2) \times \mathrm{SU}(3)$ symmetry is broken in nature, presumably in a 'spontaneous' manner. Due to the nonlocal character of quantum theories combined with the restrictions imposed by relativity on 'simultaneity' of events in different reference systems, a global or universal, 'spontaneous' symmetry breaking mechanism appears contrived, with the remaining possibility that it does however occur locally, thus resulting in quantum theories that use local approximations for broken symmetries, and thus they are not unified, as it was intended. On the other hand, in GR all interactions are local, and therefore spontaneous, local symmetry breaking may appear not to be a problem for GR, except for the major obstacle that it does severely limit the usefulness of the Lorentz group of transformations which would have to be modified accordingly to take into account the *local* $\mathrm{SU}(2) \times \mathrm{SU}(3)$ spontaneous symmetry breaking. This seems to cause problems with the GR's equivalence principle for all reference systems; the latter would give rise to an equivalence class, or possibly a set, of reference systems. On the other hand, local, spontaneous symmetry breaking generates a *groupoid of equivalence classes of reference systems*, and further, through quantization, to a category of groupoids of such reference systems, $\mathbf{Grpd}_{\mathfrak{R}}$, and their transformations defined as groupoid homomorphisms. Functor representations of $\mathbf{Grpd}_{\mathfrak{R}}$ into the category **BHilb** of rigged Hilbert spaces $\mathcal{H}_r$ would then allow the computation of *local* quantum operator eigenvalues and their eigenstates, in a manner invariant to the local, broken symmetry transformations. One might call such a theory, a locally covariant– quantized GR (lcq–GR), as it would be locally, but not necessarily, globally quantized. Obviously, such a locally covariant GR theory is consistent with AQFT and its operator nets of local quantum observables. Such an extension of the GR theory to a locally covariant GR in a quantized form may not require the 'universal' or global existence of Higgs bosons as a compelling property of the expanding Universe; thus, any lcq–GR theory can allow for the existence of inhomogeneities in spacetime caused by distinct local symmetries in the presence of very intense gravitational fields, dark matter, or other condensed quantum systems such as neutron stars and black holes (with or without 'hair'–cf. J. Wheeler). The GR principle of equivalence is then replaced in lcq–GR by the representations of the quantum fundamental groupoid functor that will be introduced in Section 9.

In view of the existing problems and limitations encountered with group quantum symmetries and their group (or group algebra) representations, current research on the geometry of state spaces of quantum operator algebras leads to extended symmetries expressed as topological groupoid representations that were shown to link back to certain C*–algebra representations. Such extended symmetries will be discussed in the next sections in terms of quantum groupoid representations involving the notion of measure Haar systems associated with locally compact quantum groupoids (as defined in the Appendix).

3. Quantum Groupoids and the Groupoid C*–Algebra

3.0.1. *Examples of Weak Hopf Algebras.*

(1) We refer here to Bais et al. (2002). Let G be a non–Abelian group and $H \subset G$ a discrete subgroup. Let $F(H)$ denote the space of functions on H and $\mathbb{C}H$ the group algebra (which consists of the linear span of group elements with the group structure). *The quantum double* $D(H)$ (Drinfel'd, 1987) is defined by

$$D(H) = F(H) \,\widetilde{\otimes}\, \mathbb{C}H \ , \tag{3.1}$$

where, for $x \in H$, the 'twisted tensor product' is specified by

$$\widetilde{\otimes} \mapsto \ (f_1 \otimes h_1)(f_2 \otimes h_2)(x) = f_1(x) f_2(h_1 x h_1^{-1}) \otimes h_1 h_2 \ . \tag{3.2}$$

The physical interpretation is often to take H as the 'electric gauge group' and $F(H)$ as the 'magnetic symmetry' generated by $\{f \otimes e\}$. In terms of the counit ε, the double $D(H)$ has a trivial representation given by $\varepsilon(f \otimes h) = f(e)$. We next look at certain features of this construction.

For the purpose of braiding relations there is an R matrix, $R \in D(H) \otimes D(H)$, leading to the operator

$$\mathcal{R} \equiv \sigma \cdot (\Pi^A_\alpha \otimes \Pi^B_\beta)(R) \ , \tag{3.3}$$

in terms of the Clebsch–Gordan series $\Pi^A_\alpha \otimes \Pi^B_\beta \cong N^{AB\gamma}_{\alpha\beta C} \ \Pi^C_\gamma$, and where σ denotes a flip operator. The operator $\mathcal{R}^2$ is sometimes called the *monodromy* or *Aharanov–Bohm phase factor.* In the case of a condensate in a state $|v\rangle$ in the carrier space of some representation Π^A_α one considers the maximal Hopf subalgebra T of a Hopf algebra A for which $|v\rangle$ is T–invariant; specifically :

$$\Pi^A_\alpha(P) \ |v\rangle = \varepsilon(P)|v\rangle \ , \ \forall P \in T \ . \tag{3.4}$$

(2) For the second example, consider $A = F(H)$. The algebra of functions on H can be broken to the algebra of functions on H/K, that is, to $F(H/K)$, where K is normal in H, that is, $HKH^{-1} = K$. Next, consider $A = D(H)$. On breaking a purely electric condensate $|v\rangle$, the magnetic symmetry remains unbroken, but the electric symmetry $\mathbb{C}H$ is broken to $\mathbb{C}N_v$, with $N_v \subset H$, the stabilizer of $|v\rangle$. From this we obtain $T = F(H)\widetilde{\otimes}\mathbb{C}N_v$.

(3) In Nikshych and Vainerman (2000) quantum groupoids (considered as weak C*–Hopf algebras, see below) were studied in relationship to the noncommutative symmetries of depth 2 von Neumann subfactors. If

$$A \subset B \subset B_1 \subset B_2 \subset \ldots \tag{3.5}$$

is the Jones extension induced by a finite index depth 2 inclusion $A \subset B$ of II_1 factors, then $Q = A' \cap B_2$ admits a quantum groupoid structure and acts on B_1, so that $B = B_1^Q$ and $B_2 = B_1 \rtimes Q$. Similarly, in Rehren (1997) 'paragroups' (derived from weak C*–Hopf algebras) comprise (quantum) groupoids of equivalence classes such as those associated with 6j–symmetry groups (relative to a fusion rules algebra). They correspond to type II von Neumann algebras in quantum mechanics, and arise as symmetries where the local subfactors (in the sense of containment of observables within fields) have depth 2 in the Jones extension. A related question is how a von Neumann algebra N, such as of finite index depth 2, sits inside a weak Hopf algebra formed as the crossed product $N \rtimes A$ (Böhm et al. 1999).

(4) Mack and Schomerus (1992) using a more general notion of the Drinfel'd construction, developed the notion of a *quasi triangular quasi–Hopf algebra* (QTQHA) with the aim of studying a range of essential symmetries with special properties, such as the quantum group algebra $\mathrm{U}_q(\mathrm{sl}_2)$ with $|q| = 1$. If $q^p = 1$, then it is shown that a QTQHA is canonically associated with $\mathrm{U}_q(\mathrm{sl}_2)$. Such QTQHAs are claimed as the true symmetries of minimal conformal field theories.

3.0.2. *The Weak Hopf C*–Algebra in Relation to Quantum Symmetry Breaking.* In our setting,a *Weak C*–Hopf algebra* is a weak *–Hopf algebra which admits a faithful *–representation on a Hilbert space. The weak C*–Hopf algebra is therefore much more likely to be closely related to a 'quantum groupoid' representation than any weak Hopf algebra. However, one can argue that locally compact groupoids equipped with a Haar measure (after quantization) come even closer to defining quantum groupoids. There are already several, significant examples that motivate the consideration of weak C*–Hopf algebras which also deserve mentioning in the context of 'standard' quantum theories. Furthermore, notions such as (proper) *weak C*–algebroids* can provide the main framework for symmetry breaking and quantum gravity that we are considering here. Thus, one may consider the quasi-group symmetries constructed by means of special transformations of the 'coordinate space' M. These transformations along with the coordinate space M define certain Lie groupoids, and also their infinitesimal version - the Lie algebroids $\mathbf{A}$, when the former are Weinstein groupoids. If one then lifts the algebroid action from M to the principal homogeneous space $\mathcal{R}$ over the cotangent bundle $T^*M \longrightarrow M$, one obtains a physically significant algebroid structure. The latter was called the Hamiltonian algebroid, $\mathcal{A}^H$, related to the Lie algebroid, $\mathbf{A}$. The Hamiltonian algebroid is an analog of the Lie algebra of symplectic vector fields with respect to the canonical symplectic structure on $\mathcal{R}$ or T^*M. In this recent example, the Hamiltonian algebroid, $\mathcal{A}^H$ over $\mathcal{R}$, was defined over the phase space of W_N–gravity, with the anchor map to Hamiltonians of canonical transformations (Levin and Olshanetsky, 2003,2008). Hamiltonian algebroids thus generalize Lie algebras of canonical transformations; canonical transformations of the Poisson sigma model phase space define a *Hamiltonian algebroid* with the Lie brackets related to such a Poisson structure on the target space. The Hamiltonian algebroid approach was utilized to analyze the symmetries of generalized deformations of complex structures on Riemann surfaces $\sum_{g,n}$ of genus g with n marked points. However, its implicit algebraic connections to von Neumann *–algebras and/or *weak C*–algebroid representations* have not yet been investigated. This example suggests that algebroid (quantum) symmetries are implicated in the foundation of relativistic quantum gravity theories and supergravity that we shall consider in further detail in Sections 6 to 9.

3.1. **Quantum Compact Groupoids (QCGd).**

4. Algebroids and Their Symmetries.

By an *algebroid structure* A we shall specifically mean also a ring, or more generally an algebra, but *with several objects* (instead of a single object), in the sense of Mitchell (1965). Thus, an algebroid has been defined in Mosa (1986) and Brown and Mosa (1986, 2008) as follows.

An *R-algebroid* A on a set of 'objects' A_0 is a directed graph over A_0 such that for each $x, y \in A_0$, $A(x, y)$ has an R-module structure and there is an R-bilinear function

$$\circ : A(x, y) \times A(y, z) \longrightarrow A(x, z) \tag{4.1}$$

where $(a, b) \mapsto a \circ b$ is the composition, that satisfies the associativity condition, and the existence of identities. A *pre-algebroid* has the same structure as an algebroid and the same axioms except for the fact that the existence of identities $1_x \in A(x, x)$ is not assumed. For example, if A_0 has exactly one object, then an R-algebroid A over A_0 is just an R-algebra. An ideal in A is then an example of a pre-algebroid. Let R be a commutative ring.

An *R-category* A is a category equipped with an R-module structure on each Hom set such that the composition is R-bilinear. More precisely, let us assume for instance that we are given a commutative ring R with identity. Then a small R-category– or equivalently an *R-algebroid*– will be defined as a category enriched in the monoidal category of R-modules, with respect to the monoidal structure of tensor product. This means simply that for all objects b, c of A, the set $A(b, c)$ is given the structure of an R-module, and composition $A(b, c) \times A(c, d) \longrightarrow A(b, d)$ is R–bilinear, or is a morphism of R-modules $A(b, c) \otimes_R A(c, d) \longrightarrow A(b, d)$.

If $\mathcal{G}$ is a groupoid (or, more generally, a category) then we can construct an *R-algebroid $R\mathcal{G}$* as follows. The object set of $R\mathcal{G}$ is the same as that of $\mathcal{G}$ and $R\mathcal{G}(b, c)$ is the free R-module on the set $\mathcal{G}(b, c)$, with composition given by the usual bilinear rule, extending the composition of $\mathcal{G}$.

Alternatively, we can define $\bar{R}\mathcal{G}(b, c)$ to be the set of functions $\mathcal{G}(b, c) \longrightarrow R$ with finite support, and then we define the *convolution product* as follows:

$$(f * g)(z) = \sum \{(fx)(gy) \mid z = x \circ y\} \ . \tag{4.2}$$

As is well known, it is the second construction which is natural for the topological case, when we need to replace 'function' by 'continuous function with compact support' (or *locally compact support* for the QFT extended symmetry sectors), and in this case $R \cong \mathbb{C}$. The point we are making here is that to make the usual construction and end up with an algebra rather than an algebroid, is a procedure analogous to replacing a groupoid $\mathcal{G}$ by a semigroup $G' = \mathcal{G} \cup \{0\}$ in which the compositions not defined in $\mathcal{G}$ are defined to be 0 in G'. We argue that this construction removes the main advantage of groupoids, namely the spatial component given by the set of objects.

At present, however, the question of how one can use categorical duality in order to find the analogue of the diagonal of a Hopf algebra remains open. Such questions require further work and also future development of the theoretical framework proposed here for extended symmetries and the related fundamental aspects of quantum field theories. Nevertheless, for Fourier–Stieltjes groupoid representations, there has already been substantial progress made (Paterson, 2003) with the specification of their dual Banach algebras (but not algebroids!), in a manner similar to the case of locally compact groups and their associated Fourier algebras. Such progress will be further discussed in Section 7.

Another related problem that we address here is to what extent the much studied theory of C^*–algebras and their representations can be naturally applied or extended to really novel applications in quantum physics such as those proposed in this report. This is indeed a moot point as the classification problem for C^*–algebra representations is more complex and appears much harder to solve than in the case of von Neumann algebra representations. On the other hand, the extended symmetry links that we shall discuss between locally compact groupoid unitary representations and their induced C^*–algebra representations warrant further careful consideration.

4.1. **The Weak C*–Hopf Algebroid and Its Symmetries.** Progressing to the next level of generality, let A denote an algebra with local identities in a commutative subalgebra $R \subset A$. We adopt the definition of a *Hopf algebroid structure on A over R* following Mrvcun (2002). Relative to a ground field $\mathbb{F}$ (typically $\mathbb{F} = \mathbb{C}$ or $\mathbb{R}$), the definition commences by taking three $\mathbb{F}$–linear maps, the *comultiplication*

$$\Delta : A \longrightarrow A \otimes_R A \ , \tag{4.3}$$

the counit

$$\varepsilon : A \longrightarrow R \ , \tag{4.4}$$

and the *antipode*

$$S : A \longrightarrow A \ , \tag{4.5}$$

such that :

(i) Δ and ε are homomorphisms of left R–modules satisfying $(\mathrm{id} \otimes \varepsilon) \circ \Delta = \mathrm{id}$ and $(\varepsilon \otimes \mathrm{id}) \circ \Delta = \mathrm{id}$.

(ii) $\varepsilon|_R = \mathrm{id}$, $\Delta|_R$ is the canonical embedding $R \cong R \otimes_R R \subset A \otimes_R A$, and the two right R–actions on $A \otimes_R A$ coincide on ΔA .

(iii) $\Delta(ab) = \Delta(a)\Delta(b)$ for any $a, b \in A$.

(iv) $S|_R = \mathrm{id}$ and $S \circ S = \mathrm{id}$.

(v) $S(ab) = S(a)S(b)$ for any $a, b \in A$.

(vi) $\mu \circ (S \otimes \mathrm{id}) \circ \Delta = \varepsilon \circ S$, where $\mu : A \otimes_R A \longrightarrow A$ denotes the multiplication.

If R is a commutative subalgebra with local identities, then a *Hopf algebroid over R* is a *quadruple* $(A, \Delta, \varepsilon, S)$ where A is an algebra which has R for a subalgebra and has local identities in R, and where (Δ, ε, S) is a Hopf algebroid structure on A over R . Our interest lies in the fact that a Hopf–algebroid comprises a (universal) enveloping algebra for a quantum 'groupoid', thus hinting either at an adjointness situation or duality between the Hopf-algebroid and such a quantum 'groupoid'.

Definition 4.1. Let $(A, \Delta, \varepsilon, S)$ be a Hopf algebroid as above. We say that $(A, \Delta, \varepsilon, S)$ is a *weak C*–Hopf algebroid* when

(1) A is a unital C*–algebra (with $\mathbf{1}$) . We set $\mathbb{F} = \mathbb{C}$.

(2) The comultiplication $\Delta : A \longrightarrow A \otimes A$ is a coassociative $*$–homomorphism. The counit is a positive linear map $\varepsilon : A \longrightarrow R$ satisfying the above compatibility condition. The antipode S is a complex–linear anti–homomorphism and anti–cohomorphism $S : A \longrightarrow A$ (that is, it reverses the order of the multiplication and comultiplication), and is inverted under the *–structure: $S^{-1}(a) = S(a^*)^*$.

(3)

$$\begin{aligned} \Delta(\mathbf{1}) &\equiv \mathbf{1}_{(1)} \otimes \mathbf{1}_{(2)} = \text{projection} \\ \varepsilon(ap) &= \varepsilon(a\mathbf{1}_{(1)}) \cdot \varepsilon(\mathbf{1}_{(2)}p) \\ S(a_{(1)})a_{(2)} \otimes a_{(3)} &= (\mathbf{1} \otimes a) \cdot \Delta(\mathbf{1}) \ . \end{aligned} \tag{4.6}$$

Here $a_{(1)} \otimes a_{(2)}$ is shorthand notation for the expansion of $\Delta(a)$.

(4) The dual $\widehat{A}$ is defined by the linear maps $\hat{a} : A \longrightarrow \mathbb{C}$. The structure of $\widehat{A}$ is canonically dualized via the pairing and $\widehat{A}$ is endowed with a dual *–structure via $\langle \hat{a}^*, a \rangle_A = \overline{\langle \hat{a}, S(a)^* \rangle_A}$. Further, $(\widehat{A}, \widehat{\Delta}, \hat{\varepsilon}, \widehat{S})$ with $*$ and $\varepsilon = \widehat{\mathbf{1}}$, is a weak C*–Hopf algebroid.

5. Comparing Groupoid and Algebroid Quantum Symmetries: Weak Hopf C*–Algebroid vs. Locally Compact Quantum Groupoid Symmetry.

At this stage we make a comparison between the Lie group 'classic' symmetries discussed in Section 2 and a schematic representation for the extended groupoid and algebroid symmetries considered in Sections 3 and 4, as follows :

Standard Classical and Quantum Group/Algebra Symmetries :

Lie Groups $\Longrightarrow$ Lie Algebras $\Longrightarrow$ Universal Enveloping Algebra $\Longrightarrow$ Quantization $\rightarrow$ Quantum Group Symmetry (or Noncommutative (quantum) Geometry).

Extended Quantum, Groupoid and Algebroid, Symmetries :

Quantum Groupoid/Algebroid $\leftarrow$ Weak Hopf Algebras $\Longleftarrow$ Representations $\leftarrow$ Quantum Groups

Our intention here is to view the latter scheme in terms of *weak Hopf C*–algebroid*– and/or other–extended symmetries, which we propose to do, for example, by incorporating the concepts of *rigged Hilbert spaces* and *sectional functions for a small category.* We note, however, that an alternative approach to quantum 'groupoids' has already been reported (Maltsiniotis, 1992), (perhaps also related to noncommutative geometry); this was later expressed in terms of deformation-quantization: the Hopf algebroid deformation of the universal enveloping algebras of Lie algebroids (Xu, 1997) as the classical limit of a quantum 'groupoid'; this also parallels the introduction of quantum 'groups' as the deformation-quantization of Lie bialgebras. Furthermore, such a Hopf algebroid approach (Lu, 1996) leads to categories of Hopf algebroid modules (Xu, 1997) which are monoidal, whereas the links between Hopf algebroids and monoidal bicategories were investigated by Day and Street (1997).

