COMPACT AND LOCALLY COMPACT GROUPS AND GROUPOIDS

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ABSTRACT. An overview of compact and locally compact quantum groups and groupoids, as well as Hopf, quasi-Hopf and weak Hopf algebras is presented with a view to applications in Quantum Field theory and especially scattering theory. New concepts that may be considered as a natural extension of Drinfeld's notion of quantum double construction are also proposed.

1. INTRODUCTION.

A salient, and well–fathomed concept from the mathematical perspective concerns that of a C^*-algebra of a (discrete) group (see e.g. Connes, 1994). The underlying vector space is that of complex valued functions with finite support, and the multiplication of the algebra is the fundamental *convolution product* which it is convenient for our purposes to write slightly differently from the common formula as

$$(f * g)(z) = \sum_{xy=z} f(x)g(y). \tag{1.1}$$

and *-operation

$$f^*(x) = \overline{f(x^{-1})}\ . \tag{1.2}$$

(The more usual expression of these formulas has a sum over the elements of the group.) For topological groups, where the underlying vector space consists of continuous complex valued functions, this product requires the availability of some structure of measure and of measurable functions, with the sum replaced by an integral. (Note that this algebra has an identity, the function δ_1, which has value 1 on the identity 1 of the group, and has zero value elsewhere.)

On the other hand, post 1955, quantum theories adopted a new lease of life when von Neumann beautifully formulated QM in the mathematically rigorous context of Hilbert spaces. The basic definition of a von Neumann algebra is outlined in the Appendix. After recalling the concept of a quantum group in relationship to a (quantum) Hopf algebra (see e.g. Majid, 1995), we shall proceed to relate these mainly algebraic concepts to symmetry and also consider their natural extensions in the context of local quantum physics and symmetry breaking. In recent years the techniques of Hopf symmetry and those of weak Hopf C*-algebras, or *quantum groupoids* as they alternatively are known (cf Böhm et al.,1999), provide important mechanisms for studying the broader relationships of the Wigner fusion rules algebra, $6j$–symmetry (Rehren, 1997) and the study of the noncommutative symmetries of subfactors within the Jones tower constructed from finite index depth 2 inclusion of factors, also from the viewpoint of related Galois correspondences

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1991 *Mathematics Subject Classification.* Quantum Algebraic Topology; Quantum Field Theory; Quantum Statistical Mechanics.
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(Nikshych and Vainerman, 2000). Quantum groupoids also figure prominently in the theory of dynamical deformations of quantum groups and the quantum Yang–Baxter equations (Etingof et al., 1999, 2001). Motivated by these examples, we introduce through several steps of increasing generality, a framework for quantum symmetry breaking in terms of a *weak Hopf C*–algebroid with convolution* set in the context of *rigged Hilbert spaces*(Bohm and Gadella, 1989).

2. The Weak Hopf C*–Algebra

Definition 2.1. A *compact quantum group,* Q_{CG} is defined as a particular case of a locally compact quantum group Q_{Glc} when the object space of the latter Q_{Glc} is a compact topological space (instead of being a locally compact one).

2.1. **Hopf Algebras.** In this section we proceed through several stages of generality by relaxing the axioms for a Hopf algebra. The motivation starts by recalling the notion of a quantum group in relation to a Hopf algebra where the former is often realized as an automorphism group for a quantum space, that is, an object in a suitable category of generally noncommutative algebras. The most common guise of a quantum group is the dual of a noncommutative, nonassociative Hopf algebra. So we commence here establishing the concept of Hopf algebras as the fundamental building blocks following e.g. Chaician and Demichev (1996), Majid (1996). Firstly, a unital associative algebra consists of a linear space A together with two linear maps

$$\begin{aligned} m &: A \otimes A \longrightarrow A \ , \ (\textit{multiplication}) \\ \eta &: \mathbb{C} \longrightarrow A \ , \ (\textit{unity}) \end{aligned} \tag{2.1}$$

satisfying the conditions

$$\begin{aligned} m(m \otimes \mathbf{1}) &= m(\mathbf{1} \otimes m) \\ m(\mathbf{1} \otimes \eta) &= m(\eta \otimes \mathbf{1}) = \mathrm{id} \ . \end{aligned} \tag{2.2}$$

