Lectures on Operator K-Theory and the Atiyah-Singer Index Theorem

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Preface

These are the notes of lectures delivered by the two of us to the Spring School on Noncommutative Geometry, held at Vanderbilt University in May, 2004. The notes were mostly written on the fly during the school. Since then some parts have been rearranged and otherwise changed, but there is quite a bit of work to be done before the notes reach their final form. Please keep this in mind while reading them!

We hope to have a close-to-final version of the notes prepared by the end of 2004.

CHAPTER 1

The Signature and Hodge Theory

In this introductory chapter we shall discuss an important differential operator to which the index theorem may be applied. At the same time we shall review some ideas in topology which we shall require when we discuss characteristic classes in Chapter 4.

1. Differential Operators

Let M be a smooth manifold and let S be a smooth vector bundle over M. A linear operator $D\colon C^\infty(M, S) \to C^\infty(M, S)$ acting on the space of smooth sections of S is a *linear partial differential operator* if:

(a) for every smooth section u and open set $U \subseteq M$, the restriction of Du to U depends only on the restriction of u to U;

(b) in any coordinate neighbourhood of M and local trivialization of S, the operator D has the form

$$Du(x) = \sum_{|\alpha|\le k} a_\alpha(x) \frac{\partial^\alpha u}{\partial x_\alpha}(x)$$

for some $k \geq 0$. Here $\alpha = (\alpha_1, \dots, \alpha_n)$ is a multi-index composed of non-negative integers, $\partial^\alpha/\partial x_\alpha$ is short hand for $\partial^{\alpha_1}/\partial x_{\alpha_1} \cdots \partial^{\alpha_n}/\partial x_{\alpha_n}$, and $|\alpha| = \alpha_1 + \cdots \alpha_n$. The quantities a_α are smooth, matrix-valued functions.

We shall be mainly interested in *order one* linear partial differential operators, which are those which have local representations, as above with $k = 1$. This is in part because the analysis of order one operators is somewhat simpler than the analysis of higher order operators, and in part because most of the fundamental examples to which the index theorem applies are order one operators.

If S is the trivial (real) vector bundle of rank one over M, then an order one partial differential operator is little more than a vector field on M and as such it will not be very interesting from our point of view. This indicates the importance of introducing operators acting on the sections of non-trivial bundles. As we shall see, if S is non-trivial, then several very interesting possibilities exist for the construction of order one partial differential operators.

1.1. EXERCISE. If D is an order one, linear partial differential operator on M, acting on the smooth sections of some bundle S, and f is a smooth function on M, acting on sections of S by pointwise multiplication, then the commutator $[D, f]\colon C^\infty(M, S) \to C^\infty(M, S)$ is the linear map induced from some endomorphism of the bundle S.

1.2. REMARK. The property of order one operators indicated in the exercise is very important and will be used repeatedly. In fact it may be shown that if a linear map $D\colon C^\infty(M, S) \to C^\infty(M, S)$ has the property that for every f the commutator $[D, f]$ is induced from a bundle endomorphism, then D is an order one linear partial differential operator.

2. De Rham Cohomology

Let M be a smooth manifold. The local differentiable structure and the global topology of M are tied together in many ways, but one of the most fundamental is the existence of a model for the *cohomology* of M based on the existence and uniqueness of solutions to certain partial differential equations. This is *de Rham cohomology*.

1.3. DEFINITION. The *de Rham complex* of M is the complex

$$\Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \Omega^2(M) \longrightarrow \cdots$$

of smooth differential forms on M, with coboundary operator given by the exterior derivative d. The *de Rham cohomology* $H^*(M; \mathbb{R})$ of M is the cohomology of the de Rham complex.

Remember that $\Omega^p(M)$ is the space of smooth sections of the pth exterior power of the cotangent bundle. So for example $\Omega^0(M)$ is the space of smooth functions on M, while $\Omega^1(M)$ is the space of sections of the cotangent bundle. The operator $d\colon \Omega^0(M) \to \Omega^1(M)$ is given by the canonical formula

$$df(X) = X(f),$$

where f is a smooth function and X is a tangent vector (acting on functions as a directional derivative). It is a basic principle that there are differential operators $d\colon \Omega^p(M) \to \Omega^{p+1}(M)$ which are uniquely determined by the requirements $d^2\alpha = 0$ for all α, and

$$d(\alpha_1 \wedge \alpha_2) = d\alpha_1 \wedge \alpha_2 + (-1)^{\deg(\alpha_1)}\alpha_1 \wedge d\alpha_2,$$

for all α_1 and α_2.