As defined in the Appendix, let (G_{lc}, τ) be a locally compact groupoid endowed with a (left) Haar system, and let $A = C^*(\mathsf{G}_{lc}, \tau)$ be the convolution C^*–algebra (we append A with $\mathbf{1}$ if necessary, so that A is unital). Then consider such a *groupoid representation* $\Lambda : (\mathsf{G}_{lc}, \tau) \longrightarrow \{\mathcal{H}_x, \sigma_x\}_{x \in X}$ that respects a compatible measure σ_x on $\mathcal{H}_x$ (cf Buneci, 2003). On taking a state ρ on A, we assume a parametrization

$$(\mathcal{H}_x, \sigma_x) := (\mathcal{H}_\rho, \sigma)_{x \in X} \ . \tag{5.1}$$

Furthermore, each $\mathcal{H}_x$ is considered as a *rigged Hilbert space* Bohm and Gadella (1989), that is, one also has the following nested inclusions:

$$\Phi_x \subset (\mathcal{H}_x, \sigma_x) \subset \Phi_x^{\times} \ , \tag{5.2}$$

in the usual manner, where Φ_x is a dense subspace of $\mathcal{H}_x$ with the appropriate locally convex topology, and $\Phi_x^{\times}$ is the space of continuous antilinear functionals of Φ . For each $x \in X$, we require Φ_x to be invariant under Λ and Im $\Lambda | \Phi_x$ is a continuous representation of G_{lc}on Φ_x . With these conditions, representations of (proper) quantum groupoids that are derived for weak C*–Hopf algebras (or algebroids) modeled on rigged Hilbert spaces could be suitable generalizations in the framework of a Hamiltonian generated semigroup of time evolution of a quantum system via integration of Schrödinger's equation $\iota\hbar\frac{\partial\psi}{\partial t} = H\psi$ as studied in the case of Lie groups (Wickramasekara and Bohm, 2006). The adoption of the rigged Hilbert spaces is also based on how the latter are recognized as reconciling the Dirac and von Neumann approaches to quantum theories (Bohm and Gadella, 1989).

Next let G_{lc} be a locally compact Hausdorff groupoid and X a locally compact Hausdorff space. In order to achieve a small C*–category we follow a suggestion of A. Seda (private communication) by using a general principle in the context of Banach bundles (Seda, 1976, 982)). Let $q = (q_1, q_2) : \mathsf{G}_{lc} \longrightarrow X \times X$ be a continuous, open and surjective map. For each $z = (x, y) \in X \times X$, consider the fibre $\mathsf{G}_{\mathsf{z}} = \mathsf{G}_{lc}(x, y) = q^{-1}(z)$, and set $\mathcal{A}_z = C_0(\mathsf{G}_{\mathsf{z}}) = C_0(\mathsf{G}_{lc})$ equipped with a uniform norm $\| \ \|_z$. Then we set $\mathcal{A} = \bigcup_z \mathcal{A}_z$. We form a Banach bundle $p : \mathcal{A} \longrightarrow X \times X$ as follows. Firstly, the projection is defined via the typical fibre $p^{-1}(z) = \mathcal{A}_z = \mathcal{A}_{(x,y)}$. Let $C_c(\mathsf{G}_{lc})$ denote the continuous complex valued functions on G_{lc} with compact support. We obtain a sectional function $\widetilde{\psi} : X \times X \longrightarrow \mathcal{A}$ defined via restriction as $\widetilde{\psi}(z) = \psi | \ \mathsf{G}_{\mathsf{z}} = \psi | \mathsf{G}_{lc}$. Commencing from the vector space $\gamma = \{\widetilde{\psi} : \psi \in C_c(\mathsf{G}_{lc})\}$, the set $\{\widetilde{\psi}(z) : \widetilde{\psi} \in \gamma\}$ is dense in $\mathcal{A}_z$. For each $\widetilde{\psi} \in \gamma$, the function $\|\widetilde{\psi}(z)\|_z$ is continuous on X, and each $\widetilde{\psi}$ is a continuous section of $p : \mathcal{A} \longrightarrow X \times X$. These facts follow from Seda (1982, Theorem 1). Furthermore, under the convolution product $f * g$, *the space* $C_c(\mathsf{G}_{lc})$ *forms an associative algebra over* $\mathbb{C}$ (cf. Seda, 1982, Theorem 3).

Definition 5.1. The data proposed for a *weak C*–Hopf symmetry* consists of:

(1) A Weak C*–Hopf Algebroid $(A, \Delta, \varepsilon, S)$, where as above, $A = C^*(\mathcal{G}, \tau)$ is constructed via sectional functions over a small category.

(2) A family of GNS representations

$$(\pi_\rho)_x : A \longrightarrow (\mathcal{H}_\rho)_x := \mathcal{H}_x \ , \tag{5.3}$$

where for each, $x \in X$, $\mathcal{H}_x$ is a rigged Hilbert space.

5.1. **Grassmann–Hopf Algebra and the Grassmann–Hopf Algebroid.** Let V be a (complex) vector space ($\dim_{\mathbb{C}} V = n$) and let $\{e_0, e_1, \ldots,\}$ with identity $e_0 \equiv 1$, be the generators of a Grassmann (exterior) algebra

$$\Lambda^* V = \Lambda^0 V \oplus \Lambda^1 V \oplus \Lambda^2 V \oplus \cdots \tag{5.4}$$

subject to the relation $e_i e_j + e_j e_i = 0$. Following Fauser (2004) we append this algebra with a Hopf structure to obtain a 'co–gebra' based on the interchange (or '*tangled duality*'):

$$(\textit{objects/points}, \ \textit{morphisms}) \mapsto (\textit{morphisms}, \ \textit{objects/points})$$

This leads to a *tangle duality* between

(i) the binary product $A \otimes A \xrightarrow{m} A$, and

(ii) the coproduct $C \xrightarrow{\Delta} C \otimes C$.

where the Sweedler notation (Sweedler, 1996), with respect to an arbitrary basis is adopted:

$$\Delta(x) = \sum_r a_r \otimes b_r = \sum_{(x)} x_{(1)} \otimes x_{(2)} = x_{(1)} \otimes x_{(2)}$$

$$\Delta(x^i) = \sum_i \Delta_i^{jk} = \sum_{(r)} a^j_{(r)} \otimes b^k_{(r)} = x_{(1)} \otimes x_{(2)}$$

Here the Δ_i^{jk} are called 'section coefficients'. We have then a generalization of associativity to coassociativity

$$\begin{array}{ccc} C & \xrightarrow{\Delta} & C \otimes C \\ \Big\downarrow{\scriptstyle \Delta} & & \Big\downarrow{\scriptstyle \mathrm{id} \otimes \Delta} \\ C \otimes C & \xrightarrow{\Delta \otimes \mathrm{id}} & C \otimes C \otimes C \end{array} \tag{5.5}$$

inducing a tangled duality between an associative (unital algebra $\mathcal{A} = (A, m)$, and an associative (unital) 'co–gebra' $\mathcal{C} = (C, \Delta)$. The idea is to take this structure and combine the Grassmann algebra $(\Lambda^* V, \wedge)$ with the 'co-gebra' $(\Lambda^* V, \Delta_\wedge)$ (the 'tangled dual') along with the Hopf algebra compatibility rules: 1) the product and the unit are 'co–gebra' morphisms, and 2) the coproduct and counit are algebra morphisms.

Next we consider the following ingredients:

(1) the graded switch $\hat{\tau}(A \otimes B) = (-1)^{\partial A \partial B} B \otimes A$

(2) the counit ε (an algebra morphism) satisfying $(\varepsilon \otimes \mathrm{id})\Delta = \mathrm{id} = (\mathrm{id} \otimes \varepsilon)\Delta$

(3) the antipode S .

The *Grassmann–Hopf algebra* $\widehat{H}$ thus consists of the *septet* $\widehat{H} = (\Lambda^* V, \wedge, \mathrm{id}, \varepsilon, \hat{\tau}, S)$.

Its generalization to a *Grassmann–Hopf algebroid* is straightforward by considering a groupoid G, and then defining a *$H^\wedge$–Algebroid* as a *quadruple* $(HG, \Delta, \varepsilon, S)$ by modifying the Hopf algebroid definition so that $\widehat{H} = (\Lambda^* V, \wedge, \mathrm{id}, \varepsilon, \hat{\tau}, S)$ satisfies the standard Grassmann–Hopf algebra axioms stated above. We may also say that $(HG, \Delta, \varepsilon, S)$ is a *weak C*–Grassmann–Hopf algebroid* when $H^\wedge$ is a unital C*–algebra (with $\mathbf{1}$) . We thus set $\mathbb{F} = \mathbb{C}$. Note however that the tangled-duals of Grassman–Hopf algebroids retain the intuitive interactions/dynamic diagram advantages of their physical, extended symmetry representations exhibited by the Grassman-Hopf al/gebras and co-gebras over those of either weak C*- Hopf algebroids or weak Hopf C*–algebras.

6. Non–Abelian Algebroid Representations of Quantum State Space Geometry in Quantum Supergravity Fields.

Supergravity, in essence, is an extended supersymmetric theory of both matter and gravitation Weinberg (1995). A first approach to supersymmetry relies on a curved 'superspace' (Wess and Bagger, 1983) and is analogous to supersymmetric gauge theories (see, for example, Sections 27.1 to 27.3 of Weinberg, 1995). Unfortunately, a complete non–linear supergravity theory might be forbiddingly complicated and furthermore, the constraints that need be made on the graviton superfield appear somewhat subjective, according to Weinberg (1995). On the other hand, the second approach to supergravity is much more transparent than the first, albeit theoretically less elegant. The physical components of the gravitational superfield can be identified in this approach based on flat-space superfield methods (Chs. 26 and 27 of Weinberg, 1995). By implementing the weak-field approximation one obtains several of the most important consequences of supergravity theory, including masses for the hypothetical gravitino and gaugino 'particles' whose existence is expected from supergravity theories. Furthermore, by adding on the higher order terms in the gravitational constant to the supersymmetric transformation, the general coordinate transformations form a closed algebra and the Lagrangian that describes the interactions of the physical fields is invariant under such transformations. Quantization of such a flat-space superfield would obviously involve its 'deformation' as discussed in Section 2 above, and as a result its corresponding *supersymmetry algebra* would become *non–commutative*.

6.1. The Metric Superfield.

Because in supergravity both spinor and tensor fields are being considered, the gravitational fields are represented in terms of *tetrads*, $e^a_\mu(x)$, rather than in terms of the general relativistic metric $g_{\mu\nu}(x)$. The connections between these two distinct representations are as follows:

$$g_{\mu\nu}(x) = \eta_{ab}\ e^a_\mu(x) e^b_\gamma(x)\ , \tag{6.1}$$

with the general coordinates being indexed by μ, ν, etc., whereas local coordinates that are being defined in a locally inertial coordinate system are labeled with superscripts a, b, etc.; η_{ab} is the diagonal matrix with elements +1, +1, +1 and -1. The tetrads are invariant to two distinct types of symmetry transformations–the local Lorentz transformations:

$$e^a_\mu(x) \longmapsto \Lambda^a_b(x) e^b_\mu(x) \ , \tag{6.2}$$

(where Λ^a_b is an arbitrary real matrix), and the general coordinate transformations:

$$x^\mu \longmapsto (x')^\mu(x) \ . \tag{6.3}$$

In a weak gravitational field the tetrad may be represented as:

$$e^a_\mu(x) = \delta^a_\mu(x) + 2\kappa \Phi^a_\mu(x) \ , \tag{6.4}$$

where $\Phi^a_\mu(x)$ is small compared with $\delta^a_\mu(x)$ for all x values, and $\kappa = \sqrt{8\pi G}$, where G is Newton's gravitational constant. As it will be discussed next, the supersymmetry algebra (SA) implies that the graviton has a fermionic superpartner, the hypothetical *gravitino*, with helicities $\pm$ 3/2. Such a self-charge-conjugate massless particle as the gravitiono with helicities $\pm$ 3/2 can only have *low-energy* interactions if it is represented by a Majorana field $\psi_\mu(x)$ which is invariant under the gauge transformations:

$$\psi_\mu(x) \longmapsto \psi_\mu(x) + \delta_\mu \psi(x) \ , \tag{6.5}$$

with $\psi(x)$ being an arbitrary Majorana field as defined by Grisaru and Pendleton (1977). The tetrad field $\Phi_{\mu\nu}(x)$ and the graviton field $\psi_\mu(x)$ are then incorporated into a term $H_\mu(x,\theta)$ defined as the *metric superfield*. The relationships between $\Phi_{\mu_\nu}(x)$ and $\psi_\mu(x)$, on the one hand, and the components of the metric superfield $H_\mu(x,\theta)$, on the other hand, can be derived from the transformations of the whole metric superfield:

$$H_\mu(x,\theta) \longmapsto H_\mu(x,\theta) + \Delta_\mu(x,\theta) \ , \tag{6.6}$$

by making the simplifying– and physically realistic– assumption of a weak gravitational field (further details can be found, for example, in Ch.31 of vol.3. of Weinberg, 1995). The interactions of the entire superfield $H_\mu(x)$ with matter would be then described by considering how a weak gravitational field, h_{μ_ν} interacts with an energy-momentum tensor $T^{\mu\nu}$ represented as a linear combination of components of a real vector superfield Θ^μ. Such interaction terms would, therefore, have the form:

$$I_{\mathcal{M}} = 2\kappa \int dx^4 [H_\mu \Theta^\mu]_D \ , \tag{6.7}$$

($\mathcal{M}$ denotes 'matter') integrated over a four-dimensional (Minkowski) spacetime with the metric defined by the superfield $H_\mu(x,\theta)$. The term Θ^μ, as defined above, is physically a *supercurrent* and satisfies the conservation conditions:

$$\gamma^\mu \mathbf{D} \Theta_\mu = \mathbf{D} \ , \tag{6.8}$$

where $\mathbf{D}$ is the four-component super-derivative and X denotes a real chiral scalar superfield. This leads immediately to the calculation of the interactions of matter with a weak gravitational field as:

$$I_{\mathcal{M}} = \kappa \int d^4x T^{\mu\nu}(x) h_{\mu\nu}(x) \ , \tag{6.9}$$

It is interesting to note that the gravitational actions for the superfield that are invariant under the generalized gauge transformations $H_\mu \longmapsto H_\mu + \Delta_\mu$ lead to solutions of the Einstein field equations for a homogeneous, non-zero vacuum energy density ρ_V that correspond to either a de Sitter space for $\rho_V > 0$, or an anti-de Sitter space for $\rho_V < 0$. Such spaces can be represented in terms of the hypersurface equation

$$x_5^2 \pm \eta_{\mu,\nu} x^\mu x^\nu = R^2 \ , \tag{6.10}$$

in a quasi-Euclidean five-dimensional space with the metric specified as:

$$ds^2 = \eta_{\mu,\nu} x^\mu x^\nu \pm dx_5^2 \ , \tag{6.11}$$

with '+' for de Sitter space and '-' for anti-de Sitter space, respectively.

The spacetime symmetry groups, or groupoids –as the case may be– are different from the 'classical' Poincaré symmetry group of translations and Lorentz transformations. Such spacetime symmetry groups, in the simplest case, are therefore the $O(4,1)$ group for the de Sitter space and the $O(3,2)$ group for the anti–de Sitter space. A detailed calculation indicates that the transition from ordinary flat space to a bubble of anti-de Sitter space is *not* favored energetically and, therefore, the ordinary (de Sitter) flat space is stable (cf. Coleman and De Luccia, 1980), even though quantum fluctuations might occur to an anti–de Sitter bubble within the limits permitted by the Heisenberg uncertainty principle.

6.2. **Supersymmetry Algebras and Lie (Graded) Superalgebras.** It is well known that *continuous symmetry transformations* can be represented in terms of a *Lie algebra* of linearly independent *symmetry generators* t_j that satisfy the commutation relations:

$$[t_j, t_k] = \iota \Sigma_l C_{jk} t_l \ , \tag{6.12}$$

Supersymmetry is similarly expressed in terms of the symmetry generators t_j of a *graded ('Lie') algebra*– which is in fact defined as a *superalgebra*– by satisfying relations of the general form:

$$t_j t_k - (-1)^{\eta_j \eta_k} t_k t_j = \iota \Sigma_l C_{jk}^l t_l \ . \tag{6.13}$$

The generators for which $\eta_j = 1$ are fermionic whereas those for which $\eta_j = 0$ are bosonic. The coefficients C_{jk}^l are structure constants satisfying the following conditions:

$$C_{jk}^l = -(-1)^{\eta_j \eta_k} C_{jk}^l \ . \tag{6.14}$$

If the generators t_j are quantum Hermitian operators, then the structure constants satisfy the reality conditions $C_{jk}^* = -C_{jk}$. Clearly, such a graded algebraic structure is a superalgebra and not a proper Lie algebra; thus graded Lie algebras are often called *'Lie superalgebras'*.

The standard computational approach in QM utilizes the S-matrix approach, and therefore, one needs to consider the general, *graded* 'Lie algebra' of *supersymmetry generators* that commute with the S-matrix. If one denotes the fermionic generators by Q, then $U^{-1}(\Lambda) Q U(\Lambda)$ will also be of the same type when $U(\Lambda)$ is the quantum operator corresponding to arbitrary, homogeneous Lorentz transformations Λ^{μ_ν} . Such a group of generators provide therefore a representation of the homogeneous Lorentz group of transformations $\mathbb{L}$. The irreducible representation of the homogeneous Lorentz group of transformations provides therefore a classification of such individual generators.

6.2.1. *Graded 'Lie Algebras'/Superalgebras.* A set of quantum operators Q^{AB}_{jk} form an $\mathbf{A}, \mathbf{B}$ representation of the group $\mathbf{L}$ defined above which satisfy the commutation relations:

$$[\mathbf{A}, Q^{AB}_{jk}] = -[\Sigma'_j J^A_{jj'}, Q^{AB}_{j'k}] \ , \tag{6.15}$$

and

$$[\mathbf{B}, Q^{AB}_{jk}] = -[\Sigma_{j'} J^A_{kk'}, Q^{AB}_{jk'}] \ , \tag{6.16}$$

with the generators $\mathbf{A}$ and $\mathbf{B}$ defined by $\mathbf{A} \equiv (1/2)(\mathbf{J} \pm i\mathbf{K})$ and $\mathbf{B} \equiv (1/2)(\mathbf{J} - i\mathbf{K})$, with $\mathbf{J}$ and $\mathbf{K}$ being the Hermitian generators of rotations and 'boosts', respectively.

In the case of the two-component Weyl-spinors Q_{jr} the Haag–Lopuszanski–Sohnius (HLS) theorem applies, and thus the fermions form a *supersymmetry algebra* defined by the anti-commutation relations:

$$\begin{aligned} [Q_{jr}, Q^*_{ks}] &= 2\delta_{rs}\sigma^{\mu}_{jk}P_{\mu} \ , \\ [Q_{jr}, Q_{ks}] &= e_{jk}Z_{rs} \ , \end{aligned} \tag{6.17}$$

where P_{μ} is the 4–momentum operator, $Z_{rs} = -Z_{sr}$ are the bosonic symmetry generators, and σ_{μ} and $\mathbf{e}$ are the usual 2×2 Pauli matrices. Furthermore, the fermionic generators commute with both energy and momentum operators:

$$[P_{\mu}, Q_{jr}] = [P_{\mu}, Q^*_{jr}] = 0 \ . \tag{6.18}$$

The bosonic symmetry generators Z_{ks} and Z^*_{ks} represent the set of *central charges* of the supersymmetric algebra:

$$[Z_{rs}, Z^*_{tn}] = [Z^*_{rs}, Q_{jt}] = [Z^*_{rs}, Q^*_{jt}] = [Z^*_{rs}, Z^*_{tn}] = 0 \ . \tag{6.19}$$

From another direction, the Poincaré symmetry mechanism of special relativity can be extended to new algebraic systems (Tanasua, 2006). In Moultaka et al. (2005) in view of such extensions, consider invariant-free Lagrangians and bosonic multiplets constituting a symmetry that interplays with (Abelian) U(1)–gauge symmetry that may possibly be described in categorical terms, in particular, within the notion of a *cubical site* (Grandis and Mauri, 2003).

We shall proceed to introduce in the next section generalizations of the concepts of Lie algebras and graded Lie algebras to the corresponding Lie *algebroids* that may also be regarded as C*–convolution representations of *quantum gravity groupoids* and superfield (or supergravity) supersymmetries. This is therefore a novel approach to the proper representation of the *non-commutative geometry of quantum spacetimes*–that are *curved* (or 'deformed') by the presence of *intense* gravitational fields–in the framework of *non-Abelian, graded Lie algebroids.* Their correspondingly *deformed quantum gravity groupoids* (QGG) should, therefore, adequately represent supersymmetries modified by the presence of such intense gravitational fields on the Planck scale. Quantum fluctuations that give rise to quantum 'foams' at the Planck scale may be then represented by *quantum homomorphisms* of such QGGs. If the corresponding graded Lie algebroids are also *integrable*, then one can reasonably expect to recover in the limit of $\hbar \to 0$ the Riemannian geometry of General Relativity and the *globally hyperbolic spacetime* of Einstein's classical gravitation theory (GR), as a result of such an integration to the *quantum gravity fundamental groupoid* (QGFG). The following subsection will define the precise mathematical concepts underlying our novel quantum supergravity and extended supersymmetry notions.