This first condition can be seen in terms of a commuting diagram :

$$\begin{array}{ccc} A \otimes A \otimes A & \xrightarrow{m \otimes \mathrm{id}} & A \otimes A \\ {\scriptstyle \mathrm{id} \otimes m}\big\downarrow & & \big\downarrow{\scriptstyle m} \\ A \otimes A & \xrightarrow{m} & A \end{array} \tag{2.3}$$

Next suppose we consider 'reversing the arrows', and take an algebra A equipped with a linear homorphisms $\Delta : A \longrightarrow A \otimes A$, satisfying, for $a, b \in A$:

$$\begin{aligned} \Delta(ab) &= \Delta(a)\Delta(b) \\ (\Delta \otimes \mathrm{id})\Delta &= (\mathrm{id} \otimes \Delta)\Delta \ . \end{aligned} \tag{2.4}$$

We call Δ a *comultiplication*, which is said to be *coaasociative* in so far that the diagram

$$\begin{array}{ccc} A \otimes A \otimes A & \xleftarrow{\Delta \otimes \mathrm{id}} & A \otimes A \\ {\scriptstyle \mathrm{id} \otimes \Delta}\big\uparrow & & \big\uparrow{\scriptstyle \Delta} \\ A \otimes A & \xleftarrow{\Delta} & A \end{array} \tag{2.5}$$

commutes. There is also a counterpart to η, the *counity* map $\varepsilon : A \longrightarrow \mathbb{C}$ satisfying

$$(\mathrm{id} \otimes \varepsilon) \circ \Delta = (\varepsilon \otimes \mathrm{id}) \circ \Delta = \mathrm{id} \ . \tag{2.6}$$

A *bialgebra* $(A, m, \Delta, \eta, \varepsilon)$ is a linear space A with maps $m, \Delta, \eta, \varepsilon$ satisfying the above properties.

Now to recover anything resembling a group structure, we must append such a bialgebra with an antihomomorphism $S : A \longrightarrow A$, satisfying $S(ab) = S(b)S(a)$, for $a, b \in A$. This map is defined implicitly via the property :

$$m(S \otimes \mathrm{id}) \circ \Delta = m(\mathrm{id} \otimes S) \circ \Delta = \eta \circ \varepsilon \ . \tag{2.7}$$

We call S the *antipode map.* A *Hopf algebra* is a bialgebra $(A, m, \eta, \Delta, \varepsilon)$ equipped with an antipode map S .

Commutative and noncommutative Hopf algebras form the backbone of quantum groups and are essential to the generalizations of symmetry. Indeed, in most respects a quantum group is identifiable with a Hopf algebra. When such algebras are associated to matrix groups there is considerable scope for representations on both finite and infinite dimensional Hilbert spaces.

2.2. **The Weak Hopf Algebra.** In order to define a *weak Hopf algebra*, we can relax certain axioms for a Hopf algebras as follows :

(1) The comultiplication is not necessarily unit–preserving.

(2) The counit ε is not necessarily a homomorphism of algebras.

(3) The axioms for the antipode map $S : A \longrightarrow A$ with respect to the counit are as follows. For all $h \in H$,

$$\begin{aligned} m(\mathrm{id} \otimes S)\Delta(h) &= (\varepsilon \otimes \mathrm{id})(\Delta(1)(h \otimes 1)) \\ m(S \otimes \mathrm{id})\Delta(h) &= (\mathrm{id} \otimes \varepsilon)((1 \otimes h)\Delta(1)) \\ S(h) &= S(h_{(1)})h_{(2)}S(h_{(3)}) \ . \end{aligned} \tag{2.8}$$

As frequently seen in the literature, a weak Hopf algebra is synonymous with a *quantum groupoid.* In our setting, a *Weak C*–Hopf algebra* is a weak *–Hopf algebra which admits a faithful *–representation on a Hilbert space. It is quite likely that other authors use the term 'quantum groupoid' in the sense of a weak C*–Hopf algebra. Eventually, the notion of a *weak C*–algebroid* will be main framework for the type of symmetry breaking we consider here. There are significant motivating examples concerning weak C*–Hopf algebras which deserve mentioning.