Differential forms pull back under smooth maps, and therefore de Rham cohomology is a contravariant functor. A fundamental theorem identifies de Rham cohomology with any other model for the cohomology for real coefficients (such as Cech or singular cohomology).

1.4. REMARK ON NOTATION. When M is not compact it is convenient to consider variants of de Rham cohomology described by various support conditions. The most usual of these is to require all the differential forms to be compactly supported, from which we obtain *cohomology with compact supports*. Because this is the variant which will be of most use to us, we shall from now on use the notation $H^*(M, \mathbb{R})$, or more briefly $H^*(M)$, to refer to de Rham cohomology with compact supports.

The simplest place to find de Rham cohomology classes is in the top dimension. Assume that M is compact and oriented, and let $n = \dim(M)$. If α is any n-form on M then we can form the integral $\int_M \alpha \in \mathbb{R}$. According to Stokes' Theorem, if $\alpha = d\beta$ then

$$\int_M \alpha = \int_M d\beta = 0,$$

and therefore $\int_M \alpha$ depends only on the class of α in $H^n(M)$ (note that $d\alpha = 0$ since there are no non-zero $(n+1)$-forms on an n-manifold). In particular if $\int_M \alpha \neq 0$ then the cohomology class of α is non-zero.

Now let $p \in M$ and let $x^1, \ldots, x^n$ be oriented local coordinates near p, with $x^1(p) = \cdots = x^n(p) = 0$. Let $\phi\colon \mathbb{R}^n \to \mathbb{R}$ be a smooth, nonnegative function with small compact support near 0 and with $\int \phi = 1$. Then the "bump n-form"

$$\alpha = \phi(x^1, \ldots, x^n) dx^1 \wedge \cdots \wedge dx^n$$

is well-defined on M (vanishing outside a neighborhood of p) and has $\int_M \alpha = 1$, so it defines a non-zero class in $H^n(M)$. In fact it can be shown that if M is connected then $H^n(M)$ is one-dimensional, generated by the cohomology class of any bump n-form. This follows from the Poincaré duality theorem which we will discuss at the end of this chapter.

In this construction we started with a point p, which is a 0-dimensional object, and finished with an n-dimensional cohomology class. We can generalize the construction by starting with a higher-dimensional submanifold instead of a point. To this end we make the following definition.

1.5. DEFINITION. Let N be a compact manifold and let V be an oriented real vector bundle over N, of fiber dimension k. A *Thom form* is a compactly supported k-form on the total space of V, which is closed and which restricts on each fiber to a k-form of integral 1.

1.6. PROPOSITION. *Let N be a compact manifold and let V be an oriented real vector bundle over N. There exists a Thom form on V.* □

Now suppose that M is a compact oriented n-manifold, as before, and that N is a compact oriented $(n-k)$-dimensional submanifold. By a standard theorem of differential topology, there is a neighborhood of N in M (a *tubular neighborhood*) which is diffeomorphic to the total space of a k-dimensional oriented vector bundle over N, namely the normal bundle $(TM)|_N/TN$. Pick a Thom form for this vector bundle, transfer it to the tubular neighborhood by the diffeomorphism, and extend it by zero to a form on the whole of M. The result is a smooth differential form on M since the Thom form was compactly supported within the tubular neighborhood. We obtain a closed k-form defining a cohomology class $\alpha_N \in H^k(M)$. When $k = n$ this is our earlier construction. It can be shown that the cohomology class obtained does not depend on the choices made in the construction.

The "dual forms" α_N are related to the geometry of submanifolds by the following proposition.

1.7. PROPOSITION. *Let N_1 and N_2 be oriented submanifolds of M having dimensions $n - k$ and k respectively, and suppose that they meet transversely in a finite set of points. The integral*

$$\int \alpha_{N_1} \wedge \alpha_{N_2}$$

is the signed count of the number of points of intersection of N_1 and N_2. □

Notice that the integral is *cohomological* in nature; it depends only on the de Rham cohomology classes of α_{N_1} and α_{N_2}, and moreover the wedge product of forms corresponds to the cohomological operation of *cup product*.