6.3. **Extending Supersymmetry in Relativistic Quantum Supergravity: Lie Bialgebroids and a Novel Graded Lie Algebroid Concept.** Whereas not all Lie algebroids are integrable to Lie groupoids, there is a subclass of the latter called sometimes 'Weinstein groupoids' that are in a one-to-one correspondence with their Lie algebroids.

6.3.1. *Lie Algebroids and Lie Bialgebroids.* One can think of a Lie algebroid as generalizing the idea of a tangent bundle where the tangent space at a point is effectively the equivalence class of curves meeting at that point (thus suggesting a groupoid approach), as well as serving as a site on which to study infinitesimal geometry (see e.g. Mackenzie, 2005). Specifically, let M be a manifold and let $\mathfrak{X}(M)$ denote the set of vector fields on M. Recall that a Lie algebroid over M consists of a vector bundle $E \longrightarrow M$, equipped with a Lie bracket $[\ ,\ ]$ on the space of sections $\gamma(E)$, and a bundle map $\Upsilon : E \longrightarrow TM$, usually called the *anchor*. Further, there is an induced map $\Upsilon : \gamma(E) \longrightarrow \mathfrak{X}(M)$, which is required to be a map of Lie algebras, such that given sections $\alpha, \beta \in \gamma(E)$ and a differentiable function f, the following Leibniz rule is satisfied :

$$[\alpha, f\beta] = f[\alpha, \beta] + (\Upsilon(\alpha))\beta \ . \tag{6.20}$$

A typical example of a Lie algebroid is when M is a Poisson manifold and $E = T^*M$ (the cotangent bundle of M).

Now suppose we have a Lie groupoid G:

$$r, s \ : \ \mathsf{G} \underset{s}{\overset{r}{\rightrightarrows}} \mathsf{G}^{(0)} \ = M \ . \tag{6.21}$$

There is an associated Lie algebroid $\mathcal{A} = \mathcal{A}(\mathsf{G})$, which in the guise of a vector bundle, is in fact the restriction to M of the bundle of tangent vectors along the fibers of s (ie. the s–vertical vector fields). Also, the space of sections $\gamma(\mathcal{A})$ can be identified with the space of s–vertical, right–invariant vector fields $\mathfrak{X}^s_{inv}(\mathsf{G})$ which can be seen to be closed under $[\ ,\ ]$, and the latter induces a bracket operation on $\gamma(A)$ thus turning $\mathcal{A}$ into a Lie algebroid. Subsequently, a Lie algebroid $\mathcal{A}$ is integrable if there exists a Lie groupoid G inducing $\mathcal{A}$.

6.3.2. *Graded Lie Bialgebroids and Symmetry Breaking.* A *Lie bialgebroid* is a Lie algebroid E such that $E^* \longrightarrow M$ also has a Lie algebroid structure. Lie bialgebroids are often thought of as the infinitesimal variations of Poisson groupoids. Specifically, with regards to a Poisson structure Λ, if $(\mathsf{G} \rightrightarrows M, \Lambda)$ is a Poisson groupoid and if $E\mathsf{G}$ denotes the Lie algebroid of G, then $(E\mathsf{G}, E^*\mathsf{G})$ is a Lie bialgebroid. Conversely, a Lie bialgebroid structure on the Lie algebroid of a Lie groupoid can be integrated to a Poisson groupoid structure. Examples are Lie bialgebras which correspond bijectively with simply connected Poisson Lie groups.

6.4. **Graded Lie Algebroids and Bialgebroids.** A grading on a Lie algebroid follows by endowing a graded Jacobi bracket on the smooth functions $C^\infty(M)$ (see Grabowski and Marmo, 2001). A Graded Jacobi bracket of degree k on a $\mathbb{Z}$–graded associative commutative algebra $\mathcal{A} = \bigoplus_{i \in \mathbb{Z}} \mathcal{A}$ consists of a graded bilinear map

$$\{\cdot\ ,\ \cdot\} : \mathcal{A} \times \mathcal{A} \longrightarrow \mathcal{A} \ , \tag{6.22}$$

of degree k (that is, $|\{a, b\}| = |a| + |b| + k$) satisfying :

1. $\{a, b\} = -(-1)^{\langle a+k, b+k \rangle} \{b, a\}$ (graded anticommutativity)
2. $\{a, bc\} = \{a, b\}c + (-1)^{\langle a+k, b \rangle}\ b\{a, c\} - \{a, \mathbf{1}\}bc$ (graded generalized Leibniz rule)

3. $\{\{a, b\}, c\} = \{a, \{b, c\}\} - (-1)^{\langle a+k, b+k\rangle}\{b, \{a, c\}\}$ (graded Jacobi identity)

where $\langle \cdot, \cdot\rangle$ denotes the usual pairing in $\mathbb{Z}^n$. Item 2. says that $\{\ ,\ \}$ corresponds to a first–order bidifferential operator on $\mathcal{A}$, and an odd Jacobi structure corresponds to a generalized *graded Lie bialgebroid.*

Having considered and also introduced several extended quantum symmetries, we are summarizing in the following diagram the key links between such quantum symmetry related concepts; included here also are the groupoid/algebroid representations of quantum symmetry and QG supersymmetry breaking. Such interconnections between quantum symmetries and supersymmetry are depicted in the following diagram in a manner suggestive of novel physical applications that will be reported in further detail in a subsequent paper (Baianu, Glazebrook, and Brown, 2008).

See also: "Supersymmetry and Supergroups in Stochastic Quantum Physics" by Thomas Guhr from the Max P

The extended quantum symmetries formalized in the next section are defined as representations of the groupoid, algebroid and categorical structures considered in the above sections.

7. Extended Quantum Symmetries as Algebroid and Groupoid Representations.

7.1. Algebroid Representations. A definition of a *vector bundle representation* (VBR), (ρ, V), of a Lie algebroid Λ over a manifold M was published by Levin and Oshanetsky (2001) as a vector bundle $V \longrightarrow M$ and a bundle map ρ from Λ to the bundle of order ≤ 1 differential operators $\mathsf{D} : \Gamma(V) \longrightarrow \Gamma(V)$ on sections of V compatible with the anchor map and commutator such that:

(i) for any $\epsilon_1, \epsilon_2 \in \gamma$ the symbol $\mathrm{Symb}(\rho(\epsilon))$ is a scalar equal to the anchor of ϵ:

$$\mathrm{Symb}(\rho(\epsilon)) = \delta_\epsilon Id_V$$

(ii) for any $\epsilon_1, \epsilon_2 \in \gamma(\Lambda)$ and $f \in C^\infty(M)$ we have $[\rho(\epsilon_1), \rho(\epsilon_2)] = \rho([\epsilon_1, \epsilon_2])$.

In (ii) $C^\infty(M)$ is the algebra of $\Re$-valued functions on M.

7.2. Hopf and Weak Hopf C*– Algebroid Representations. We shall begin in this section with a consideration of the Hopf algebra representations that are known to have additional structure to that of a Hopf algebra. If H is a Hopf algebra and A is an algebra with the product operation $\mu : A \otimes\ A \longrightarrow A$, then a linear map $\rho : H \otimes A \longrightarrow A$ is an *algebra representation* of H if in addition to being a (vector space) representation of H, μ is also an H–intertwiner. If A happens to be unital, it will also be required that there is an H-intertwiner from ϵ_H to A such that the unity of ϵ_H maps to the unit of A.

On the other hand, the Hopf–algebroid H_A over $C_c^\infty(M)$, with M a smooth manifold, is sometimes considered as a quantum groupoid because one can construct its spectral *étale* Lie groupoid $\mathsf{G}_s \to p(H_A)$ representation beginning with the groupoid algebra $C_c(\mathsf{G})$ of smooth functions with compact support on G_{lc}; this is an *étale* Lie groupoid for M's that are not necessarily Hausdorff (cf. Mrvcun, 2002). Recently, Konno (2008) reported a systematic construction of both finite and infinite–dimensional *dynamical representations of a H–Hopf algebroid* (introduced by Etingof and Varchenko, 1996x) and their parallel structures to the *quantum affine algebra* $U_q(\hat{sl}_2)$. Such generally non–Abelian structures are constructed in terms of the Drinfel'd generators of the quantum affine algebra $U_q(\hat{sl}_2)$ and a Heisenberg algebra. The structure of the tensor product of two evaluation representations was also provided by Konno, and an elliptic analogue of the Clebsch-Gordan coefficients was expressed by using certain balanced elliptic hypergeometric series ${}_{12}V_{11}$.

7.3. **Groupoid Representations.** Whereas group representations of quantum unitary operators are extensively employed in standard quantum mechanics, the applications of groupoid representations are still under development. For example, a description of stochastic quantum mechanics in curved spacetime (Drechsler and Tuckey, 1996) involving a Hilbert bundle is possible in terms of *groupoid representations* which can indeed be defined on such a Hilbert bundle $(X*\mathcal{H}, \pi)$, but cannot be expressed as the simpler group representations on a Hilbert space $\mathcal{H}$. On the other hand, as in the case of group representations, unitary groupoid representations induce associated C*-algebra representations. In the next subsection we recall some of the basic results concerning groupoid representations and their associated groupoid *–algebra representations. For further details and recent results in the mathematical theory of groupoid representations one has also available the succint monograph by Buneci (2003) and references cited therein (*www.utgjiu.ro/math/mbuneci/preprint.html*; [42, 43, 44, 45, 46, 47, 48, 49, 50]).

7.4. **Equivalent Groupoid and Algebroid Representations: The Correspondence between Groupoid Unitary Representations and the Associated C*–Algebra Representations.** We shall briefly consider here a main result due to Hahn (1978) that relates groupoid and associated groupoid algebra representations:

Theorem 7.1. (source: Theorem 3.4 on p. 50 of P. Hahn, 1978.) *Any representation of a groupoid* G_{lc} *with Haar measure* (ν, μ) *in a separable Hilbert space* $\mathcal{H}$ *induces a *–algebra representation* $f \mapsto X_f$ *of the associated groupoid algebra* $\Pi(\mathsf{G}_{lc}, \nu)$ *in* $L^2(U_{\mathsf{G}_{lc}}, \mu, \mathcal{H})$ *with the following properties:*

(1) *For any* $l, m \in \mathcal{H}$ *, one has that* $|< X_f(u \mapsto l), (u \mapsto m) >| \leq \|f_l\| \, \|l\| \, \|m\|$ *and*

(2) $M_r(\alpha)X_f = X_{f\alpha\circ r}$, *where* $M_r : L^\infty(U_{\mathcal{G}}, \mu, \mathcal{H}) \longrightarrow \mathcal{L}(L^2(U_{\mathcal{G}}, \mu, \mathcal{H}))$, *with* $M_r(\alpha)j = \alpha \cdot j$.

*Conversely, any *–algebra representation with the above two properties induces a groupoid representation, X, as follows:*

$$\langle X_f, j, k\rangle \; = \; \int f(x)[X(x)j(d(x)), k(r(x))d\nu(x)], \tag{7.1}$$

(cf. p. 50 of Hahn, 1978).

Furthermore, according to Seda (1986, on p.116) the continuity of a Haar system is equivalent to the continuity of the convolution product $f * g$ for any pair f, g of continuous functions with compact support. One may thus conjecture that similar results could be obtained for functions with *locally compact* support in dealing with convolution products of either locally compact groupoids or quantum groupoids. Seda's result also implies that the convolution algebra $C_c(\mathsf{G})$ of a groupoid G is closed with respect to convolution if and only if the fixed Haar system associated with the measured groupoid G is *continuous* (Buneci, 2003).

In the case of groupoid algebras of transitive groupoids, Buneci (2003) and in related refs.([42, 43, 44, 45, 46, 47, 48, 49, 50]) showed that representations of a measured groupoid $(\mathsf{G}, [\int \nu^u d\tilde{\lambda}(u)] = [\lambda])$ on a separable Hilbert space $\mathcal{H}$ induce *non-degenerate* *–representations $f \mapsto X_f$ of the associated groupoid algebra $\Pi(\mathsf{G}, \nu, \tilde{\lambda})$ with properties formally similar to (1) and (2) above ([51]). Moreover, as in the case of groups, *there is a correspondence between the unitary representations of a groupoid and its associated C*–convolution algebra representations* (p.182 of Buneci, 2003), the latter involving however fiber bundles of Hilbert spaces instead of single Hilbert spaces. Therefore, groupoid representations appear as the natural construct for Algebraic Quantum Field theories in which nets of local observable operators in Hilbert space fiber bundles were introduced by Rovelli (1998).

7.5. **Generalized Fourier–Stieltjes Transforms of Groupoids. Fourier–Stieltjes Algebras of Locally Compact Groupoids and Quantum Groupoids. Left Regular Groupoid Representations and the Fourier Algebra of a Measured Groupoid.** We shall recall first that the *Fourier–Stieltjes algebra* $\mathbf{B}(G_{lc})$ of a locally compact group G_{lc} is defined by the space of coefficients (ξ, η) of *Hilbert space representations* of G_{lc}. In the special case of left regular representations and a measured groupoid, G, the Fourier–Stieltjes algebra $\mathbf{B}(\mathsf{G},\ \nu^u, \mu)$ –defined as an involutive subalgebra of $L^\infty(\mathsf{G})$ becomes the *Fourier algebra* $\mathbf{A}(\mathsf{G})$ defined by Renault (1997); such algebras are thus defined as a set of *representation coefficients* $(\mu, U_{\mathsf{G}} * H, L)$, which are effectively realized as a function $(\xi, \eta) : \mathsf{G} \longrightarrow \mathbb{C}$, defined by

$$(\xi, \eta)(x) := < \xi(r(x)), \hat{L}(x)\eta(d(x)) > , \tag{7.2}$$

(see pp.196–197 of Buneci, 2003).

The Fourier–Stieltjes (FS) and Fourier (FR) algebras, respectively, $\mathbf{B}(G_{lc}), \mathbf{A}(G_{lc})$, were first studied by P. Eymard for a general locally compact group G_{lc}, and have since played ever increasing roles in harmonic analysis and in the study of the operator algebras generated by G_{lc}.

Recently, there is also a considerable interest in developing extensions of these two types of algebras for *locally compact groupoids* because, as in the group case, such algebras play a useful role both in the study of the theory of quantum operator algebras and that of groupoid operator algebras. Furthermore, as discussed in the Introduction, there are new links between (physical) scattering theories for paracrystals, or other systems with local/partial ordering such as glasses/ 'non-crystalline' solids, and the generalizations of Fourier transforms that realize the well–established duality between the physical space, S, and the 'diffraction', or *reciprocal*, space, $\mathcal{R} = \tilde{S}$. On the other hand, the duality between the real time of quantum dynamics/resonant processes, T, and the 'spectral space', $\mathcal{F} = \tilde{T}$, of resonance frequencies (and the corresponding quanta of energies, $h\nu$) for electrons, nucleons and other particles in bound configurations is just as well–established by comparison with that occurring between the 'real' and reciprocal spaces in the case of electrons, neutron or emf/X–ray diffraction and scattering by periodic and aperiodic solids. The deep quantum connection between these two fundamental dualities, or symmetries, that seem to be ubiquitous in nature, can possibly lead to an unified quantum theory of *dispersion* in solids, liquids, superfluids and plasmas.

Let X be a locally compact Hausdorff space and $C(X)$ the algebra of bounded, continuous, complex-valued functions on X. Then denote the space of continuous functions in $C(X)$ that vanish at infinity by $C_0(X)$, while $C_c(X)$ is the space of functions in $C(X)$ with compact support. The space of complex, bounded, regular Borel measures on X is then denoted by $M(X)$. The Banach spaces $\mathbf{B}(\mathsf{G}_{lc}), \mathbf{A}(\mathsf{G}_{lc})$ (where G_{lc}denotes a locally compact groupoid) as considered here occur naturally in the group case in both non-commutative harmonic analysis and duality theory. Thus, in the case when G is a locally compact group, $\mathbf{B}(G_{lc})$ and $\mathbf{A}(G_{lc})$ are just the well known Fourier–Stieltjes and Fourier algebras discussed above. The need to have available generalizations of these Banach algebras for the case of a *locally compact groupoid* stems from the fact that many of the operator algebras of current interest– as for example in non-commutative geometry and quantum operator algebras–originate from *groupoid*, rather than group, representations, so that one needs to develop the notions of $\mathbf{B}(\mathsf{G}_{lc}), \mathbf{A}(\mathsf{G}_{lc})$ in the groupoid case for groupoid operator algebras (or indeed for *algebroids*) that are much more general than $\mathbf{B}(G_{lc}), \mathbf{A}(G_{lc})$. One notes also that in the operator space context, $\mathbf{A}(G_{lc})$ is regarded as the *convolution algebra of the dual quantum group* (Paterson,2003).

However, for groupoids and more general structures (e.g., categories and toposes of ŁM–algebras), such an extension of Banach space duality still needs further investigation. Thus, one can also conceive the notion of a measure theory based on Łukasiewicz-Moisil (ŁM) N-valued logic algebras (Georgescu, 2006 and references cited therein), and a corresponding ŁM–topos generalization of harmonic (or anharmonic) analysis by defining extended Haar–LM measures, ŁM– topos representations and $\mathcal{F}_{S-L-M}$ transforms. This raises the natural question of duality for the catgeory of ŁM-algebras that was introduced by Georgescu and Vraciu (1970).

Let us consider first the algebra involved in the simple example of the Fourier transform and then note that its extension to the Fourier-Stieltjes transform involves a convolution, just as it did in the case of the paracrystal scattering theory.

Thus, consider as in Paterson (2003) the Fourier algebra in the locally compact group case and further assume that G_{lc} is a locally compact abelian group with character space $\hat{G_{lc}}$; then an element of $\hat{G_{lc}}$ is a continuous homomorphism $t : G_{lc} \to T$, with $\hat{G_{lc}}$ being a locally compact abelian group with pointwise product and the topology of uniform convergence on compacta. Then, the Fourier transform $f \to \hat{f}$ takes $f \in L^1(G_{lc})$ into $C_0(\hat{G_{lc}})$, with $\hat{f}(t) = \int f(x)\overline{t(x)}dx$, where dx is defined as a left Haar measure on G_{lc}. On the other hand, its inverse Fourier transform $\mu \longrightarrow \check{\mu}$ reverses the process by taking $M(\hat{G_{lc}})$ back into $C(G_{lc})$, with $\check{\mu}$ being defined by the (inverse Fourier transform) integral: $\check{\mu}(x) = \int \check{x}(t)d\mu(t)$. For example, when $G_{lc} = \Re$, one also has that $\hat{G_{lc}} = \Re$ so that $t \in \hat{G_{lc}}$ is associated with the character $x \longrightarrow e^{ixt}$. Therefore, one obtains in this case the usual Fourier transform $\hat{f}(t) = \int f(x)e^{-ixt}dx$ and its inverse (or dual) $\check{\mu}(x) = \int e^{itx}d\mu(t)$. By considering $M(\hat{G_{lc}})$ as a *convolution* Banach algebra (which contains $L^1(\hat{G_{lc}})$ as a closed ideal) one can define the Fourier–Stieltjes algebra $\mathbf{B}(G_{lc})$ by $M(\hat{G_{lc}})\,\check{}$, whereas the simpler Fourier algebra, $\mathbf{A}(G_{lc})$, is defined as $L^1(\hat{G_{lc}})\,\check{}$.

Remark 7.1. In the case of a *discrete* Fourier transform, the integral is replaced by summation of the terms of a Fourier series. The discrete Fourier (transform) summation has by far the widest and most numerous applications in digital computations in both science and engineering. Thus, one represents a continuous function by an infinite Fourier series of 'harmonic' components that can be either real or complex, depending on the *symmetry* properties of the represented function; the latter is then approximated to any level of desired precision by truncating the Fourier series to a finite number of terms and then neglecting the remainder. To avoid spurious 'truncation errors' one then applies a 'smoothing' function, such as a negative exponential, that is digitized at closely spaced sample points so that the Nyquist's theorem criterion is met in order to both obtain the highest possible resolution and to drastically reduce the noise in the final, computed fast Fourier transform (FFT). Thus, for example, in the simpler case of a centrosymmetric electron density of a unit cell in a crystalline lattice, the diffracted X-ray, electron or neutron intensity can be shown to be proportional to the modulus squared of the *real* Fourier transform of the (centrosymmetric) electron density of the lattice. In a (digital) FFT computation, the approximate electron density reconstruction of the lattice structure is obtained through truncation to the highest order(s) of diffraction observed, and thus the spatial resolution obtained is limited to a corresponding value in real 3–D space.