2.3. **Examples.**

(1) We refer here to Bais et al. (2002). Let G be a nonabelian group and $H \subset G$ a discrete subgroup. Let $F(H)$ denote the space of functions on H and $\mathbb{C}H$ the group algebra (which consists of the linear span of group elements with the group structure). *The quantum double* $D(H)$ (Drinfeld, 1987) is defined by

$$D(H) = F(H) \ \widetilde{\otimes} \ \mathbb{C}H \ , \tag{2.9}$$

where, for $x \in H$, the 'twisted tensor product' is specified by

$$\widetilde{\otimes} \mapsto \ (f_1 \otimes h_1)(f_2 \otimes h_2)(x) = f_1(x)f_2(h_1 x h_1^{-1}) \otimes h_1 h_2 \ . \tag{2.10}$$

The physical interpretation is often to take H as the 'electric gauge group' and $F(H)$ as the 'magnetic symmetry' generated by $\{f \otimes e\}$. In terms of the counit ε, the double $D(H)$ has a trivial representation given by $\varepsilon(f \otimes h) = f(e)$. We next look at certain features of this construction.

For the purpose of braiding relations there is an R matrix, $R \in D(H) \otimes D(H)$, leading to the operator

$$\mathcal{R} \equiv \sigma \cdot (\Pi^A_\alpha \otimes \Pi^B_\beta)(R) \ , \tag{2.11}$$

in terms of the Clebsch–Gordan series $\Pi^A_\alpha \otimes \Pi^B_\beta \cong N^{AB\gamma}_{\alpha\beta C} \ \Pi^C_\gamma$, and where σ denotes a flip operator. The operator $\mathcal{R}^2$ is sometimes called the *monodromy* or *Aharanov–Bohm phase factor.* In the case of a condensate in a state $|v\rangle$ in the carrier space of some representation

Π^A_α . One considers the maximal Hopf subalgebra T of a Hopf algebra A for which $|v\rangle$ is T–invariant; specifically :

$$\Pi^A_\alpha(P)\ |v\rangle = \varepsilon(P)|v\rangle\ ,\ \forall P \in T\ . \tag{2.12}$$

(2) For the second example, consider $A = F(H)$. The algebra of functions on H can be broken to the algebra of functions on H/K, that is, to $F(H/K)$, where K is normal in H, that is, $HKH^{-1} = K$. Next, consider $A = D(H)$. On breaking a purely electric condensate $|v\rangle$, the magnetic symmetry remains unbroken, but the electric symmetry $\mathbb{C}H$ is broken to $\mathbb{C}N_v$, with $N_v \subset H$, the stabilizer of $|v\rangle$. From this we obtain $T = F(H)\widetilde{\otimes}\mathbb{C}N_v$.

(3) In Nikshych and Vainerman (2000) quantum groupoids (as weak C*–Hopf algebras, see below) were studied in relationship to the noncommutative symmetries of depth 2 von Neumann subfactors. If

$$A \subset B \subset B_1 \subset B_2 \subset \dots \tag{2.13}$$

is the Jones extension induced by a finite index depth 2 inclusion $A \subset B$ of II_1 factors, then $Q = A' \cap B_2$ admits a quantum groupoid structure and acts on B_1, so that $B = B_1^Q$ and $B_2 = B_1 \rtimes Q$. Similarly, in Rehren (1997) ‘paragroups’ (derived from weak C*–Hopf algebras) comprise (quantum) groupoids of equivalence classes such as associated with 6j–symmetry groups (relative to a fusion rules algebra). They correspond to type II von Neumann algebras in quantum mechanics, and arise as symmetries where the local subfactors (in the sense of containment of observables within fields) have depth 2 in the Jones extension. Related is how a von Neumann algebra N, such as of finite index depth 2, sits inside a weak Hopf algebra formed as the crossed product $N \rtimes A$ (Böhm et al. 1999).