3. The Signature

Let us assume now that $\dim(M)$ is a multiple of 4. If M is a compact, oriented $4k$-manifold, then the formula

$$Q([\alpha],[\beta]) = \int \alpha \wedge \beta$$

defines a symmetric bilinear form on the middle-dimensional cohomology group $H^{2k}(M;\mathbb{R})$.

1.8. DEFINITION. Because of the geometric interpretation given in Proposition 1.7, the symmetric bilinear form Q is called the *intersection form* for the manifold M.

Elementary linear algebra tells us that there is a basis $\{[\alpha_i]\}$ for the vector space $H^{2k}(M)$ such that $Q([\alpha_i],[\alpha_j]) = 0$ when $i \neq j$. Moreover the difference

$$\#\{j : Q([\alpha_j],[\alpha_j]) > 0\} - \#\{j : Q([\alpha_j],[\alpha_j]) < 0\}$$

is an invariant of Q, and hence of M. It is therefore natural to make the following definition.

1.9. DEFINITION. The *signature* $\operatorname{Sign}(M)$ of a $4k$-dimensional compact oriented manifold M is the signature of its intersection form.

1.10. REMARK. It follows from the Poincaré duality theorem that the intersection form Q is non-degenerate, that is, if $[\alpha] \neq 0$ then there exists β such that $Q([\alpha], [\beta]) \neq 0$. This quite deep result implies that the "diagonal values" $Q([\alpha_j], [\alpha_j])$ are all non-zero. So they are all either positive or negative, and they all contribute to $\operatorname{Sign}(M)$.

1.11. EXERCISE. Show that the signature for the complex projective plane, $\mathbb{CP}^2$ is ± 1 (the sign depends on the choice of orientation on $\mathbb{CP}^2$).

During and after the 1950's it became clear that the signature invariant (and its extensions and generalizations) includes much of the most important topological information about the manifold M. In 1956 Hirzebruch proved his *signature theorem* which identified the signature with certain differential invariants of M, so-called *characteristic numbers*. This result was a precursor to the index theorem which is the subject of these notes.

The construction of characteristic numbers depends on an important classification principle for vector bundles.

1.12. DEFINITION. The *Grassmannian* $G_k(\mathbb{C}^N)$ of the vector space $\mathbb{C}^N$ is the space of k-dimensional subspaces of $\mathbb{C}^N$. The *canonical bundle* over the Grassmannian $G_k(\mathbb{C}^N)$ is the k-dimensional vector bundle E whose fiber over a point p of the Grassmannian is the k-dimensional subspace represented by that point. The *real* Grassmannian $G_k(\mathbb{R}^N)$ and its canonical bundle are defined similarly.

In the following theorem, $\mathbb{F}$ will denote $\mathbb{R}$ or $\mathbb{C}$.

1.13. THEOREM. *Let V be an $\mathbb{F}$-vector bundle over a compact manifold M. Then for sufficiently large N:*

(i) *V is isomorphic to a sub-bundle of the trivial bundle $M \times \mathbb{F}^N$.*

(ii) *There is a* classifying map $\phi_V \colon M \to G_k(\mathbb{F}^N)$, *the Grassmannian of k-planes, such that V is isomorphic to the pull-back $\phi_V^* U$ of the universal bundle E over $G_k(\mathbb{F}^N)$.*

(iii) *The set of isomorphism classes of complex, rank k vector bundles on M is isomorphic to the set of homotopy classes of maps from M into $G_k(\mathbb{F}^N)$, in such a way that to a map $\phi \colon M \to G_k(\mathbb{F}^N)$ there corresponds the pullback $\phi^* E$.* □

Suppose now that c is a class in the cohomology of the Grassmanian $G_k(\mathbb{F}^N)$: thus $c \in H^*(G_k(\mathbb{F}^N))$. If V is a vector bundle on M, and if $\phi_V\colon M \to G_k(\mathbb{F}^N)$ is its classifying map, then the cohomology class $\phi_V^*(c) \in H^*(M)$ depends only on the isomorphism class of the vector bundle V. Such a class, which measures the "twistedness" of V, is called a *characteristic class* for V and is usually denoted $c(V)$. These classes will be studied in some detail in Chapter 4.