Remark 7.2. : Laplace vs 1-D and 2-D Fourier transforms. On the other hand, Laplace transforms although used in some engineering applications to calculate transfer functions, are much less utilized in the experimental sciences than the Fourier transforms even though the former may have advantages over FFT for obtaining both improved resolution and increased signal-to-noise. It seems that the major reason for this strong preference for FFT is the much shorter computation time on digital computers, and perhaps also FFT's relative simplicity when compared with Laplace transforms; the latter may also be one of the main reasons for the presence of very few digital applications in experimental science of the Fourier–Stieltjes transforms which generalize Fourier transforms. Somewhat surprising, however, is the use of FFT also in *algebraic quantum field computations on a lattice* where both FS or Laplace transforms could provide superior results, albeit at the expense of increased digital computation time and substantially more complex programming. On the other hand, one also notes the increasing use of 'two–dimensional' FFT in comparison with one-dimensional FFT in both experimental science and medicine (for example, in 2D–NMR, 2D–chemical (IR/NIR) imaging and MRI cross-section computations, respectively), even though the former require both significantly longer computation times and more complex programming.

7.5.1. *Fourier-Stieltjes Transforms as Generalizations of Classical Fourier Transforms in Harmonic Analysis to Extended Anharmonic Analysis in Quantum Theories.* Not surprisingly, there are several versions of the near-'harmonic' F-S algebras for the locally compact groupoid case that appear at least in three related theories:

(1) the *measured groupoid* theory of J. Renault (1976),

(2) a Borel theory of A. Ramsay and M. Walter (2003), and

(3) a continuity-based theory of A. Paterson (2003).

Ramsay and Walter (1997) made a first step towards extending the theory of Fourier – Stieltjes algebras from groups to groupoids, thus paving the way to the extension of $F-S$ applications to generalized anharmonic analysis in Quantum theories *via* quantum algebra and quantum groupoid representations. Thus, if G_{lc} is a locally compact (second countable) groupoid, Ramsay and Walter showed that $\mathbf{B}(\mathsf{G}_{lc})$, which was defined as the linear span of the Borel positive definite functions on G_{lc}, is a *Banach algebra* when represented as an algebra of completely bounded maps on a C*–algebra associated with G_{lc}involving equivalent elements of $\mathbf{B}(\mathsf{G}_{lc})$; positive definite functions will be defined in the next paragraph using the notation of Paterson (2003). Corresponding to the universal C*-algebra, C*(G), in the group case is the universal $C^*_\mu(\mathsf{G})$ in the measured groupoid G case. The latter is the completion of $C_c(\mathsf{G}_{lc})$ under the largest C*-norm coming from some measurable $\mathsf{G_{lc}}$–Hilbert bundle $(\mu, \Re, L)$. In the group case, it is known that $\mathbf{B}(G)$ is isometric to the Banach space *dual* of C*(G). On the other hand, for groupoids, one can consider a representation of $\mathbf{B}(\mathsf{G}_{lc})$ as a Banach space of completely bounded maps from a C*–algebra associated with $\mathsf{G}_{lc}$to a C*–algebra associated with the equivalence relation induced by G_{lc}. Obviously, any Hilbert space $\mathcal{H}$ can also be regarded as an operator space by identifying it with a subspace of $\mathbf{B}(\mathbb{C}, \mathcal{H})$: each $\xi \in \mathcal{H}$ is identified with the map $a \longrightarrow a\xi$ for $a \in \mathbb{C}$; thus, $\mathcal{H}^*$ is an operator space as a subspace of $\mathbf{B}(\mathcal{H}, \mathbb{C})$. Renault showed for measured groupoids that the operator space $C^*_\mu(\mathsf{G}_{lc})$ is a completely contractive left $L^\infty(\mathsf{G}^0_{lc})$ module. If E is a right, and F is a left, A–operator module, with A being a C*–algebra, then a Haagerup tensor norm is determined on the algebraic tensor product $E \otimes_A F$ by setting $\| \ u\| = \sum_{i=1}^n \|e_i\| \, \|f_i\|$ over all representations $u = \sum_{i=1}^n e_i \otimes_A f_i$.

According to Paterson (2003), the completion $E \otimes_A F$ of E is called the *module Haagerup tensor product of E and F over A.* With this definition, the module Haagerup tensor product is:

$$X(\mathsf{G}_{lc}) = L^2(\mathsf{G}^0_{lc})^* \otimes C^*_\mu(\mathsf{G}_{lc}) \otimes L^2(\mathsf{G}^0_{lc}) \ , \tag{7.3}$$

taken over $L^\infty(\mathsf{G}^0_{lc})$. Then, with this tensor product construction, Renault was able to prove that

$$X(\mathsf{G}_{lc})^* = \mathbf{B}_\mu(\mathsf{G}_{lc}) \ . \tag{7.4}$$

Thus, each $\phi = (\xi, \eta)$ can be expressed by the linear functional $a^* \otimes f \otimes b \longrightarrow \int \overline{a \circ r}(\phi f) b \circ s \, d\nu$ with $f \in C_c(\mathsf{G}_{lc})$.

We shall also briefly discuss here Paterson's generalization to the groupoid case in the form of a *Fourier–Stieltjes algebra of a groupoid*, $\mathbf{B}_\mu(\mathsf{G}_{lc})$, which was defined (e.g., Paterson, 2003) as the space of coefficients $\phi = (\xi, \eta)$, where ξ, η are L^∞–sections for some measurable G–Hilbert bundle $(\mu, \Re, L)$. Thus, for $x \in \mathsf{G}_{lc}$,

$$\phi(x) = (L(x)\xi(s(x)), \eta(r(x))). \tag{7.5}$$

Therefore, ϕ belongs to $L^\infty(\mathsf{G}_{lc}) = L^\infty(\mathsf{G}_{lc}, \nu)$.

Both in the groupoid and group case, the set $P_\mu(\mathsf{G}_{lc})$ of *positive definite functions* in $L^\infty(\mathsf{G}_{lc})$ plays the central role. Thus, a function $\phi \in L^\infty(\mathsf{G}_{lc})$ is called *positive definite* if and only if for all $u \in (\mathsf{G}_{lc})_0$,

$$\int \int \phi(y^{-1}x) f(y) \overline{f(x)} d\lambda^u(x) d\lambda^u(y) \geq 0 \ . \tag{7.6}$$

Now, one can define the notion of a *Fourier–Stieltjes Transform* as follows:

Definition 7.5.1 : The Fourier–Stieltjes Transform.

Given a *positive definite, measurable function* $f(x)$ on the interval $(-\infty, \infty)$ there exists a monotone increasing, real-valued bounded function $\alpha(t)$ such that:

$$f(x) = \int_{-\infty}^{\infty} e^{itx} d\alpha(t) \ , \tag{7.7}$$

for all x except a small set. When $f(x)$ is defined as above and if $\alpha(t)$ is nondecreasing and bounded then the measurable function defined by the above integral is called *the Fourier-Stieltjes transform of* $\alpha(t)$, and it is *continuous* in addition to being positive definite in the sense defined above.

Paterson (2003) also defined the *continuous* Fourier–Stieltjes algebra $\mathbf{B}(\mathsf{G})$ as follows. Let us consider a *continuous* G–Hilbert bundle $\mathcal{H}_\Re$, and the Banach space Δ_b of continuous, bounded sections of $\mathcal{H}_\Re$. For $\xi, \eta \in \Delta_b$, the coefficient $(\xi, \eta) \in C(\mathsf{G})$ is defined by:

$$(\xi, \eta)(u) = (L_x \xi(s(x)), \eta(r(x))) \ , \tag{7.8}$$

where $x \longrightarrow L_x$ is the G–action on $\mathcal{H}_\Re$. Then, the *continuous Fourier–Stieltjes algebra* $\mathbf{B}(\mathsf{G})$ is defined to be the set of all such coefficients, coming from all possible continuous G–Hilbert bundles. Thus, $\mathbf{B}(\mathsf{G}$ is an algebra over $\mathbb{C}$ and the norm of $\phi \in \mathbf{B}(\mathsf{G}$ is defined to be $inf \, \|\xi\| \, \|\eta\|$, with the *infimum* inf being taken over all G representations $\phi = (\xi, \eta)$. Then $\mathbf{B}(\mathsf{G}) \subset C(\mathsf{G})$, and $\|\cdot\|_\infty = \|\cdot\|$.

Paterson (2003) showed that $\mathbf{B}(\mathsf{G}$ thus defined–just as in the group case– is a *commutative Banach algebra*. He also defined for a general group G the *left regular representation* π_2 of G on $L^2(G)$ by: $\pi_2(x) f(t) = f(x^{-1}t)$. One also has the *universal representation* $\pi_{2,univ}$ of G which is defined on a Hilbert space $\mathcal{H}_{univ}$. Moreover, every *unitary* representation of G determines by integration a *non-degenerate* π_2–representation of $C_c(G)$. The norm closure of $\pi_2(C_c(G))$ then defines the *reduced* C*–algebra $C^*_{red}(G)$ of G, whereas the norm closure of $\pi_{2,univ}(C_c(G))$ was defined as the *universal C*–algebra* of G (*loc.cit.*). The algebra $C^*_{red}(G) \subset B(L^2(G))$ generates a von Neumann algebra denoted by $V_N(G)$. Thus, $C^*_{red}(G)$ representations generate $V_N(G)$ representations that have a much simpler classification through their V_N factors than the representations of general C*–algebras; consequently, the classification of $C^*_{red}(G)$ representations is closer linked to that of V_N factors than in the general case of C*–algebras. One would expect that a similar simplification may not be available when group G symmetries (and, respectively, their associated $C^*_{red}(G)$ representations) are extended to the more general groupoid symmetries (and their associated C*-convolution (groupoid) algebra representations on Hilbert space *bundles*). Recently, however, Ros (2006, 2008) reported that one can extend–with appropriate modifications and conditions added– the Schur's Lemma and Peter-Weyl theorems from group representations to corresponding theorems for (continuous) internally irreducible representations of *continuous* groupoids in the case of Schur's Lemma, and restriction maps in the case of two Peter-Weyl theorems, (one of the latter theorems being applicable only to *compact*, proper groupoids and their isomorphism classes of irreducible unitary (or internally irreducible) representations ($IrRep(\mathsf{G})$ and $IrRep^i(\mathsf{G})$, respectively)). It is well established that using Schur's Lemma for groups one can prove that if a matrix commutes with every element in an *irreducible* representation of a group that matrix must be a multiple of the *identity*. A continuous groupoid representation $(\pi, \mathcal{H}, \Delta)$ of a continuous groupoid $\mathsf{G}\rightrightarrows M$ was called *internally irreducible* by Ros if the restriction of π to each of the isotropy groups is an irreducible representation. Thus, in the case of *continuous* groupoids $\mathsf{G}\rightrightarrows M$ (endowed with a Haar system), *irreducible representations are also internally irreducible* but the converse does not hold in general (see preprints of R.D. Bos at : http://www.math.ru.nl/ rdbos/ContinReps.pdf).

Bos (2006, 2008) also introduced the *universal enveloping C^*-category* of a Banach *–category, and then used this to define the C^*-category, $\mathbf{C}^*(\mathsf{G},\mathsf{G})$, of a groupoid. Then, he found that *there exists a bijection between the continuous representations of* $\mathbf{C}^*(\mathsf{G},\mathsf{G})$ *and the continuous representations of* $\mathsf{G}\rightrightarrows M$.

7.6. **General Definition of Extended Symmetries as Representations.** We aim here to define extended quantum symmetries as general representations of mathematical structures that have as many as possible physical realizations, i.e. *via* unified quantum theories. In order to be able to extend this approach to very large ensembles of composite or complex quantum systems one requires general procedures for quantum 'coupling' of component quantum systems; we propose to approach this important 'composition', or scale up/assembly problem in a formal manner as described in the next section.

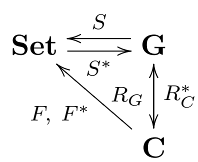
Because a group G can be viewed as a category with a single object, whose morphisms are just the elements of G, a *general representation* of G in an arbitrary category $\mathbf{C}$ is a functor R_G from G to $\mathbf{C}$ that selects an object X in $\mathbf{C}$ and a group homomorphism from γ to $\mathrm{Aut}(X)$, the automorphism group of X. Let us also define an *adjoint representation* by the functor $R^*_{\mathbf{C}} : \mathbf{C} \longrightarrow \mathbf{G}$. If $\mathbf{C}$ is chosen as the category **Top** of topological spaces and homeomorphisms then representations of G in **Top** are homomorphisms from G to the homeomorphism group of a topological space X. Similarly, a *general representation of a groupoid* $\mathcal{G}$ (considered as a category of invertible morphisms) in an arbitrary category $\mathbf{C}$ is a functor $\mathbf{R}_{\mathcal{G}}$ from $\mathcal{G}$ to $\mathbf{C}$, defined as above simply by substituting $\mathcal{G}$ for G. In the special case of a Hilbert space, this categorical definition is consistent with the representation of the groupoid on a bundle of Hilbert spaces.

Remark 7.3. Unless one is operating in super–categories, such as 2-categories and higher dimensional categories, one needs to distinguish between the *representations of an (algebraic) object*– as defined above– and the *representation of a functor S* (from $\mathbf{C}$ to the category of sets, **Set**) by an object in an arbitrary category $\mathbf{C}$ as defined next. Thus, in the latter case, a *functor representation* will be defined by a certain *natural equivalence between functors*. Furthermore, one needs consider also the following sequence of functors:

$$R_{\mathbf{G}} : \mathbf{G} \longrightarrow \mathbf{C} \; ,$$
$$(7.9) \qquad R^*_{\mathbf{C}} : \mathbf{C} \longrightarrow \mathbf{G} \; ,$$
$$S : \mathbf{G} \longrightarrow \mathbf{Set} \; ,$$

where $R_{\mathbf{G}}$ and $R^*_{\mathbf{C}}$ are adjoint representations as defined above, and S is the forgetful functor which forgets the group structure; the latter also has a right adjoint $S*$. With these notations one obtains the following commutative diagram of adjoint representations and adjoint functors that can be expanded to a square diagram to include either **Top**–the category of topological spaces and homeomorphisms, or **TGrpd**, and/or $\mathbf{C}_{\mathcal{G}} = \mathbf{CM}$ (respectively, the category of topological groupoids, and/or the category of categorical groups and homomorphisms) in a manner analogous to Diagrams 9.10 and 9.13 that will be discussed in Section 9 (with the additional, unique adjunction situations to be added in accordingly).

(7.10)



7.7. **Representable Functors and Their Representations.** The key notion of *representable functor* was first reported by Grothendieck (1960–1962). This is a functor $S : \mathbf{C} \longrightarrow \mathbf{Set}$, from an arbitrary category $\mathbf{C}$ to the category of sets, **Set**, if it admits a (functor) *representation* defined as follows. A *functor representation* of S is a pair, (R, ϕ), which consists of an object R of $\mathbf{C}$ and a family ϕ of equivalences $\phi(C) : \mathrm{Hom}_C(R, C) \cong S(C)$, which is natural in C. When the functor S has such a representation, it is also said to be *represented by the object* R of $\mathbf{C}$. For each object R of $\mathbf{C}$ one writes $h_R : \mathbf{C} \longrightarrow \mathbf{Set}$ for the covariant Hom–functor $h_R(C) \cong \mathrm{Hom}_{\mathbf{C}}(R, C)$. A *representation* (R, ϕ) of S is therefore *a natural equivalence of functors*

$$(7.11) \qquad \phi : h_R \cong S \; .$$

The equivalence classes of such functor representations (defined as natural equivalences) obviously determine an *algebraic groupoid* structure. As a simple example of an *algebraic* functor representation, let us also consider (cf. MacLane, 1965) the functor $N : \mathbf{Gr} \longrightarrow \mathbf{Set}$ which assigns to each group G its underlying set and to each group homomorphism f the same morphism but regarded just as a function on the underlying sets; such a functor N is called a *forgetful* functor because it "forgets" the group structure. N is a representable functor as it is represented by the additive group $\mathbb{Z}$ of integers and one has the well–known bijection $\mathrm{Hom}_{Gx}(Z, G) \cong S(G)$ which assigns to each homomorphism $f : \mathbb{Z} \longrightarrow G$ the image $f(1)$ of the generator 1 of $\mathbb{Z}$.

In the case of groupoids there is also a forgetful functor $F : \mathbf{Grpd} \longrightarrow \mathbf{Set}$ which has an unique right adjoint defined for freely generated groupoids.

Is F representable, and if so, what is the object that represents F?

One can also describe (cf. MacLane, 1965) representable functors in terms of certain universal elements called *universal points*. Thus, consider $S : \mathbf{C} \rightarrow \mathbf{Set}$ and let $\mathbf{C}_{s*}$ be the category whose objects are those pairs (A, x) for which $x \in S(A)$ and with morphisms $f : (A, x) \longrightarrow (B, y)$ specified as those morphisms $f : A \rightarrow B$ of $\mathbf{C}$ such that $S(f)x = y$; this category $\mathbf{C}_{s*}$ will be called the *category of S–pointed objects* of $\mathbf{C}$. Then one defines a *universal point* for a functor $S : \mathbf{C} \longrightarrow \mathbf{Set}$ to be an initial object (R, u) in the category $\mathbf{C}_{s*}$. At this point, a general connection between representable functors/functor representations and *algebraic topology* is established by the following, fundamental functor representation theorem (MacLane, 1965).

Theorem 7.2. (Theorem 7.1 of MacLane, 1965) *For each functor* $S : \mathbf{C} \longrightarrow \mathbf{Set}$*, the formulas* $u = (\phi R)1_R$*, and* $(\phi c)h = (Sh)u$*, (with the latter holding for any morphism* $h : R \longrightarrow C$*), establish a one–to–one correspondence between the functor representations* (R, ϕ) *of* S *and the universal points* (R, u) *for* S*.*

8. Algebraic Categories and Their Representations in the Category of Hilbert Spaces. Generalization of Tensor Products.

Quantum theories of quasi-particle, or multi-particle, systems are well known to require not just products of Hilbert spaces but instead their tensor products. On the other hand, symmetries are usually built through representations of products of groups such as $\mathrm{U}(1) \times \mathrm{SU}(2) \times \mathrm{SU}(3)$ in the current 'Standard Model'; the corresponding Lie algebras are of course $\mathfrak{u}(1), \mathfrak{su}(2)$ and $\mathfrak{su}(3)$. To represent the more complex symmetries involving quantum groups that have underlying Hopf algebras, or in general Grassman–Hopf algebras, associated with many-particle or quasi-particle systems, one is therefore in need of considering new notions of generalized tensor products.

8.1. **Introducing Tensor Products of Algebroids and Categories.** Firstly, we note that tensor products of ω–groupoids have been considered by Brown and Higgins (1987) as giving rise to a crossed complex, and indeed this has been used by Baues and Conduche to define the *'tensor algebra'* of a group. Also Day and Street (1997) have subsequently considered *Hopf algebras with many objects* in tensor categories. Further work is however needed to explore possible links of these ideas with the functional analysis and operator algebras considered earlier. Thus, in attempting to generalize the notion of Hopf algebra to the many object case, one also needs to consider what could be the notion of tensor product of two R–algebroids C and D. If this can be properly defined one can then expect to see the composition in C as some partial functor $m : C \otimes C \longrightarrow C$ and a diagonal as some partial functor $\Delta : C \longrightarrow C \otimes C$. The definition of $C \otimes D$ is readily obtained for categories C, D by modifying slightly the definition of the tensor product of groupoids, regarded as crossed complexes in Brown and Higgins (1987). So we define $C \otimes D$ as the pushout of categories

$$
\begin{array}{ccc}
C_0 \times D_0 & \longrightarrow & C_1 \times D_0 \\
\downarrow & & \downarrow \\
C_0 \times D_1 & \longrightarrow & C \otimes D
\end{array}
\tag{8.1}
$$

This category may be seen also as generated by the symbols

$$\{c \otimes y \mid c \in C_1\} \cup \{x \otimes d \mid d \in D_1\}$$

for all $x \in C_0$ and $y \in D_0$ subject to the relations given by the compositions in C_1 and on D_1.