(4) In Mack and Schomerus (1992) using a more general notion of the Drinfeld construction, develop the notion of a *quasi triangular quasi–Hopf algebra* (QTQHA) is developed with the aim of studying a range of essential symmetries with special properties, such the quantum group algebra $\mathrm{U}_q(\mathrm{sl}_2)$ with $|q| = 1$. If $q^p = 1$, then it is shown that a QTQHA is canonically associated with $\mathrm{U}_q(\mathrm{sl}_2)$. Such QTQHAs are claimed as the true symmetries of minimal conformal field theories.

3. Quantum Compact Groupoids.

Let G be a (topological) groupoid. We denote by $C_c(\mathsf{G})$ the space of smooth complex–valued functions with compact support on G . In particular, for all $f, g \in C_c(\mathsf{G})$, the function defined via convolution

$$(f\ *\ g)(\gamma) = \int_{\gamma_1 \circ \gamma_2 = \gamma} f(\gamma_1)g(\gamma_2)\ , \tag{3.1}$$

is again an element of $C_c(\mathsf{G})$, where the convolution product defines the composition law on $C_c(\mathsf{G})$. We can turn $C_c(\mathsf{G})$ into a *–algebra once we have defined the involution $*$, and this is done by specifying $f^*(\gamma) = \overline{f(\gamma^{-1})}$. We recall following Landsman (1998) that a *representation* of a groupoid G, consists of a family (or field) of Hilbert spaces $\{\mathcal{H}_x\}_{x \in X}$ indexed by $X = \mathrm{Ob}\ \mathsf{G}$, along with a collection of maps $\{U(\gamma)\}_{\gamma \in \mathsf{G}}$, satisfying:

1. $U(\gamma) : \mathcal{H}_{s(\gamma)} \longrightarrow \mathcal{H}_{r(\gamma)}$, is unitary.
2. $U(\gamma_1\gamma_2) = U(\gamma_1)U(\gamma_2)$, whenever $(\gamma_1, \gamma_2) \in \mathsf{G}^{(2)}$.

3. $U(\gamma^{-1}) = U(\gamma)^*$, for all $\gamma \in \mathsf{G}$.

Suppose now G is a Lie groupoid. Then the isotropy group G_x is a Lie group, and for a (left or right) Haar measure μ_x on G_x, we can consider the Hilbert spaces $\mathcal{H}_x = L^2(\mathsf{G}_x, \mu_x)$ as exemplifying the above sense of a representation. Putting aside some technical details which can be found in Connes (1994), Landsman (2006), the overall idea is to define an operator of Hilbert spaces

$$\pi_x(f) : L^2(\mathsf{G}_x, \mu_x) \longrightarrow L^2(\mathsf{G}_x, \mu_x) \ , \tag{3.2}$$

given by

$$(\pi_x(f)\xi)(\gamma) = \int f(\gamma_1)\xi(\gamma_1^{-1}\gamma) \ d\mu_x \ , \tag{3.3}$$

for all $\gamma \in \mathsf{G}_x$, and $\xi \in \mathcal{H}_x$. For each $x \in X = \mathrm{Ob}\ \mathsf{G}$, π_x defines an involutive representation $\pi_x : C_c(\mathsf{G}) \longrightarrow \mathcal{H}_x$. We can define a norm on $C_c(\mathsf{G})$ given by

$$\|f\| = \sup_{x \in X} \|\pi_x(f)\| \ , \tag{3.4}$$

whereby the completion of $C_c(\mathsf{G})$ in this norm, defines *the reduced C*–algebra $C_r^*(\mathsf{G})$ of* G. It is perhaps the most commonly used C*–algebra for Lie groupoids (groups) in noncommutative geometry.