The category $G \# H$ is generated by all elements $(1_x, h), (g, 1_y)$ where $g \in G, h \in H, x \in G_0, y \in H_0$. We will sometimes write g for $(g, 1_y)$ and h for $(1_x, h)$. This may seem to be willful ambiguity, but when composites are specified in $G \# H$, the ambiguity is resolved; for example, if gh is defined in $G \# H$, then g must refer to $(g, 1_y)$, where $y = sh$, and h must refer to $(1_x, h)$, where $x = tg$. This convention simplifies the notation and there is an easily stated solution to the word problem for $G \# H$. Every element of $G \# H$ is uniquely expressible in one of the following forms:

(i) an identity element $(1_x, 1_y)$;

(ii) a generating element $(g, 1_y)$ or $(1_x, h)$, where $x \in G_0, y \in H_0, g \in G, h \in H$ and g, h are not identities;

(iii) a composite $k_1 k_2 \cdots k_n (n \geq 2)$ of non–identity elements of G or H in which the k_i lie alternately in G and H, and the odd and even products $k_1 k_3 k_5 \cdots$ and $k_2 k_4 k_6 \cdots$ are defined in G or H.

For example, if $g_1 : x \longrightarrow y, g_2 : y \longrightarrow z$, in G, g_2 is invertible, and $h_1 : u \longrightarrow v, h_2 : v \longrightarrow w$ in H, then the word $g_1 h_1 g_2 h_2 g_2^{-1}$ represents an element of $G \# H$ from (x, u) to (y, w). Note that the two occurrences of g_2 refer to different elements of $G \# H$, namely $(g_2, 1_v)$ and $(g_2, 1_w)$. This can be represented as a path in a 2-dimensional grid as follows

$$\begin{array}{ccccc}
(x,u) & & (x,v) & & (x,w) \\
\downarrow{\scriptstyle g_1} & & & & \\
(y,u) & \xrightarrow{h_1} & (y,v) & & (y,w) \\
& & \downarrow{\scriptstyle g_2} & & \uparrow{\scriptstyle g_2^{-1}} \\
(z,u) & & (z,v) & \xrightarrow{h_2} & (z,w)
\end{array}$$

The similarity with the free product of monoids is obvious and the normal form can be verified in the same way; for example, one can use 'van der Waerden's trick'. In the case when C and D are R–algebroids one may consider the pushout in the category of R–algebroids.

Now if $\mathbf{C}$ is a category, we can consider the possibility of a diagonal morphism

$$\Delta : \mathbf{C} \longrightarrow \mathbf{C} \# \mathbf{C} \; . \tag{8.2}$$

We may also include the possibility of a morphism

$$\mu : \mathbf{C} \# C \longrightarrow \mathbf{C} \; . \tag{8.3}$$

This seems possible in the algebroid case, namely the sum of the odd and even products. Or at least, μ could be defined on $\mathbf{C} \; \# \mathbf{C}((x, x), (y, y))$.

It can be argued that a most significant effect of the use of categories as algebraic structures is to allow for algebraic structures with operations that are partially defined. These were early considered by Higgins in 'Algebras with a scheme of operators' (1968). In general, *'Higher Dimensional Algebra'* (HDA) may be defined as the study of algebraic structures with operations whose domains of definitions are defined by geometric considerations. This allows for a splendid interplay of algebra and geometry, which early appeared in category theory with the use of complex commutative diagrams. What is needed next is a corresponding interplay with analysis and functional analysis that would extend also to quantum operator algebras, their representations and symmetries.

8.2. **Construction of Weak Hopf Algebras via Tensor Category Classification (Ostrik, 2006).** If $\mathbf{k}$ denotes an algebraically closed field, let $\mathbf{C}$ be a tensor category over $\mathbf{k}$. The classification of all semisimple module categories over $\mathbf{C}$ would then allow in principle the construction of all weak Hopf algebras H so that the category of comodules over H is tensor equivalent to $\mathbf{C}$, that is, as realizations of $\mathbf{C}$. There are at least three published cases where such a classification is possible :

(1) when $\mathbf{C}$ is a group theoretical fusion category (as an example when $\mathbf{C}_\gamma$ is the category of representations of a finite group γ, or a Drinfel'd quantum double of a finite group), (see [O2]);

(2) when **k** is a fusion category attached to quantum Sl(2) (see [Oc, BEK, KO, O1, EO]);

(3) when $k = \mathbf{C}_q$ is the category of representations of quantum $\mathrm{Sl}_q(2)$ Hopf algebras and q is not a root of unity (see [EO]).

This approach was further developed recently for module categories over quantum $SL(2)$ representations in the non-simple case (see also **Example 2.1.2** regarding the quantum $\mathrm{Sl}_q(2)$ Hopf algebras for further details), thus establishing a link between weak Hopf algebras and module categories over quantum Sl(2) representations (Ostrik, 2006).

Remark 8.1. One notes the condition imposed here of an *algebraically closed* field which is essential for remaining within the bounds of algebraic structures, as fields– in general– are not algebraic.

9. Double Algebroids and Double Groupoids.

There is a body of recent non-Abelian algebraic topology results giving a form of "higher dimensional group (HDG) theory" which is based on intuitive ideas of composing squares or n-cubes rather than just paths as in the case of groups Such an HDG theory yielded important results in homotopy theory and the homology of discrete groups, and seems also to be connected to a generalized categorical Galois theory (Janelidze and Brown, 2004). It also has suggested other new constructions in group theory, for example a non-Abelian tensor product of groups. One of the aims of this paper is to proceed towards a corresponding theory for associative algebras and algebroids rather than groups. One also finds that there are many results and methods in HDG theories that are analogous to those in the lower dimensional group theory, but with a corresponding increase in technical sophistication for the former. Such complications occur mainly at the step of increasing dimension from one to dimension two; thus, we shall deal in this section only with the latter case. The general, n-dimensional case of such results will be presented in subsequent reports.

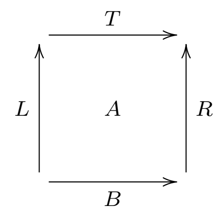
Thus, in developing a corresponding theory for algebras we expect that in order to obtain a non-trivial theory we shall have to replace, for example, R-algebras by R-algebroids, by which is meant just an R-category for a commutative ring R ; in the case when R is the ring of integers, an R-algebroid is just a *'ring with many objects'* in the sense of Mitchell (1972,1985) (for further details see for example Section 4 and other references cited therein). The necessary algebroid concepts were already presented in Section 4. In the following subsections we shall briefly introduce the other key concepts needed for such HGD developments. Thus, we begin by considering the simpler structure of double algebras and then proceed to their natural extension to double algebroids.

9.1. **Double Algebras.** Next we approach convolution and the various Hopf structures so far discussed from the point of view of 'double structures'. So to this extent, let A be taken to denote one of the following structures: a Hopf, a weak Hopf algebra or a Hopf algebroid (whose base rings need not be commutative). Starting with a Frobenius homomorphism $i : A \longrightarrow A^*$, we consider as in Szlachányi (2004) the horizontal (H) and vertical (V) components of the algebra along with a convolution product (*). Specifically, we take unital algebra structures $V = \langle A, \circ, e\rangle$ and $H = \langle A, *, i\rangle$ as leading to a double algebra structure with axioms as given in Szlachányi (2004). Thus the basic framework starts with a quadruple $(V, H, *, i)$. With respect to k–linear maps $\varphi : A \longrightarrow A$, we consider sublagebras $L, R \subset V$ and $B, T \subset H$ in accordance with the Frobenius homorphisms (for $a \in A$):

$$\varphi_L(a) := a * e \ , \ \varphi_R(a) := e * a$$
$$\varphi_B(a) := a \circ i \ , \ \varphi_T(a) := i \circ a \tag{9.1}$$

Comultiplication of the 'quantum groupoid' arises from the dual bases of φ_B and φ_T with a D_4–symmetry:

(9.2)



9.2. **Double Algebroids and Crossed Modules of Algebroids.** In recent work by Brown and Mosa (1986, 2008) the notion of *double algebroid* was introduced and its relationship to crossed modules of algebroids was investigated. Here we summarize the main results reported so far, but without providing the proofs that can be found either in Mosa (1986) or in Brown and Mosa (1986, 2008).

9.2.1. *Crossed Modules.* Let A be an R-algebroid over A_0 and let M be a pre-algebroid over A_0. One can define an *action* of A on M as follows:

Definition 9.1. A left action of A on M assigns to each $m \in M(x, y)$ and $a \in A(w, x)$ an element ${}^a m \in M(w, y)$, satisfying the axioms:

i) ${}^c({}^a m) = {}^{(ca)}m,\ {}^1 m = m,$
ii) ${}^a(mn) = {}^a mn,$
iii) ${}^a(m + m_1) = {}^a m + {}^a m_1,$
iv) ${}^{a+b}(m) = {}^a m + {}^b m,$
v) ${}^a(rm) = r({}^a m) = {}^{ra}(m),$

for all $m, m_1 \in M(x, y), n \in M(y, z), a, b \in A(w, x), c \in A(u, w)$ and $r \in R$.

Definition 9.2. A right action of A on M assigns to each $m \in M(x, y), a \in A(y, z)$ an element $m^a \in M(x, z)$ satisfying the axioms:

i) $(m^a)^c = m^{(ac)},\ m^1 = m,$
ii) $(mn)^a = mn^a,$
iii) $(m + m_1)^a = m^a + m_1^a,$
iv) $m^{(a+b)} = m^a + m^b,$
v) $(rm)^a = rm^a = m^{ra}$

for all $m, m_1 \in M(x, y), n \in M(y, z), a, b \in A(y, u), c \in A(u, v)$ and $r \in R$.

Left and right actions of A on M *commute* if $({}^a m)^b = {}^a(m^b)$, for all $m \in M(x, y), a \in A(w, x), b \in A(y, u)$.

A *crossed module of algebroids* consists of an R-algebroid A, a pre–algebroid M, both over the same set of objects, and commuting left and right actions of A on M, together with a pre–algebroid morphism $\mu : M \to A$ over the identity on A_0. These must also satisfy the following axioms:

i) $\mu({}^a m) = a(\mu m), \mu(m^b) = (\mu m)b$

ii) $mn = m^{(\mu n)} = {}^{(\mu m)}n$,

, and for all $m \in M(x, y)$, $n \in M(y, z)$, $a \in A(w, x)$, $b \in A(y, u)$.

A *morphism* $(\alpha, \beta) : (A, M, \mu) \longrightarrow (A', M', \mu')$ *of crossed modules* all over the same set of objects is an algebroid morphism $\alpha : A \longrightarrow A'$ and a pre–algebroid morphism $\beta : M \longrightarrow M'$ such that $\alpha\mu = \mu'\beta$ and $\beta({}^a m) = {}^{\alpha a}(\beta m), \beta(m^b) = (\beta m)^{\alpha b}$ for all $a, b \in A, m \in M$. Thus one constructs a *category* **CM** *of crossed modules of algebroids.*

Two basic examples of crossed modules are as follows.

(1) Let A be an R-algebroid over A_0 and suppose I is a two-sided ideal in A. Let $i : I \longrightarrow A$ be the inclusion morphism and let A operate on I by $a^c = ac$, ${}^b a = ba$ for all $a \in I$ and $b, c \in A$ such that these products ac, ba are defined. Then $i : I \to A$ is a crossed module.

(2) A two-sided module over the algebroid A is defined to be a crossed module $\mu : M \to A$ in which $\mu m = 0_{xy}$ for all $m \in M(x, y), x, y \in A_0$.

Similar to the case of categorical groups discussed above, a key feature of double groupoids is their relation to crossed modules "of groupoids" [40]. One can thus establish relations between double algebroids with thin structure and crossed modules "of algebroids" analogous to those already found for double groupoids, and also for categorical groups. Thus, it was recently reported that the *category of double algebroids with connections* is equivalent to the *category of crossed modules over algebroids* (Brown and Mosa, 1986; 2008).

9.2.2. *Double Algebroids.* In this subsection we recall the definition of a double algebroid introduced by Brown and Mosa (1986). Two functors are then constructed, one from the category of double algebroids to the category of crossed modules of algebroids, whereas the other is its unique adjoint functor.

A *double R–algebroid* consists of a double category D such that each category structure has the additional structure of an R–algebroid. More precisely, a double R–algebroid D involves four related R–algebroids:

$$
\begin{aligned}
&(D, D_1, \partial_1^0, \partial_1^1, \varepsilon_1, +_1, \circ_1, ._1), \qquad (D, D_2, \partial_2^0, \partial_2^1, \varepsilon_2, +_2, \circ_2, ._2) \\
&(D_1, D_0, \delta_1^0, \delta_1^1, \varepsilon, +, \circ, .), \qquad (D_2, D_0, \delta_2^0, \delta_2^1, \varepsilon, +, \circ, .)
\end{aligned} \tag{9.3}
$$

that satisfy the following rules:

i) $\delta_2^i \partial_2^j = \delta_1^j \partial_1^i$ for $i, j \in \{0, 1\}$

ii)

$$
\begin{aligned}
\partial_2^i(\alpha +_1 \beta) = \partial_2^i \alpha + \partial_2^i \beta, \qquad \partial_1^i(\alpha +_2 \beta) = \partial_1^i \alpha + \partial_1^i \beta \\
\partial_2^i(\alpha \circ_1 \beta) = \partial_2^i \alpha \circ \partial_2^i \beta, \qquad \partial_1^i(\alpha \circ_2 \beta) = \partial_1^i \alpha \circ \partial_1^i \beta
\end{aligned} \tag{9.4}
$$

for $i = 0, 1, \alpha, \beta \in D$ and both sides are defined.

iii)

$$
\begin{aligned}
r._1(\alpha +_2 \beta) = (r._1\alpha) +_2 (r._1\beta), \qquad r._2(\alpha +_1 \beta) = (r._2\alpha) +_1 (r._2\beta) \\
r._1(\alpha \circ_2 \beta) = (r._1\alpha) \circ_2 (r._1\beta), \qquad r._2(\alpha \circ_1 \beta) = (r._2\alpha) \circ_1 (r._2\beta) \\
r._1(s._2\beta) = s._2(r._1\beta)
\end{aligned} \tag{9.5}
$$

for all $\alpha, \beta \in D,\ r, s \in R$ and both sides are defined.

iv)

$$(\alpha +_1 \beta) +_2 (\gamma +_1 \lambda) = (\alpha +_2 \gamma) +_1 (\beta +_2 \lambda),$$
$$(\alpha \circ_1 \beta) \circ_2 (\gamma \circ_1 \lambda) = (\alpha \circ_2 \gamma) \circ_1 (\beta \circ_2 \lambda) \qquad (9.6)$$
$$(\alpha +_i \beta) \circ_j (\gamma +_i \lambda) = (\alpha \circ_j \gamma) +_i (\beta \circ_j \lambda)$$

for $i \neq j$, whenever both sides are defined.

A *morphism* $f : \mathsf{D} \to \mathcal{E}$ *of double algebroids* is then defined as a morphism of truncated cubical sets which commutes with all the algebroid structures. Thus, one can construct a category **DA** of double algebroids and their morphisms.

9.3. **Double Groupoids.** We can take further advantage of the above procedures by reconsidering the earlier, double groupoid case (Brown and Spencer, 1976) in relationship to a C*–convolution algebroid that links both 'horizontal' and 'vertical' structures in an internally consistent manner. The geometry of squares and their compositions leads to a common representation of a *double groupoid* in the following form:

$$\mathsf{D} = \begin{array}{ccc} S & \overset{s^1}{\underset{t^1}{\rightrightarrows}} & H \\ s_2 \downarrow\uparrow\downarrow t_2 & & s \downarrow\uparrow\downarrow t \\ V & \overset{s}{\underset{t}{\rightrightarrows}} & M \end{array} \qquad (9.7)$$

where M is a set of 'points', H, V are 'horizontal' and 'vertical' groupoids, and S is a set of 'squares' with two compositions. The laws for a double groupoid make it also describable as a groupoid internal to the category of groupoids. Furthermore, because in a groupoid, any composition of commutative squares is also commutative, several groupoid square diagrams of the type shown above can be composed to yield larger square diagrams that are naturally commutative.

Given two groupoids H, V over a set M, there is a double groupoid $\square(H, V)$ with H, V as horizontal and vertical edge groupoids, and squares given by quadruples

$$\begin{pmatrix} & h & \\ v & & v' \\ & h' & \end{pmatrix} \qquad (9.8)$$

for which we assume always that $h, h' \in H$, $v, v' \in V$ and that the initial and final points of these edges match in M as suggested by the notation, that is for example $sh = sv, th = sv', \ldots$, etc. The compositions are to be inherited from those of H, V, that is:

(9.9)
$$\begin{pmatrix} & h & \\ v & & v' \\ & h' & \end{pmatrix} \circ_1 \begin{pmatrix} & h' & \\ w & & w' \\ & h'' & \end{pmatrix} = \begin{pmatrix} & h & \\ vw & & v'w' \\ & h'' & \end{pmatrix}, \quad \begin{pmatrix} & h & \\ v & & v' \\ & h' & \end{pmatrix} \circ_2 \begin{pmatrix} & k & \\ v' & & v'' \\ & k' & \end{pmatrix} = \begin{pmatrix} & hk & \\ v & & v'' \\ & h'k' & \end{pmatrix}.$$

This construction is defined by the right adjoint R to the forgetful functor L which takes the double groupoid as above, to the pair of groupoids (H, V) over M. Furthermore, this right adjoint functor can be utilized to relate *double groupoid representations* to the corresponding pairs of groupoid representations induced by L. Thus, one can obtain a functorial construction of certain double groupoid representations from those of the groupoid pairs (H, V) over M. Further uses of adjointness to classifying groupoid representations related to extended quantum symmetries can also be made through the generalized Galois theory presented in the next subsection; therefore, Galois groupoids constructed with a pair of adjoint functors and their representations may play a central role in such future developments of the mathematical theory of groupoid representations and their applications in quantum physics.

Given a general double groupoid as above, one can define $S\begin{pmatrix} & h & \\ v & & v' \\ & h' & \end{pmatrix}$ to be the set of squares with these as horizontal and vertical edges.

$$\mathsf{AD} = \begin{array}{ccc} AS & \overset{s^1}{\underset{t^1}{\rightleftarrows}} & AH \\ s_2 \downarrow\uparrow\downarrow t_2 & & s \downarrow\uparrow\downarrow t \\ AV & \overset{s}{\underset{t}{\rightleftarrows}} & M \end{array} \tag{9.10}$$

for which

$$AS\begin{pmatrix} & h & \\ v & & v' \\ & h' & \end{pmatrix} \tag{9.11}$$

is the free A-module on the set of squares with the given boundary. The two compositions are then bilinear in the obvious sense.

Alternatively, we can use the convolution construction $\bar{A}\mathsf{D}$ induced by the convolution C*–algebra over H and V. This allows us to construct for at least a commutative C*–algebra A a double algebroid (i.e. a set with two algebroid structures), as discussed in the previous subsection. These novel ideas need further development in the light of the algebra of crossed modules of algebroids, developed in (Mosa, 1986, Brown and Mosa, 1986), crossed cubes of (C*)–algebras following Ellis (1988), as well as crossed complexes of groupoids (Brown, 2006).

The next, natural extension of this *quantum algebroid* approach to QFT generalized symmetries can now be formulated in terms of *graded Lie algebroids*, or supersymmetry algebroids, for the supersymmetry-based theories of *Quantum Gravity/ Supergravity* that were discussed above.

10. Conclusions and Discussion.

Extended quantum symmetries, recent Quantum Operator Algebra developments and also Non-Abelian Algebraic Topology (NAT) results were here discussed with a view to physical applications in Quantum Field theories, general molecular and nuclear scattering theories, symmetry breaking, as well as Supergravity/Supersymmetry based on a *locally covariant* approach to General Relativity theories in Quantum Gravity.