Compact quantum groupoids were introduced in Landsman (1998) as a simultaneous generalization of a compact groupoid and a quantum group. Since the construction is relevant to that which we propose, it deserves some exposition before we step to the next level of generality. Firstly, let $\mathfrak{A}$ and $\mathfrak{B}$ denote C*–algebras equipped with a *–homomorphism $\eta_s : \mathfrak{B} \longrightarrow \mathfrak{A}$, and a *–antihomomorphism $\eta_t : \mathfrak{B} \longrightarrow \mathfrak{A}$ whose images in $\mathfrak{A}$ commute. A noncommutative Haar measure is defined as a completely positive map $P : \mathfrak{A} \longrightarrow \mathfrak{B}$ which satisfies $P(A\eta_s(B)) = P(A)B$. Alternatively, the composition $\mathcal{E} = \eta_s \circ P : \mathfrak{A} \longrightarrow \eta_s(B) \subset \mathfrak{A}$ is a faithful conditional expectation.

The next step requires a little familiarity with the theory of Hilbert modules (see e.g. Lance, 1995). We define a left $\mathfrak{B}$–action λ and a right $\mathfrak{B}$–action ρ on $\mathfrak{A}$ by $\lambda(B)A = A\eta_t(B)$ and $\rho(B)A = A\eta_s(B)$. For the sake of localization of the intended Hilbert module, we implant a $\mathfrak{B}$–valued inner product on $\mathfrak{A}$ given by $\langle A, C\rangle_{\mathfrak{B}} = P(A^*C)$. Since P is faithful, we fit a new norm on $\mathfrak{A}$ given by $\|A\|^2 = \|P(A^*A)\|_{\mathfrak{B}}$. The completion of $\mathfrak{A}$ in this new norm is denoted by $\mathfrak{A}^-$ leading then to a Hilbert module over $\mathfrak{B}$.

The tensor product $\mathfrak{A}^- \otimes_{\mathfrak{B}} \mathfrak{A}^-$ can be shown to be a Hilbert bimodule over $\mathfrak{B}$, which for $i = 1, 2$, leads to *–homorphisms $\varphi^i : \mathfrak{A} \longrightarrow \mathcal{L}_{\mathfrak{B}}(\mathfrak{A}^- \otimes \mathfrak{A}^-)$. Next is to define the (unital) C*–algebra $\mathfrak{A} \otimes_{\mathfrak{B}} \mathfrak{A}$ as the C*–algebra contained in $\mathcal{L}_{\mathfrak{B}}(\mathfrak{A}^- \otimes \mathfrak{A}^-)$ that is generated by $\varphi^1(\mathfrak{A})$ and $\varphi^2(\mathfrak{A})$. The last stage of the recipe for defining a compact quantum groupoid entails considering a certain coproduct operation $\Delta : \mathfrak{A} \longrightarrow \mathfrak{A} \otimes_{\mathfrak{B}} \mathfrak{A}$, together with a coinverse $Q : \mathfrak{A} \longrightarrow \mathfrak{A}$ that it is both an algebra and bimodule antihomomorphism. Finally, the following axiomatic relationships are observed :

$$\begin{aligned} (\mathrm{id} \otimes_{\mathfrak{B}} \Delta) \circ \Delta &= (\Delta \otimes_{\mathfrak{B}} \mathrm{id}) \circ \Delta \\ (\mathrm{id} \otimes_{\mathfrak{B}} P) \circ \Delta &= P \\ \tau \circ (\Delta \otimes_{\mathfrak{B}} Q) \circ \Delta &= \Delta \circ Q \end{aligned} \tag{3.5}$$

where τ is a flip map : $\tau(a \otimes b) = (b \otimes a)$.

4. Appendix

4.1. **von Neumann Algebras.** Let $\mathcal{H}$ denote a complex (separable) Hilbert space. A *von Neumann algebra* $\mathcal{A}$ acting on $\mathcal{H}$ is a subset of the algebra of all bounded operators $\mathcal{L}(\mathcal{H})$ such that:

(1) $\mathcal{A}$ is closed under the adjoint operation (with the adjoint of an element T denoted by T^*).

(2) $\mathcal{A}$ equals its bicommutant, namely:
$$\mathcal{A} = \{A \in \mathcal{L}(\mathcal{H}) : \forall B \in \mathcal{L}(\mathcal{H}), \forall C \in \mathcal{A},\ (BC = CB) \Rightarrow (AB = BA)\}\ .$$

If one calls a *commutant* of a set $\mathcal{A}$ the special set of bounded operators on $\mathcal{L}(\mathcal{H})$ which commute with all elements in $\mathcal{A}$, then this second condition implies that the commutant of the commutant of $\mathcal{A}$ is again the set $\mathcal{A}$.

On the other hand, a von Neumann algebra $\mathcal{A}$ inherits a *unital* subalgebra from $\mathcal{L}(\mathcal{H})$, and according to the first condition in its definition $\mathcal{A}$ does indeed inherit a **-subalgebra* structure, as further explained in the next section on C*-algebras. Furthermore, we have notable *Bicommutant Theorem* which states that $\mathcal{A}$ *is a von Neumann algebra if and only if* $\mathcal{A}$ *is a *-subalgebra of* $\mathcal{L}(\mathcal{H})$*, closed for the smallest topology defined by continuous maps* $(\xi, \eta) \longmapsto (A\xi, \eta)$ *for all* $< A\xi, \eta >$ *where* $< .,. >$ *denotes the inner product defined on* $\mathcal{H}$. For further instruction on this subject, see e.g. Aflsen and Schultz (2003), Connes (1994).

4.2. **Groupoids.** Recall that a groupoid G is, loosely speaking, a small category with inverses over its set of objects $X = \mathrm{Ob}(\mathsf{G})$. One often writes G_x^y for the set of morphisms in G from x to y . *A topological groupoid* consists of a space G, a distinguished subspace $\mathsf{G}^{(0)} = \mathrm{Ob}(\mathsf{G}) \subset \mathsf{G}$, called *the space of objects* of G, together with maps

$$r, s\ :\ \mathsf{G} \overset{r}{\underset{s}{\rightrightarrows}} \mathsf{G}^{(0)} \tag{4.1}$$

called the *range* and *source maps* respectively, together with a law of composition

$$\circ\ :\ \mathsf{G}^{(2)} := \mathsf{G} \times_{\mathsf{G}^{(0)}} \mathsf{G} = \{\ (\gamma_1, \gamma_2) \in \mathsf{G} \times \mathsf{G}\ :\ s(\gamma_1) = r(\gamma_2)\ \} \longrightarrow \mathsf{G}\ , \tag{4.2}$$

such that the following hold :

(1) $s(\gamma_1 \circ \gamma_2) = r(\gamma_2)$, $r(\gamma_1 \circ \gamma_2) = r(\gamma_1)$, for all $(\gamma_1, \gamma_2) \in \mathsf{G}^{(2)}$.

(2) $s(x) = r(x) = x$, for all $x \in \mathsf{G}^{(0)}$.

(3) $\gamma \circ s(\gamma) = \gamma$, $r(\gamma) \circ \gamma = \gamma$, for all $\gamma \in \mathsf{G}$.

(4) $(\gamma_1 \circ \gamma_2) \circ \gamma_3 = \gamma_1 \circ (\gamma_2 \circ \gamma_3)$.

(5) Each γ has a two–sided inverse γ^{-1} with $\gamma\gamma^{-1} = r(\gamma)$, $\gamma^{-1}\gamma = s(\gamma)$.

It is usual to call $\mathsf{G}^{(0)} = \mathrm{Ob}(\mathsf{G})$ *the set of objects* of G . For $u \in \mathrm{Ob}(\mathsf{G})$, the set of arrows $u \longrightarrow u$ forms a group G_u, called the *isotropy group of* G *at* u. For a further study of groupoids we refer to Brown (2006), Connes (1994) .