Fundamental concepts of Quantum Operator Algebra and Quantum Algebraic Topology, such as C*-algebras, Quantum Groups, von Neumann/Hopf Algebras, Quantum Groupoids, Quantum Groupoid/Algebroid Representations and so on, were here considered first with a view to their possible extensions and future applications in Quantum Field theories and beyond.

Recently published mathematical generalizations that represent extended quantum symmetries range from quantum group algebras to 'quantum groupoids', and then further, to quantum topological/Lie groupoids/Lie algebroids and Hamiltonian algebroids in W_N-gravity theories. Algebraically simpler representations of quantum spacetime than QAT have also been proposed in terms of causal sets, quantized causal sets, and quantum toposes (Nishimura, 1996; Raptis, 2000a,b; Raptis and Zapatrin, 2000; Butterfield and Isham, 2000-2005; Heunen, Landsman, and Spitters, 2008). However, the consistency of such 'quantum' toposes with the real quantum logic is yet to be validated; the 'quantum' toposes that have been proposed so far are all clearly inconsistent with the Birkhoff-von Neumann Quantum Logic (see for example, Heunen, Landsman, and Spitters, 2008). An alternative, Generalized Łukasiewicz topos (GLT) that may allow us avoid such major logical inconsistences with quantum logics has also been developed (Baianu, 2004a,b; Baianu, Brown, Glazebrook and Georgescu, 2006,2008; Georgescu, 1970; 2006).

We have suggested here several new applications of Grassmann-Hopf algebras/ algebroids, graded 'Lie' algebroids, weak Hopf C*-algebroids, quantum locally compact groupoids to interacting quasiparticle and many-particle quantum systems. These concepts lead to higher dimensional symmetries represented by double groupoids, as well as other higher dimensional algebraic topology structures (Brown and Mosa,1986; Mosa,1986); they also have potential applications to spacetime structure determination using Higher dimensional Algebra tools and its powerful results to uncover universal, topological invariants of 'hidden' quantum symmetries. New, non-abelian results may thus be obtained through Higher Homotopy, Generalized van Kampen theorems (Brown et al., 2002; Brown and Janelidze, 1997), Lie Groupoids/Algebroids and Groupoid Atlases, possibly with novel applications to Quantum Dynamics and local-to-global problems, as well as Quantum Logic Algebras. Novel mathematical representations in the form of Higher Homotopy Quantum Field (HHQFT) and Quantum Non-Abelian Algebraic Topology (QNAT) theories have the potential to develop a self-consistent Quantum–General Relativity Theory (QGRT) in the context of Supersymmetry/Supergravity/Supersymmetry Algebroids and metric superfields in the Planck limit of spacetime (Baianu, Brown, Glazebrook and Georgescu, 2008). Especially interesting in QGRT are global representations of fluctuating spacetime structures in the presence of intense, fluctuating quantum gravitational fields. The development of such mathematical representations of extended quantum symmetries and supersymmetry appears as a logical requirement for the unification of quantum field (and especially AQFT) with general relativity theories in QGRT *via* quantum supergravity and NAAT approaches to determining supersymmetry invariants of quantum spacetime geometry.

11. Appendix.

11.1. **Hilbert spaces.**

11.1.1. *Vector spaces.* **Vector space: Definition**

Let F be a field (or, more generally, a division ring). A *vector space* V over F is a set with two operations, $+ : V \times V \longrightarrow V$ and $\cdot : F \times V \longrightarrow V$, such that

1. $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$ for all $\mathbf{u},\mathbf{v},\mathbf{w}\in V$
2. $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$ for all $\mathbf{u},\mathbf{v}\in V$
3. There exists an element $\mathbf{0}\in V$ such that $\mathbf{u}+\mathbf{0}=\mathbf{u}$ for all $\mathbf{u}\in V$
4. For any $\mathbf{u}\in V$, there exists an element $\mathbf{v}\in V$ such that $\mathbf{u}+\mathbf{v}=\mathbf{0}$
5. $a\cdot(b\cdot\mathbf{u})=(a\cdot b)\cdot\mathbf{u}$ for all $a,b\in F$ and $\mathbf{u}\in V$
6. $1\cdot\mathbf{u}=\mathbf{u}$ for all $\mathbf{u}\in V$
7. $a\cdot(\mathbf{u}+\mathbf{v})=(a\cdot\mathbf{u})+(a\cdot\mathbf{v})$ for all $a\in F$ and $\mathbf{u},\mathbf{v}\in V$
8. $(a+b)\cdot\mathbf{u}=(a\cdot\mathbf{u})+(b\cdot\mathbf{u})$ for all $a,b\in F$ and $\mathbf{u}\in V$

Equivalently, a vector space is a module V over a ring F which is a field (or, more generally, a division ring).

The elements of V are called *vectors*, and the element $\mathbf{0}\in V$ is called the *zero vector* of V.

("vector space" is defined at PM by djao).

11.1.2. *Inner Product space or* pre-Hilbert space*.*

An *inner product space* (or *pre-Hilbert space*) is a vector space (over $\mathbb{R}$ or $\mathbb{C}$) with an inner product $\cdot,\cdot$.

For example, $\mathbb{R}^n$ with the familiar dot product forms an inner product space.

Every inner product space is also a normed vector space, with the norm defined by $\|x\| := \sqrt{x,\, x}$. This norm satisfies the parallelogram law.

If the metric $\|x-y\|$ induced by the norm is completeComplete, then the inner product space is called a Hilbert space.

The Cauchy–Schwarz inequality

$$|x,\, y| \leq \|x\|\cdot\|y\| \tag{11.1}$$

holds in any inner product space.

According to (1), one can define the *angle between two non-zero vectors* x and y:

$$\cos(x,\, y) := \frac{x,\, y}{\|x\|\cdot\|y\|}. \tag{11.2}$$

This provides that the scalars are the real numbers. In any case, the *perpendiculatity* of the vectors may be defined with the condition

$$x,\, y = 0.$$

["inner product space" is defined at PM by Mr. Chi Woo].

11.2. **Hilbert space: DEFINITION.**

A **Hilbert space** is an inner product space which is complete under the induced metric.

In particular, a Hilbert space is a Banach space in the norm induced by the inner product, since the norm and the inner product both induce the same metric. Any finite-dimensional inner product space is a Hilbert space, but it is worth mentioning that some authors require the space to be infinite dimensional for it to be called a Hilbert space.

11.3. **Von Neumann Algebras.** Let $\mathcal{H}$ denote a complex (separable) Hilbert space. A *von Neumann algebra* $\mathcal{A}$ acting on $\mathcal{H}$ is a subset of the algebra of all bounded operators $\mathcal{L}(\mathcal{H})$ such that:

(1) $\mathcal{A}$ is closed under the adjoint operation (with the adjoint of an element T denoted by T^*).

(2) $\mathcal{A}$ equals its bicommutant, namely:

$$\mathcal{A} = \{A \in \mathcal{L}(\mathcal{H}) : \forall B \in \mathcal{L}(\mathcal{H}), \forall C \in \mathcal{A},\ (BC = CB) \Rightarrow (AB = BA)\}\ . \tag{11.3}$$

If one calls a *commutant* of a set $\mathcal{A}$ the special set of bounded operators on $\mathcal{L}(\mathcal{H})$ which commute with all elements in $\mathcal{A}$, then this second condition implies that the commutant of the commutant of $\mathcal{A}$ is again the set $\mathcal{A}$.

On the other hand, a von Neumann algebra $\mathcal{A}$ inherits a *unital* subalgebra from $\mathcal{L}(\mathcal{H})$, and according to the first condition in its definition $\mathcal{A}$ does indeed inherit a **-subalgebra* structure, as further explained in the next section on C*-algebras. Furthermore, we have notable *Bicommutant Theorem* which states that $\mathcal{A}$ *is a von Neumann algebra if and only if* $\mathcal{A}$ *is a *-subalgebra of* $\mathcal{L}(\mathcal{H})$, *closed for the smallest topology defined by continuous maps* $(\xi, \eta) \longmapsto (A\xi, \eta)$ *for all* $< A\xi, \eta) >$ *where* $< .,. >$ *denotes the inner product defined on* $\mathcal{H}$. For a well-presented treatment of the geometry of the state spaces of quantum operator algebras, see e.g. Aflsen and Schultz (2003).

11.4. **Groupoids.** Recall that a groupoid $\mathcal{G}$ is, loosely speaking, a small category with inverses over its set of objects $X = \mathrm{Ob}(\mathcal{G})$. One often writes $\mathcal{G}_x^y$ for the set of morphisms in $\mathcal{G}$ from x to y . *A topological groupoid* consists of a space G, a distinguished subspace $\mathsf{G}^{(0)} = \mathrm{Ob}(\mathsf{G}) \subset \mathsf{G}$, called *the space of objects* of G, together with maps

$$r, s\ :\ \ \mathsf{G} \underset{s}{\overset{r}{\rightrightarrows}} \mathsf{G}^{(0)} \tag{11.4}$$

called the *range* and *source maps* respectively, together with a law of composition

$$\circ\ :\ \mathsf{G}^{(2)} := \mathsf{G} \times_{\mathsf{G}^{(0)}} \mathsf{G} = \{\ (\gamma_1, \gamma_2) \in \mathsf{G} \times \mathsf{G}\ :\ s(\gamma_1) = r(\gamma_2)\ \} \longrightarrow \mathsf{G}\ , \tag{11.5}$$

such that the following hold :

(1) $s(\gamma_1 \circ \gamma_2) = r(\gamma_2)\ ,\ r(\gamma_1 \circ \gamma_2) = r(\gamma_1)$, for all $(\gamma_1, \gamma_2) \in \mathsf{G}^{(2)}$.

(2) $s(x) = r(x) = x$, for all $x \in \mathsf{G}^{(0)}$.

(3) $\gamma \circ s(\gamma) = \gamma\ ,\ r(\gamma) \circ \gamma = \gamma$, for all $\gamma \in \mathsf{G}$.

(4) $(\gamma_1 \circ \gamma_2) \circ \gamma_3 = \gamma_1 \circ (\gamma_2 \circ \gamma_3)$.

(5) Each γ has a two–sided inverse γ^{-1} with $\gamma\gamma^{-1} = r(\gamma)\ ,\ \gamma^{-1}\gamma = s(\gamma)$. Furthermore, only for topological groupoids the inverse map needs be continuous.

It is usual to call $\mathsf{G}^{(0)} = \mathrm{Ob}(\mathsf{G})$ *the set of objects* of G . For $u \in \mathrm{Ob}(\mathsf{G})$, the set of arrows $u \longrightarrow u$ forms a group $\mathsf{G_u}$, called the *isotropy group of* G *at* u.

Thus, as is well kown, a topological groupoid is just a groupoid internal to the category of topological spaces and continuous maps. The notion of internal groupoid has proved significant in a number of fields, since groupoids generalise bundles of groups, group actions, and equivalence relations. For a further study of groupoids we refer the reader to Brown (2006).

Examples of groupoids are often encountered; the following are just a few specialized groupoid structures: (a) locally compact groups, transformation groups , and any group in general, (b) equivalence relations, (c) tangent bundles, (d) the tangent groupoid, (e) holonomy groupoids for foliations, (f) Poisson groupoids, and (g) graph groupoids.

As a simple, helpful example of a groupoid, consider (b) above. Thus, let R be an *equivalence relation* on a set X. Then R is a groupoid under the following operations: $(x, y)(y, z) = (x, z), (x, y)^{-1} = (y, x)$. Here, $\mathcal{G}^0 = X$, (the diagonal of $X \times X$) and $r((x, y)) = x, s((x, y)) = y$.

So $R^2 = \{((x, y), (y, z)) : (x, y), (y, z) \in R\}$. When $R = X \times X$, R is called a *trivial* groupoid. A special case of a trivial groupoid is $R = R_n = \{1, 2, ..., n\} \times \{1, 2, ..., n\}$. (So every i is equivalent to every j). Identify $(i, j) \in R_n$ with the matrix unit e_{ij}. Then the groupoid R_n is just matrix multiplication except that we only multiply e_{ij}, e_{kl} when $k = j$, and $(e_{ij})^{-1} = e_{ji}$. We do not really lose anything by restricting the multiplication, since the pairs e_{ij}, e_{kl} excluded from groupoid multiplication just give the 0 product in normal algebra anyway. For a groupoid $\mathsf{G}_{lc}$to be a *locally compact groupoid* means that G_{lc} is required to be a (second countable) *locally compact Hausdorff* space, and the product and also inversion maps are required to be *continuous*. Each G^u_{lc} as well as the unit space G^0_{lc} is closed in G_{lc}. What replaces the left Haar measure on G_{lc} is a system of measures λ^u $(u \in \mathsf{G}^0_{lc})$, where λ^u is a positive regular Borel measure on G^u_{lc} with dense support. In addition, the λ^u s are required to vary continuously (when integrated against $f \in C_c(\mathsf{G}_{lc})$ and to form an invariant family in the sense that for each x, the map $y \mapsto xy$ is a measure preserving homeomorphism from $\mathsf{G}^s_{lc}(x)$ onto $\mathsf{G}^r_{lc}(x)$. Such a system $\{\lambda^u\}$ is called a *left Haar system* for the locally compact groupoid G_{lc}.

This is defined more precisely next.

11.5. **Haar systems for locally compact topological groupoids.** Let

$$\mathsf{G}_{lc} \underset{s}{\overset{r}{\rightrightarrows}} \mathsf{G}^{(0)}_{lc} = X \tag{11.6}$$

be a locally compact, locally trivial topological groupoid with its transposition into transitive (connected) components. Recall that for $x \in X$, the *costar of* x denoted CO*(x) is defined as the closed set $\bigcup\{\mathsf{G}_{lc}(y, x) : y \in \mathsf{G}_{lc}\}$, whereby

$$\mathsf{G}_{lc}(x_0, y_0) \hookrightarrow \mathrm{CO}^*(\mathrm{x}) \longrightarrow \mathrm{X} \ , \tag{11.7}$$

is a principal $\mathsf{G}_{lc}(x_0, y_0)$–bundle relative to fixed base points (x_0, y_0) . Assuming all relevant sets are locally compact, then following Seda (1976), a *(left) Haar system on* G_{lc} denoted (G_{lc}, τ) (for later purposes), is defined to comprise of i) a measure κ on G_{lc}, ii) a measure μ on X and iii) a measure μ_x on CO*(x) such that for every Baire set E of G_{lc}, the following hold on setting $E_x = E \cap \mathrm{CO}^*(\mathrm{x})$:

(1) $x \mapsto \mu_x(E_x)$ is measurable.

(2) $\kappa(E) = \int_x \mu_x(E_x)\ d\mu_x$.

(3) $\mu_z(tE_x) = \mu_x(E_x)$, for all $t \in \mathsf{G}_{lc}(x, z)$ and $x, z \in \mathsf{G}_{lc}$.

The presence of a left Haar system on G_{lc} has important topological implications: it requires that the range map $r : \mathsf{G}_{lc} \to \mathsf{G}^0_{lc}$ is open. For such a G_{lc} with a left Haar system, the vector space $C_c(\mathsf{G}_{lc})$ is a *convolution *–algebra*, where for $f, g \in C_c(\mathsf{G}_{lc})$:

$f * g(x) = \int f(t)g(t^{-1}x)d\lambda^{r(x)}(t)$, with f*(x) = $\overline{f(x^{-1})}$.

One has $C^*(\mathsf{G}_{lc})$ to be the *enveloping* C^**–algebra* of $C_c(\mathsf{G}_{lc})$ (and also representations are required to be continuous in the inductive limit topology). Equivalently, it is the completion of $\pi_{univ}(C_c(\mathsf{G}_{lc}))$ where π_{univ} is the *universal representation* of G_{lc}. For example, if $\mathsf{G}_{lc} = R_n$, then $C^*(\mathsf{G}_{lc})$ is just the finite dimensional algebra $C_c(\mathsf{G}_{lc}) = M_n$, the span of the e_{ij} ' s.

There exists (e.g.[63, p.91]) a *measurable Hilbert bundle* $(\mathsf{G}^0_{lc}, \mathcal{H}, \mu)$ with $\mathcal{H} = \left\{\mathcal{H}^u_{u\in\mathsf{G}^0_{lc}}\right\}$ and a G-representation L on $\mathcal{H}$. Then, for every pair ξ, η of square integrable sections of $\mathcal{H}$, it is required that the function $x \mapsto (L(x)\xi(s(x)), \eta(r(x)))$ be ν–measurable. The representation Φ of $C_c(\mathsf{G}_{lc})$ is then given by:
$\langle\Phi(f)\xi|, \eta\rangle = \int f(x)(L(x)\xi(s(x)), \eta(r(x)))d\nu_0(x)$.

The triple $(\mu, \mathcal{H}, L)$ is called a *measurable* G_{lc}*–Hilbert bundle.*

12. C*-algebras and Compact Quantum Groupoids (CGQd' s)

12.1. **Von Neumann and C*-algebras: Quantum Operator Algebra and Quantum Theories.** C*-algebra has evolved as a key concept in Quantum Operator Algebra after the introduction of the von Neumann algebra for the mathematical foundation of Quantum Mechanics. The von Neumann algebra classification is simpler and studied in greater depth than that of general C*-algebra classification theory. The importance of C*-algebras for understanding the geometry of quantum state spaces (Alfsen and Schultz, 2003 [2]) cannot be overestimated. Moreover, the introduction of non-commutative C*-algebras in Noncommutative Geometry has already played important roles in expanding the Hilbert space perspective of Quantum Mechanics developed by von Neumann. Furthermore, extended quantum symmetries are currently being approached in terms of groupoid C*- convolution algebra and their representations; the latter also enter into the construction of compact quantum groupoids as developed in the Bibliography cited, and also briefly outlined here in the second section. The fundamental connections that exist between categories of C^*-algebras and those of von Neumann and other quantum operator algebras, such as JB- or JBL- algebras are yet to be completed and are the subject of in depth studies [2].

A **C*-algebra** is simultaneously a *–algebra and a Banach space -with additional conditions- as defined next.

Let us consider first the definition of an *involution* on a complex algebra $\mathfrak{A}$.

Definition 12.1. An *involution* on a complex algebra $\mathfrak{A}$ is a *real–linear map* $T \mapsto T^*$ such that for all

$S, T \in \mathfrak{A}$ and $\lambda \in \mathbb{C}$, we have $T^{**} = T$, $(ST)^* = T^*S^*$, $(\lambda T)^* = \bar{\lambda}T^*$.

A **-algebra* is said to be a complex associative algebra together with an involution $*$.

Definition 12.2. A *C*-algebra* is simultaneously a *-algebra and a Banach space $\mathfrak{A}$, satisfying for all $S, T \in \mathfrak{A}$ the following conditions:

$\|S \circ T\| \le \|S\| \, \|T\|$,
$\|T^*T\|^2 = \|T\|^2$.

One can easily verify that $\|A^*\| = \|A\|$.

By the above axioms a C*–algebra is a special case of a Banach algebra where the latter requires the above C*-norm property, but not the involution (*) property.

Given Banach spaces E, F the space $\mathcal{L}(E, F)$ of (bounded) linear operators from E to F forms a Banach space, where for $E = F$, the space $\mathcal{L}(E) = \mathcal{L}(E, E)$ is a Banach algebra with respect to the norm

$$\|T\| := \sup\{\|Tu\| : u \in E \ , \ \|u\| = 1\} \ .$$

In quantum field theory one may start with a Hilbert space H, and consider the Banach algebra of bounded linear operators $\mathcal{L}(H)$ which given to be closed under the usual algebraic operations and taking adjoints, forms a $*$–algebra of bounded operators, where the adjoint operation functions as the involution, and for $T \in \mathcal{L}(H)$ we have :

$$\|T\| := \sup\{(Tu, Tu) : u \in H \ , \ (u, u) = 1\} \ , \text{and } \|Tu\|^2 = (Tu, Tu) = (u, T^*Tu) \leq \|T^*T\| \ \|u\|^2 \ .$$

By a *morphism between C^*-algebras* $\mathfrak{A}, \mathfrak{B}$ we mean a linear map $\phi : \mathfrak{A} \longrightarrow \mathfrak{B}$, such that for all $S, T \in \mathfrak{A}$, the following hold :

$$\phi(ST) = \phi(S)\phi(T) \ , \ \phi(T^*) = \phi(T)^* \ ,$$

where a bijective morphism is said to be an isomorphism (in which case it is then an isometry). A fundamental relation is that any norm-closed $*$-algebra $\mathcal{A}$ in $\mathcal{L}(H)$ is a *C^*-algebra*, and conversely, any C*-algebra is isomorphic to a norm–closed $*$-algebra in $\mathcal{L}(H)$ for some Hilbert space H . One can thus also define *the category $\mathcal{C}^*$ of C^*-algebras and morphisms between C^*-algebras.*

For a *C^*-algebra* $\mathfrak{A}$, we say that $T \in \mathfrak{A}$ is *self–adjoint* if $T = T^*$. Accordingly, the self–adjoint part $\mathfrak{A}^{sa}$ of $\mathfrak{A}$ is a real vector space since we can decompose $T \in \mathfrak{A}^{sa}$ as :

$$T = T' + T'' := \tfrac{1}{2}(T + T^*) + \iota(\tfrac{-\iota}{2})(T - T^*) \ .$$

A *commutative* C*–algebra is one for which the associative multiplication is commutative. Given a commutative C*–algebra $\mathfrak{A}$, we have $\mathfrak{A} \cong C(Y)$, the algebra of continuous functions on a compact Hausdorff space Y .

The classification of C^*-algebras is far more complex than that of von Neumann algebras that provide the fundamental algebraic content of quantum state and operator spaces in quantum theories.

12.2. **Quantum Groupoids and the Groupoid C*–Algebra.** Quantum 'groupoid' (e.g., weak Hopf algebras) and algebroid symmetries figure prominently both in the theory of dynamical deformations of quantum 'groups' (e.g., Hopf algebras) and the quantum Yang–Baxter equations (Etingof et al., 1999, 2001). On the other hand, one can also consider the natural extension of locally compact (quantum) groups to locally compact (proper) *groupoids* equipped with a Haar measure and a corresponding groupoid representation theory (Buneci, 2003) as a major, potentially interesting source for locally compact (but generally *non-Abelian*) *quantum groupoids.* The corresponding quantum groupoid representations on bundles of Hilbert spaces extend quantum symmetries well beyond those of quantum 'groups'/Hopf algebras and simpler operator algebra representations, and are also consistent with the locally compact quantum group representations that were recently studied in some detail by Kustermans and Vaes (2000, and references cited therein). The latter quantum groups are neither Hopf algebras, nor are they equivalent to Hopf algebras or their dual coalgebras. As pointed out in the previous section, quantum groupoid representations are, however, the next important step towards unifying quantum field theories with

General Relativity in a locally covariant and quantized form. Such representations need not however be restricted to weak Hopf algebra representations, as the latter have no known connection to any type of GR theory and also appear to be inconsistent with GR.

In Nikshych and Vainerman (2000) quantum groupoids (considered as weak C*–Hopf algebras) were studied in relationship to the noncommutative symmetries of depth 2 von Neumann subfactors. If

$$A \subset B \subset B_1 \subset B_2 \subset \dots \tag{12.1}$$

is the Jones extension induced by a finite index depth 2 inclusion $A \subset B$ of II_1 factors, then $Q = A' \cap B_2$ admits a quantum groupoid structure and acts on B_1, so that $B = B_1^Q$ and $B_2 = B_1 \rtimes Q$. Similarly, in Rehren (1997) 'paragroups' (derived from weak C*–Hopf algebras) comprise (quantum) groupoids of equivalence classes such as those associated with 6j–symmetry groups (relative to a fusion rules algebra). They correspond to type II von Neumann algebras in quantum mechanics, and arise as symmetries where the local subfactors (in the sense of containment of observables within fields) have depth 2 in the Jones extension. A related question is how a von Neumann algebra N, such as of finite index depth 2, sits inside a weak Hopf algebra formed as the crossed product $N \rtimes A$ (Böhm et al. 1999).

12.3. **Quantum Compact Groupoids.** Compact quantum groupoids were introduced in Landsman (1998) as a simultaneous generalization of a compact groupoid and a quantum group. Since this construction is relevant to the definition of locally compact quantum groupoids and their representations investigated here, its exposition is required before we can step up to the next level of generality. Firstly, let $\mathfrak{A}$ and $\mathfrak{B}$ denote C*–algebras equipped with a *–homomorphism $\eta_s : \mathfrak{B} \longrightarrow \mathfrak{A}$, and a *–antihomomorphism $\eta_t : \mathfrak{B} \longrightarrow \mathfrak{A}$ whose images in $\mathfrak{A}$ commute. A non–commutative Haar measure is defined as a completely positive map $P : \mathfrak{A} \longrightarrow \mathfrak{B}$ which satisfies $P(A\eta_s(B)) = P(A)B$. Alternatively, the composition $\mathcal{E} = \eta_s \circ P : \mathfrak{A} \longrightarrow \eta_s(B) \subset \mathfrak{A}$ is a faithful conditional expectation.

Next consider G to be a (topological) groupoid as defined in the Appendix. We denote by $C_c(\mathsf{G})$ the space of smooth complex–valued functions with compact support on G . In particular, for all $f, g \in C_c(\mathsf{G})$, the function defined via convolution

$$(f \ * \ g)(\gamma) = \int_{\gamma_1 \circ \gamma_2 = \gamma} f(\gamma_1) g(\gamma_2) \ , \tag{12.2}$$

is again an element of $C_c(\mathsf{G})$, where the convolution product defines the composition law on $C_c(\mathsf{G})$. We can turn $C_c(\mathsf{G})$ into a *–algebra once we have defined the involution $*$, and this is done by specifying $f^*(\gamma) = \overline{f(\gamma^{-1})}$.

We recall that following Landsman (1998) a *representation* of a groupoid $\mathcal{G}$, consists of a family (or field) of Hilbert spaces $\{\mathcal{H}_x\}_{x \in X}$ indexed by $X = \text{Ob } \mathcal{G}$, along with a collection of maps $\{U(\gamma)\}_{\gamma \in \mathcal{G}}$, satisfying:

1. $U(\gamma) : \mathcal{H}_{s(\gamma)} \longrightarrow \mathcal{H}_{r(\gamma)}$, is unitary.
2. $U(\gamma_1 \gamma_2) = U(\gamma_1) U(\gamma_2)$, whenever $(\gamma_1, \gamma_2) \in \mathcal{G}^{(2)}$ (the set of arrows).
3. $U(\gamma^{-1}) = U(\gamma)^*$, for all $\gamma \in \mathcal{G}$.

Suppose now G_{lc} is a Lie groupoid. Then the isotropy group G_x is a Lie group, and for a (left or right) Haar measure μ_x on G_x, we can consider the Hilbert spaces $\mathcal{H}_x = L^2(\mathsf{G}_x, \mu_x)$ as exemplifying the above sense of a representation. Putting aside some technical details which can be found in Connes (1994) and Landsman (2006), the overall idea is to define an operator of Hilbert spaces

$$\pi_x(f) : L^2(\mathsf{G}_{\mathsf{x}}, \mu_x) \longrightarrow L^2(\mathsf{G}_x, \mu_x) \ , \tag{12.3}$$

given by

$$(\pi_x(f)\xi)(\gamma) = \int f(\gamma_1)\xi(\gamma_1^{-1}\gamma) \ d\mu_x \ , \tag{12.4}$$

for all $\gamma \in \mathsf{G}_x$, and $\xi \in \mathcal{H}_x$. For each $x \in X = \mathrm{Ob}\ \mathsf{G}$, π_x defines an involutive representation $\pi_x : C_c(\mathsf{G}) \longrightarrow \mathcal{H}_x$. We can define a norm on $C_c(\mathsf{G})$ given by

$$\|f\| = \sup_{x \in X} \|\pi_x(f)\| \ , \tag{12.5}$$

whereby the completion of $C_c(\mathsf{G})$ in this norm, defines *the reduced C^*–algebra $C_r^*(\mathsf{G})$ of G_{lc}*. It is perhaps the most commonly used C*–algebra for Lie groupoids (groups) in noncommutative geometry.

The next step requires a little familiarity with the theory of Hilbert modules (see e.g. Lance, 1995). We define a left $\mathfrak{B}$–action λ and a right $\mathfrak{B}$–action ρ on $\mathfrak{A}$ by $\lambda(B)A = A\eta_t(B)$ and $\rho(B)A = A\eta_s(B)$. For the sake of localization of the intended Hilbert module, we implant a $\mathfrak{B}$–valued inner product on $\mathfrak{A}$ given by $\langle A, C\rangle_{\mathfrak{B}} = P(A^*C)$. Let us recall that P is defined as a *completely positive map*. Since P is faithful, we fit a new norm on $\mathfrak{A}$ given by $\|A\|^2 = \|P(A^*A)\|_{\mathfrak{B}}$. The completion of $\mathfrak{A}$ in this new norm is denoted by $\mathfrak{A}^-$ leading then to a Hilbert module over $\mathfrak{B}$.

The tensor product $\mathfrak{A}^- \otimes_{\mathfrak{B}} \mathfrak{A}^-$ can be shown to be a Hilbert bimodule over $\mathfrak{B}$, which for $i = 1, 2$, leads to *–homorphisms $\varphi^i : \mathfrak{A} \longrightarrow \mathcal{L}_{\mathfrak{B}}(\mathfrak{A}^- \otimes \mathfrak{A}^-)$. Next is to define the (unital) C*–algebra $\mathfrak{A} \otimes_{\mathfrak{B}} \mathfrak{A}$ as the C*–algebra contained in $\mathcal{L}_{\mathfrak{B}}(\mathfrak{A}^- \otimes \mathfrak{A}^-)$ that is generated by $\varphi^1(\mathfrak{A})$ and $\varphi^2(\mathfrak{A})$. The last stage of the recipe for defining a compact quantum groupoid entails considering a certain coproduct operation $\Delta : \mathfrak{A} \longrightarrow \mathfrak{A} \otimes_{\mathfrak{B}} \mathfrak{A}$, together with a coinverse $Q : \mathfrak{A} \longrightarrow \mathfrak{A}$ that it is both an algebra and bimodule antihomomorphism. Finally, the following axiomatic relationships are observed :

$$\begin{aligned} (\mathrm{id} \otimes_{\mathfrak{B}} \Delta) \circ \Delta &= (\Delta \otimes_{\mathfrak{B}} \mathrm{id}) \circ \Delta \\ (\mathrm{id} \otimes_{\mathfrak{B}} P) \circ \Delta &= P \\ \tau \circ (\Delta \otimes_{\mathfrak{B}} Q) \circ \Delta &= \Delta \circ Q \end{aligned} \tag{12.6}$$

where τ is a flip map : $\tau(a \otimes b) = (b \otimes a)$.

There is a natural extension of the above definition of quantum compact groupoids to *locally compact* quantum groupoids by taking G_{lc} to be a locally compact groupoid (instead of a compact groupoid), and then following the steps in the above construction with the topological groupoid G being replaced by G_{lc}. Additional integrability and Haar measure system conditions need however be also satisfied as in the general case of locally compact groupoid *representations* (for further details, see for example the monograph by Buneci (2003).

12.4. **'Alternative' Definition of C*** $-$ *algebra*. C^*-algebras are a type of involutive Banach algebras which arise in the study of operators on Hilbert spaces, Lie group representations, locally compact topological spaces, knots, *noncommutative geometry*, among other topics in mathematics and theoretical physics . Their study was initiated in the 1930's with the purpose of axiomatizing quantum mechanics, and still today, C^*-algebras play a decisive role in formulations of quantum statistical mechanics and *quantum field theory.*

The defining property of these algebras is that the norm and the involution are related in a very special way.

Definition 1 - *A* C^****-algebra*** $\mathcal{A}$ *is a Banach *-algebra such that* $a^*a = a^2$ *for all* $a \in \mathcal{A}$.

The equality in Definition 1 is sometimes called the C^* **axiom**. It turns out that one can weaken this condition and still specify the same *class* of algebras.

Definition 2 - *A* C^****-algebra*** $\mathcal{A}$ *is a Banach algebra with an antilinear involution* $*$ *such that* $a^2 \leq a^*a$ *for all* $a \in \mathcal{A}$.

Definition 3 - *A* C^****-algebra*** $\mathcal{A}$ *is a Banach algebra with an antilinear involution* $*$ *such that* $a^*a = a^*a$

12.4.1. *C^* Norm.* C^*-algebras are a very peculiar type of topological algebras. The C^* axiom, deceptively simple, imposes severe restrictions on the the algebraic and topological structure of a C^*-algebra.

A most striking consequence of the C^* axiom is that the norm is solely determined by the algebraic structure of the algebra. More specifically,

$$\|a\| = \sqrt{R_\sigma(a^*a)}$$

where $R_\sigma(x)$ denotes the spectral radius of the element $x \in \mathcal{A}$. For C^* algebras with an identity element e we can specify even further: the norm of an element $a \in \mathcal{A}$ is determined by

$$\|a\|^2 = \sup\{|\lambda| : \lambda \in \mathbb{C} \text{ and } a^*a - \lambda e \text{ is not invertible}\}$$

This also implies that the norm in a C^*-algebra is unique, in the sense that there is no other norm in the algebra that satisfies that C^* axiom, i.e. that turns the algebra into a C^*-algebra. This is a stark contrast to the case of general normed algebras, where one may find many norms which are compatible with the algebraic structure.

Moreover, the C^* norm occupies a unique place amongst the possible norms for an involutive algebra. Suppose that $\mathcal{A}$ is a C^* algebra with norm $\cdot_{C^*}$. If $\cdot_B$ is any other norm for which $\mathcal{A}$ is a Banach *-algebra, then we must have

$$\|a\|_{C^*} \leq \|a\|_B\,, \qquad \forall a \in \mathcal{A}$$

Hence we see that the C^* norm enjoys an extremal property — it is the smallest possible norm for which $\mathcal{A}$ is a Banach *-algebra.

There are many other surprising consequences of the C^* axiom, like: *-homomorphisms between C^*-algebras are automatically continuous and every C^*-algebra is semi-simple, which again are not true for general involutive algebras.

12.4.2. *Elements of a C^*-algebra.* Like in involutory rings, there are some special elements in C^*-algebras that deserve some attention. We recall some definitions here:

Let $\mathcal{A}$ be a C^*-algebra with identity element e. An element $a \in \mathcal{A}$ is said to be

- **self-adjoint** if $a^* = a$
- **unitary** if $a^*a = aa^* = e$
- **positive** if $a = b^*b$ for some element $b \in \mathcal{A}$

It is many times useful to have some interpretation for this elements. One of this interpretations comes from complex analysis: we regard the elements of a C^*-algebra as functions with values in $\mathbb{C}$ and the involution as complex conjugation.

In this frame, self-adjoint elements correspond to real functions, unitary elements correspond to functions whose values lie in the unit circle in $\mathbb{C}$ and positive elements correspond to positive functions (functions with values in $\mathbb{R}^+_{\not{}}$).

It is easily seen that self-adjoint elements are closed under addition, multiplication and multiplication by real numbers. It can be proven the same for positive elements (with multiplication by positive numbers).

There are some decompositions of elements in a C^*-algebra analogous to some decompositions in complex analysis. For instance, every element a in a C^*-algebra has a unique decomposition of the form

$$a = x + iy$$

where x, y are self-adjoint. This is similar to the decomposition of a complex valued function in its real and imaginary parts.

Moreover, every self-adjoint element a is of the form

$$a = x - y$$

where x, y are positive elements. This is similar to the decomposition of real valued functions in its positive and negative parts.

There are many other aspects of the theory of C^*-algebras for which this kind of interpretation proves to be very insightful.

For example, C^*-algebras happen to have a natural partial ordering. One can define an ordering by declaring that $x > y$ when $x - y$ is positive. Given this ordering, one can then speak of such things as monotonic functions, monotonic sequences, and positive linear functionals on the algebra. These notions, in turn, prove to be extremely useful in the study of C^*-algebras.

12.4.3. *Examples.* Having discussed these algebras in general terms, it is high time that we illustrate the definition with some examples.

Example 1

As our first class of examples, we consider algebras of functions. Let X be a compact Hausdorff topological space and let $C(X)$ be the algebra of continuous functions from X to $\mathbb{C}$. For the involution operation, we take pointwise complex conjugation and for the norm we take the norm of uniform convergence:

$$f = \sup_{x \in X} |f(x)|$$

It is a routine matter to check that the norm and involution satisfy the appropriate algebraic requirements. Completeness under this norm follows from the fact that the uniform limit of continuous functions on a locally compact Hausdorff topological space is continuous.

More generally, instead of a compact space, we can take a locally compact Hausdorff space X and consider the algebra $C_0(X)$ of continuous functions $X \to \mathbb{C}$ that vanish at infinity, endowed with the same norm and involution. These are important examples of C^*-algebras.

Example 2

As our second class of examples, we consider operator algebras. Let H be a complex Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and let $B(H)$ be the algebra of bounded operators on H. For the involution, we take the adjoint operation and as a norm we take the usual operator norm:

$$T = \sup_{\|\xi\|=1} \|T\xi\|$$

Again, it is straightforward to verify that the norm and involution satisfy the appropriate algebraic requirements, as is done in an attachment to this entry. Completeness under the norm follows from a well-known theorem of functional analysis.

12.4.4. *Commutative vs. Non–commutative C^*-algebras.* The algebras $C_0(X)$ in Example 1 above are more than just an example. In fact, all commutative C^*-algebras are *-isomorphic to $C_0(X)$ for some locally compact Hausdorff space X.

Moreover, *X is compact if and only if the C^*-algebra has an identity element.* This is the content of **the Gelfand-Naimark theorem.**

Furthermore, there is a correspondence between properties of the topological space and properties of the C^*-algebra. For example: a compactification of the space corresponds to a unitization of the C^*-algebra; the space is connected if and only if the C^*-algebra has no non-trivial projections, among many other interesting correspondences. For this reason, the *theory of non–commutative C^*-algebras* is sometimes called **non–commutative topology**, or non–commutative geometry.
The second example is also more than just an example of C^*-algebras. In fact, by the Gelfand-Naimark representation theorem, all C^*-algebras are *-isomorphic to a norm closed *-subalgebra of $B(H)$, for some Hilbert space H. Note, however, that this does not provide a "classification" of C^*-algebras since we do not know in general what are the closed *-subalgebras of $B(H)$. This is merely a very-important structural theorem. The classification problem for C^*-algebras is still open.

Additional Examples of C^*–algebras

Example 3 Compact operators in a Hilbert space H form a closed ideal of $B(H)$. Moreover, this ideal is also closed for the involution of operators. Hence, the algebra of compact operators, $K(H)$, is a C^*-algebra.

Example 4 Let $(X, \mathfrak{B}, \mu)$ be a measure space. The space $L^\infty(X)$ (LpSpace) is an algebra under pointwise operations. We can define an involution again by complex conjugation and we consider the essential supremum norm $\|\cdot\|_\infty$. It can be readily verified that, under these operations and norm, $L^\infty(X)$ is a C^*-algebra. The algebras $L^\infty(X)$ are also particularly important since they are examples of von Neumann algebras, which are a specific kind of C^*-algebras.

Example 5

12.4.5. *Reduced C*–algebra.* Consider G to be a topological groupoid. We denote by $C_c(\mathsf{G})$ the space of smooth complex–valued functions with compact support on G . In particular, for all $f, g \in C_c(\mathsf{G})$, the function defined via convolution

$$(f \ * \ g)(\gamma) = \int_{\gamma_1 \circ \gamma_2 = \gamma} f(\gamma_1) g(\gamma_2) \ , \tag{12.7}$$

is again an element of $C_c(\mathsf{G})$, where the convolution product defines the composition law on $C_c(\mathsf{G})$. We can turn $C_c(\mathsf{G})$ into a *–algebra once we have defined the involution $*$, and this is done by specifying $f^*(\gamma) = \overline{f(\gamma^{-1})}$.

It is perhaps the most commonly used C*–algebra for Lie groupoids (groups) in noncommutative geometry.

12.5. **Lie Groups and Orthogonal groups. Lie group Definition:**

*A **Lie group** is a group endowed with a compatible analytic structure.* To be more precise, Lie group structure consists of two kinds of data a **finite-dimensional, real-analytic manifold** , and **two analytic maps**, one for *multiplicatio*n and one for *inversion* , which obey the appropriate group axioms. Thus, a homomorphism in the category of Lie groups is a *group homomorphism* that is simultaneously an analytic mapping between two real-analytic manifolds.