4.3. **Haar Systems for Locally Compact Topological Groupoids.** Let

$$\mathsf{G} \underset{s}{\overset{r}{\rightrightarrows}} \mathsf{G}^{(0)} = X \tag{4.3}$$

be a locally compact, locally trivial topological groupoid with its transposition into transitive (connected) components. Recall that for $x \in X$, the *costar of* x denoted CO*(x) is defined as the closed set $\bigcup\{\mathsf{G}(y, x) : y \in \mathsf{G}\}$, whereby

$$\mathsf{G}(x_0, y_0) \hookrightarrow \mathrm{CO}^*(\mathrm{x}) \longrightarrow \mathrm{X} \ ,$$

is a principal $\mathsf{G}(x_0, y_0)$–bundle relative to fixed base points (x_0, y_0) . Assuming all relevant sets are locally compact, then following Seda (1976), a *(left) Haar system on* G denoted (G, τ) (for later purposes), is defined to comprise of i) a measure κ on G, ii) a measure μ on X and iii) a measure μ_x on CO*(x) such that for every Baire set E of G, the following hold on setting $E_x = E \cap \mathrm{CO}^*(\mathrm{x})$:

(1) $x \mapsto \mu_x(E_x)$ is measurable.
(2) $\kappa(E) = \int_x \mu_x(E_x)\ d\mu_x$.
(3) $\mu_z(tE_x) = \mu_x(E_x)$, for all $t \in \mathsf{G}(x, z)$ and $x, z \in \mathsf{G}$.

4.4. **Compact quantum group proceedings.** Locally Compact Quantum Groups and Groupoids: Proceedings of the Meeting of Theoretical Physicists and Mathematicians, Strasbourg, February 21-23, 2002

Authors: Leonid Vainerman (Editor)

Comments: 247 pages, year 2003, 8 chapters, intermediate to advanced operator, commutative and non-commutative, algebra level and beyond Description: Two major papers by Leonid Vainerman, one with Stefaan Vaes, and the other his editorial, algebraically detailed, Introduction provide the background and advances in the field of locally compact quantum groups up to 2002. The previous work on Kac algebras is also explained and the locally compact quantum groups are defined in terms of C*– or von Neumann algebras equipped with a co-associative multiplication and also associated left- and right-Haar measures defined by two semi-finite normal weights. The remaining five papers deal respectively with: Quantum groupoids (by Michael Enock, pp. 17 to 48; Peter Schoenburg, pp.79 to 104; Galois actions by finite QG's by Korne'l Szlachanyi, pp 106-126; J. M. Vallin, isometries and QG's, pp. 229), plus two chapters on specific cases of quantum groups. The concluding chapter by Alfons Van Deele is on Multiplier Hopf *-algebras representations of locally compact quantum groups (pp.230-247).

Editorial Book Description verbatim quote:

"This volume contains seven papers written by participants of the 69th meeting of theoretical physicists and mathematicians held in Strasbourg (February 21-23, 2002)... The book contains seven refereed research papers on locally compact quantum groups and groupoids by leading experts in the respective fields. These contributions are based on talks presented on the occasion of the meeting between mathematicians and theoretical physicists held in Strasbourg from February 21 to February 23, 2002. Topics covered are: various constructions of locally compact quantum groups and their multiplicative unitaries; duality theory for locally compact quantum groups; combinatorial quantization of flat connections associated with SL(2,c); quantum groupoids, especially coming from Depth 2 Extensions of von Neumann algebras, C*-algebras and Rings. Many mathematical results are motivated by problems in theoretical physics. Historical remarks set the results presented in perspective.

Directed at research mathematicians and theoretical physicists as well as graduate students, the

volume will give an overview of a field of research in which great progress has been achieved in the last few years, with new ties to many other areas of mathematics and physics."
Keywords:locally compact quantum groups, quantum groupoids, discrete quantum groups, algebraic quantum groups, monoidal category structure, locally compact quantum groupoids, Hopf algebra, von Neumann algebra, duality, unitary corepresentations, weak bialgebras, relative tensor product, multiplicative unitary, corresponding matched pair, faithful weight, multiplicative unitaries, quantum subgroup, involutive homomorphism, face algebra, grouplike elements, partial isometrics, partial isometry, infinitesimal objects, quantum group theory, phase change, monoidal functor, canonical trace, preprint math, dense ideal, Rights: Copyright@ 2003 by Walter de Gruyter Gmbh & Co, 10785 Berlin http://www-irma.u-strasbg.fr/rubrique83.html Links:http://www-irma.u-strasbg.fr/rubrique8

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