To be more precise, Lie group structure consists of two kinds of data

- a finite-dimensional, real-analytic manifold G, and
- two analytic maps, one for multiplication $G \times G \to G$ and one for inversion $G \to G$, which obey the appropriate group axioms.

Thus, a homomorphism in the category of Lie groups is a group homomorphism that is simultaneously an analytic mapping between two real-analytic manifolds.

Next, we describe a natural construction that associates a certain Lie algebra to every Lie group G. Let $e \in G$ denote the identity element of G. For $g \in G$ let $_g\colon G \to G$ denote the diffeomorphisms corresponding to left multiplication by g.

Definition 12.3. A vector field V on G is called left-invariant if V is invariant with respect to all left multiplications. To be more precise, V is left-invariant if and only if

$$(_g)_*(V) = V$$

(see push-forward of a vector-field) for all $g \in G$.

Proposition 12.1. *The Lie bracket of two left-invariant vector fields is again, a left-invariant vector field.*

Proof. Let V_1, V_2 be left-invariant vector fields, and let $g \in G$. The bracket operation is covariant with respect to diffeomorphism, and in particular

$$(_g)_*[V_1, V_2] = [(_g)_*V_1, (_g)_*V_2] = [V_1, V_2].$$

Definition 12.4. The **Lie algebra of G, denoted hereafter by , is the vector space of all left-invariant vector fields equipped with the vector-field bracket.**

Now a right multiplication is invariant with respect to all left multiplications, and it turns out that we can characterize a left-invariant vector field as being an infinitesimal right multiplication.

Proposition 12.2. *Let $a \in T_eG$ and let V be a left-invariant vector-field such that $V_e = a$. Then for all $g \in G$ we have*

$$V_g = (_g)_*(a).$$

The intuition here is that a gives an infinitesimal displacement from the identity element and that V_g gives a corresponding infinitesimal right displacement away from g. Indeed consider a curve

$$\gamma\colon (-\epsilon, \epsilon) \to G$$

passing through the identity element with velocity a; i.e.

$$\gamma(0) = e, \quad \gamma'(0) = a.$$

The above proposition is then saying that the curve

$$t \mapsto g\gamma(t), \quad t \in (-\epsilon, \epsilon)$$

passes through g at $t = 0$ with velocity V_g.

Thus we see that a left-invariant vector field is completely determined by the value it takes at e, and that therefore is isomorphic, as a vector space to T_eG.

Of course, we can also consider the Lie algebra of right-invariant vector fields. The resulting Lie-algebra is anti-isomorphic (the order in the bracket is reversed) to the Lie algebra of left-invariant vector fields. Now it is a general principle that the group inverse operation gives an anti-isomorphism between left and right group actions. So, as one may well expect, the anti-isomorphism between the Lie algebras of left and right-invariant vector fields can be realized by considering the linear action of the inverse operation on T_eG.

Finally, let us remark that one can induce the Lie algebra structure directly on T_eG by considering adjoint action of G on T_eG.

Notes.

(1) No generality is lost in assuming that a Lie group has analytic, rather than C^∞ or even $C^k,\ k = 1, 2, \ldots$ structure. Indeed, given a C^1 differential manifold with a C^1 multiplication rule, one can show that the exponential mapping endows this manifold with a compatible real-analytic structure.

Indeed, one can go even further and show that even C^0 suffices. In other words, a topological group that is also a finite-dimensional topological manifold possesses a compatible analytic structure. This result was formulated by Hilbert as his *fifth problem* (http://www.reed.edu/ wieting/essays/LieHilbert.pdf), and proved in the 1950's by Montgomery and Zippin.

(2) One can also speak of a complex Lie group, in which case G and the multiplication mapping are both complex-analytic. The theory of *complex* Lie groups requires the notion of a *holomorphic vector-field.* Not withstanding this complication, most of the essential features of the real theory carry over to the complex case.

(3) The name "Lie group" honours the Norwegian mathematician Sophus Lie who pioneered and developed the theory of continuous transformation groups and the corresponding theory of Lie algebras of vector fields (the group's infinitesimal generators, as Lie termed them). Lie's original impetus was the study of continuous symmetry of geometric objects and differential equations.

The scope of the theory has grown enormously in the 100+ years of its existence. The contributions of Elie Cartan and Claude Chevalley figure prominently in this evolution. Cartan is responsible for the celebrated ADE classification of simple Lie algebras, as well as for charting the essential role played by Lie groups in differential geometry and mathematical physics. Chevalley made key foundational contributions to the analytic theory, and did much to pioneer the related theory of algebraic groups. Armand Borel's book "Essays in the History of Lie groups and algebraic groups" is the definitive source on the evolution of the Lie group concept. Sophus Lie's contributions are the subject of a number of excellent articles by T. Hawkins.

12.5.1. *The Orthogonal Group O(2).* An elementary example of a Lie group is afforded by O(2), the orthogonal group in two dimensions. This is the set of transformations of the plane which fix the origin and preserve the distance between points. It may be shown that a transform has this property if and only if it is of the form

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto M \begin{pmatrix} x \\ y \end{pmatrix},$$

where M is a 2×2 matrix such that $M^T M = I$. (Such a matrix is called orthogonal.) It is easy enough to check that this is a group. To see that it is a Lie group, we first need to make sure that it is a *manifold*. To that end, we will parameterize it. Calling the entries of the matrix a, b, c, d, the condition becomes

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^T \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^2 + c^2 & ab + cd \\ ab + cd & b^2 + d^2 \end{pmatrix}$$

which is equivalent to the following system of equations:

$$a^2 + c^2 = 1$$
$$ab + cd = 0$$
$$b^2 + d^2 = 1$$

The first of these equations can be solved by introducing a parameter θ and writing $a = \cos\theta$ and $c = \sin\theta$. Then the second equation becomes $b\cos\theta + d\sin\theta = 0$, which can be solved by introducing a parameter r:

$$b = -r\sin\theta$$
$$d = r\cos\theta$$

Substituting this into the third equation results in $r^2 = 1$, so $r = -1$ or $r = +1$. This means we have two matrices for each value of θ:

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \qquad \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$$

Since more than one value of θ will produce the same matrix, we must restrict the range in order to obtain a bona fide coordinate. Thus, we may cover $O(2)$ with an atlas consisting of four neighborhoods:

$$\left\{ \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \mid -\frac{3}{4}\pi < \theta < \frac{3}{4}\pi \right\}$$
$$\left\{ \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \mid \frac{1}{4}\pi < \theta < \frac{7}{4}\pi \right\}$$
$$\left\{ \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix} \mid -\frac{3}{4}\pi < \theta < \frac{3}{4}\pi \right\}$$
$$\left\{ \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix} \mid \frac{1}{4}\pi < \theta < \frac{7}{4}\pi \right\}$$

Every element of $O(2)$ must belong to at least one of these neighborhoods.

12.6. **Examples of 'non-matrix' Lie groups.** Whereas most well-known Lie groups are matrix groups, there do exist Lie groups which are not matrix groups, that is, *they have no faithful finite dimensional representations.*

For example, let H be the real Heisenberg group

$$H = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \mid a, b, c \in \mathcal{R} \right\},$$

and Γ the discrete subgroup

$$\Gamma = \left\{ \begin{pmatrix} 1 & 0 & n \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mid n \in Z \right\}.$$

The subgroup Γ is central, and thus normal. The Lie group H/Γ has no faithful finite dimensional representations over $\mathcal{R}$ or C^*.

Another example is the universal cover of $\mathrm{Sl}2\mathcal{R}$. $\mathrm{Sl}2\mathcal{R}$ is homotopy equivalent to a circle, and thus $\pi(\mathrm{Sl}2\mathcal{R}) \cong Z$, and thus has an infinite-sheeted cover. Any real or complex representation of this group factors through the projection map to $\mathrm{Sl}2\mathcal{R}$.

12.7. **Spinors.** A *spinor* is a mathematical object introduced to expand the notion of **spatial vector**. Spinors are needed for example because the full structure of rotations in a given number of dimensions requires some extra number of dimensions to exhibit it. More formally, spinors can be defined as geometrical objects constructed from a given vector space endowed with an inner product by means of an algebraic or quantization procedure. The rotation group acts upon the space of spinors, but for an ambiguity in the sign of the action. Spinors thus form a projective representation of the rotation group. One can remove this sign ambiguity by regarding the space of spinors as a (linear) group representation of the spin group **Spin(n)**. In this alternative point of view, many of the intrinsic and algebraic properties of spinors are more clearly visible, but the connection with the original spatial geometry is more obscure. On the other hand the use of complex number scalars can be kept to a minimum.

Historically, spinors in general were discovered by Élie Cartan in 1913. Later spinors were adopted by quantum mechanics in order to study the properties of the intrinsic angular momentum of the electron and other fermions. Today spinors enjoy a wide range of physics applications. Classically, spinors in three dimensions are used to describe the spin of the non-relativistic electron. Via the Dirac equation, Dirac spinors are required in the mathematical description of the quantum state of the relativistic electron. In quantum field theory, spinors describe the state of relativistic many-particle systems.

In the classical geometry of space, a vector exhibits a certain behavior when it is acted upon by a rotation or reflected in a hyperplane. However, in a certain sense rotations and reflections contain finer geometrical information than can be expressed in terms of their actions on vectors. Spinors are objects constructed in order to encompass more fully this geometry.

There are essentially two frameworks for viewing the notion of a spinor.

One is representation theoretic. In this point of view, one knows a priori that there are some representations of the Lie algebra of the orthogonal group which cannot be formed by the usual tensor constructions. These missing representations are then labeled the "spin representations", and their constituents spinors. In this view, a spinor must belong to a group representation—representation of the covering space—double cover of the rotation group $SO(n, R)$, or more generally of the generalized special orthogonal group $SO(p, q, R)$ on spaces with metric signature (p, q). These double-covers are Lie groups, called the spin groups $Spin(p, q)$. All the properties of spinors, and their applications and derived objects, are manifested first in the spin group.

The other point of view is geometrical. One can explicitly construct the spinors, and then examine how they behave under the action of the relevant Lie groups. This latter approach has the advantage of being able to say precisely what a spinor is, without invoking some non-constructive theorem from representation theory. Representation theory must eventually supplement the geometrical machinery once the latter becomes too unwieldy.

The most general mathematical form of spinors was discovered by Élie Cartan in 1913. *The word "spinor" was coined by Paul Ehrenfest in his work on quantum physics.*

Spinors were first applied to mathematical physics by Wolfgang Pauli in 1927, when he introduced Pauli matrices. or spin matrices. The following 1928—year, Paul Dirac discovered the fully special relativity—relativistic theory of electron spin (physics)—spin by showing the connection between spinors and the Lorentz group. By the 1930s, Dirac, Piet Hein and others at the Niels Bohr Institute created games such as "Tangloids" to teach and model the calculus of spinors.

Some important simple examples of spinors in low dimensions arise from considering the even-graded subalgebras of the Clifford algebra $Cl_{p,q}(R)$. This is an algebra built up from an orthonormal basis of $n = p + q$ mutually orthogonal vectors under addition and multiplication, p of which have norm $+1$ and q of which have norm -1, with the product rule for the basis vectors

$$e_i e_j = \begin{cases} +1 & i = j,\, i \in (1 \dots p) \\ -1 & i = j,\, i \in (p+1 \dots n) \\ -e_j e_i & i \neq j \end{cases}$$

A space of spinors can be constructed explicitly. For a complete example in dimension 3, see spinors in three dimensions. There are two different, but essentially equivalent, ways to proceed. One approach seeks to identify the minimal ideals for the left action of $Cl(V, g)$ on itself. These are subspaces of the Clifford algebra of the form $Cl(V, g)\omega$, admitting the evident action of $Cl(V, g)$ by left-multiplication: $c : x\omega \to cx\omega$. There are two variations on this theme: one can either find a primitive element ω which is a nilpotent element of the Clifford algebra, or one which is an idempotent. The construction via nilpotent elements is more fundamental in the sense that an idempotent may then be produced from it. In this way, the spinor representations are identified with certain subspaces of the Clifford algebra itself. The second approach is to construct a vector space using a distinguished subspace of V, and then specify the action of the Clifford algebra externally to that vector space.

In either approach, the fundamental notion is that of an isotropic line-the isotropic subspace W. Each construction depends on an initial freedom in choosing this subspace. In physical terms, this corresponds to the fact that **there is no measurement protocol which can specify a basis of the spin space**, even should a preferred basis of V already be given.

This entry was adapted from the Wikipedia article Spinor *http://en.wikipedia.org/wiki/Spinor as of May 10, 2007.*

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View style: TeX source **See Also:** spin groups, Dirac equation, Pauli matrices, spin networks and spin foams, spin groups, lie Joseph Cartan **Keywords:** spinor, spin group, spin

12.8. **Spin groups.** Spins and spin group mathematics are important subjects both in theoretical physics and mathematics. In physics, the term *spin 'groups'* is often used with the broad meaning of a collection of coupled, or interacting spins, and thus covers the broad 'spectrum' of spin clusters ranging from gravitons (as in spin networks and spin foams, for example) to 'up' (u) and 'down' (d) quark spins (fermions) coupled by gluons in nuclei (as treated in quantum chromodynamics or theoretical nuclear physics), and electron spin Cooper pairs (regarded as bosons) in low-temperature superconductivity. On the other hand, in relation to *quantum symmetry*, *spin groups* are defined in quantum mechanics and quantum field theories (QFT) in a precise, mathematical (algebraic) sense as properly defined groups, as introduced next. (In a semi-classical approach, the related concept of a *spinor* has been introduced and studied in depth by É. Cartan, who found that with his definition of spinors the (special) relativistic Lorentz covariance properties were not recovered, or applicable.)

Definition 12.5. In the mathematical, precise sense of the term, a *spin group* –as for example the Lie group $Spin(n)$– is defined as a *double cover of the special orthogonal (Lie) group* $SO(n)$ satisfying the additional condition that there exists the *short exact sequence of Lie groups*:

$$1 \to \mathbb{Z}_2 \to Spin(n) \to SO(n) \to 1$$

Alternatively one can say that the above exact sequence of Lie groups defines the spin group $Spin(n)$. Furthermore, $Spin(n)$ can also be defined as the *proper subgroup* (or groupoid) of the invertible elements in the *Clifford algebra* $\mathbb{C}l(n)$; (when defined as a double cover this should be $Cl_{p,q}(R)$, a *Clifford algebra* built up from an orthonormal basis of $n = p + q$ mutually orthogonal vectors under addition and multiplication, p of which have norm $+1$ and q of which have norm -1, as further explained in the *spinor definition* (Spinor).

Note also that other spin groups such as *Spin d* (ref. [4]) are mathematically defined, and also important, in QFT(Quantum Field Theories).

Important examples of $Spin(n)$ and quantum symmetries: there exist the following isomorphisms:

(1) $Spin(1) \cong O(1)$
(2) $Spin(2) \cong U(1) \cong SO(2)$
(3) $Spin(3) \cong Sp(1) \cong SU(2)$
(4) $Spin(4) \cong Sp(1) \times Sp(1)$
(5) $Spin(5) \cong Sp(2)$
(6) $Spin(6) \cong SU(4)$

Thus, the symmetry groups in the Standard Model (SUSY) of current Physics can also be written as : $Spin(2) \times Spin(3) \times SU(3)$.

Remarks

- In modern Physics, non-Abelian spin groups are also defined, as for example, spin quantum groups and spin quantum groupoids.
- An extension of the concepts of spin group and spinor, is the notion of a 'twistor', a mathematical concept introduced by Sir Roger Penrose, generally with distinct symmetry/mathematical properties from those of spin groups, such as those defined above.

12.9. **The Fundamental Groups of $Spin(p,q)$.** With the usual notation, the fundamental groups $\pi_1(Spin(p,q))$ are as follows:

(1) $\{0\}$, for $(p,q) = (1,1)$ and $(p,q) = (1,0)$;
(2) $\{0\}$, if $p > 2$ and $q = 0, 1$;
(3) $\mathbb{Z}$ for $(p,q) = (2,0)$ and $(p,q) = (2,1)$;
(4) $\mathbb{Z} \times \mathbb{Z}$ for $(p,q) = (2,2)$;
(5) $\mathbb{Z}$ for $p > 2, q = 2$
(6) $\mathbb{Z}_2$ for $p > 2, q > 2$

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See Also: spinor, Clifford algebra, Clifford algebra, exact sequence, categorical sequence, examples of groups, noncommutative geometry, lie Joseph Cartan

Other names: spinor in quantum physics **Also defines:** spin group, spin symmetry, $Spin(n)$, $SO(n)$,$Spin(3)$, $Spin(4)$, $Sp(1)$, short exact sequence of Lie groups **Keywords:** spin groups, spin symmetry, spinor in quantum physics, SUSY, Standard Model in current physics, Clifford algebra, Spinor and twistor methods, Newman-Penrose formalism, Pauli spin matrices, Dirac matrix, Spin and Spin geometry, spin glasses, spin waves and spin wave excitations in ferromagnetic metallic glasses

Cross-references: fundamental groups, extension, quantum groupoids, quantum groups, non-Abelian, isomorphic, current, isomorphisms, norm, multiplication, addition, orthogonal vectors, orthonormal basis, invertible, groupoid, proper subgroup, exact sequence, orthogonal, Lie group, properties, covariance, depth, spinor, groups, algebraic, QFT, quantum field theories, symmetry, relation, nuclear, spin networks and spin foams, spectrum, covers, collection, term

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"C*-algebras and quantum compact groupoids"

View style: TeX source

See Also: groupoid C*-dynamical system, groupoid and group representations related to quantum symmetries, quantum algebraic topology, Grassmann-Hopf algebras and coalgebras, noncommutative geometry, groupoid C*-convolution algebras, Jordan-Banach and Jordan-Lie algebras, algebra classification, classes of algebras, groupoid C*-dynamical system, -Clifford algebra, weak Hopf C*-algebra, -algebra, nuclear C*-algebra, von Neumann algebra, topic entry on applied mathematics, -algebra homomorphisms are continuous, -algebra, compact quantum group, category of C*-algebras, Gelfand-Naimark-Segal construction, Gelfand transform

Other names: weak Hopf algebra, quantized locally compact groupoids with left Haar measure *Also defines:* commutative C*-algebra, C*-algebra, C*-norm, -norm, morphism between C*-algebras, category of C*-algebras, quantum compact groupoid Keywords: algebra of quantum operators, morphism of C*-algebras, C*-algebra, -algebra groupoid and group representations related to quantum symmetries, C*-algebra definition, von Neumann Algebras, Grassman-Hopf algebra, coalgebra and tangled G-H algebras

Attachments: Compact Quantum Group (CQG) (Definition)

Cross-references: topological groupoid, locally compact groupoid, axiomatic, coproduct, generated by, contained, unital, bimodule, tensor product, inner product, localization, Hilbert modules, reduced, completion, involutive, right, Lie group, isotropy group, Lie groupoid, unitary, collection, indexed by, convolution product, convolution, support, smooth, conditional expectation, faithful, composition, positive, images, level, quantum group, product, fields, equivalence classes, acts on, structure, factors, inclusion, index, finite, induced, noncommutative, inconsistent, type, restricted, coalgebras, equivalent, locally compact quantum group, consistent, operator, non-Abelian, source, groupoid representation, Haar measure, groups, locally compact, extension, equations, quantum groups, deformations, symmetries, quantum theories, and operator, quantum state, Hausdorff space, continuous functions, multiplication, commutative, vector space, real, isomorphic, conversely, relation, isometry, isomorphism, morphism, bijective, linear map, mean, bounded operators, adjoints, operations, algebraic, closed under, bounded linear operators, quantum field theory, norm, linear operators, bounded, property, Banach algebra, Axiom As, *-algebra, associative, map, complex, involution, Banach space, algebras, quantum operator algebras, categories, connections, section, Bibliography, quantum groupoids, compact, representations, convolution algebra, groupoid, terms, extended quantum symmetries, Hilbert space, noncommutative geometry, non-commutative, quantum state spaces, geometry, theory, depth, mathematical foundation, von Neumann algebra

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