1.14. DEFINITION. Let M be a compact n-dimensional, oriented manifold. The *characteristic numbers* of M are the numbers

$$c(M) = \int_M c(TM)$$

obtained by integrating the n-dimensional characteristic classes $c(TM) \in H^n(M)$ of M.

Thom showed that any invariant of oriented manifolds $f(M) \in \mathbb{R}$ which satisfies $f(M_1 \cup M_2) = f(M_1) + f(M_2)$, and which vanishes whenever M is the boundary of an oriented manifold, is necessarily a characteristic number. The signature of M has precisely these properties, and Hirzebruch was able to determine the characteristic classes L_{4k} for real vector bundles needed to represent the signature:

1.15. THEOREM. *Let M be a compact oriented $4k$-manifold. Then*

$$\text{Sign}(M) = \int_M L_{4k}(TM).$$

□

In later chapters we will explain in detail the construction of these and other characteristic classes. For now, notice the very different nature of the terms appearing on the two sides of the signature theorem. The left hand side is obviously an integer, the right hand side is not; the left hand side is an invariant of homotopy type, the right hand side *a priori* only of diffeomorphism (since it involves the tangent bundle); the left hand side involves the global topology of M, whereas the right side involves an invariant which (it turns out) depends only on the local geometry of TM. Much of the power of Hirzebruch's theorem comes from these contrasts.

4. Hodge Theory and the de Rham Operator

Let M be a smooth, closed, oriented manifold. A Riemannian metric on M is of course an inner product on the tangent bundle TM. It gives rise to inner products on all the vector bundles associated to TM — in particular on the bundles $\bigwedge^p(T^*M)$ whose sections are differential forms. The metric also provides a canonical choice of volume form on M. Using these two notions we can define the $\star$-*operator* $\star\colon \Omega^p(M) \to \Omega^{n-p}(M)$ as follows.

1.16. DEFINITION. Let α be a p-form. Define $\star\,\alpha$ to be the unique $(n-p)$-form such that for all p-forms β

$$(\alpha, \beta)\,\mathrm{Vol} = \beta \wedge \star\alpha,$$

where (α, β) is the pointwise inner product of α and β (and is thus a function on M).

1.17. EXAMPLE. On $\mathbb{R}^3$, where $\mathrm{Vol} = dx\,dy\,dz$, $\star\,dx = dy\,dz$, $\star\,dy = -dx\,dz$, $\star\,dx\,dy = dz$, $\star\,dx\,dz = -dy$, and so on.

The Hodge $\star$-operator is "almost" an involution:

1.18. LEMMA. *If α is a p-form, then $\star\star\alpha = (-1)^{pn+p}\alpha$.*

PROOF. Exercise. □

1.19. DEFINITION. If α is a p-form, define

$$\delta\alpha = (-1)^{np+n+1} \star d \star \alpha.$$

Thus $\delta\alpha$ is a $(p-1)$-form. Clearly, $\delta^2 = 0$, since $d^2 = 0$. The importance of δ lies in the fact that it is the *formal adjoint* d^* of d. Specifically, let α and β be forms of the same degree. Define their global inner product by

$$\langle \alpha, \beta \rangle = \int_M (\alpha, \beta)\,\mathrm{Vol} = \int_M \beta \wedge \star\alpha = \int_M \alpha \wedge \star\beta.$$

Then:

1.20. PROPOSITION. *If α, β are smooth forms of degrees p and $p-1$ on M, then*

$$\langle \alpha, d\beta \rangle = \langle \delta\alpha, \beta \rangle.$$

PROOF. This is proved using integration by parts. By Stokes' theorem, if α is a p-form and β is a $(p-1)$-form, then

$$\begin{aligned} 0 = \int d(\beta \wedge \star\alpha) &= \int d\beta \wedge \star\alpha + (-1)^{p-1} \int \beta \wedge d \star \alpha \\ &= \langle d\beta, \alpha \rangle \ + \ (-1)^{np+n} \langle \beta, \star d \star \alpha \rangle \end{aligned}$$

(Note that the differential form $\gamma = d \star \alpha$ is an $(n-p+1)$-form, so that $\star\star\gamma = (-1)^{np+n+p+1}\gamma$ by Lemma 1.18. We have used this formula to convert the second integral into an inner product.) It now follows from our definition of d^* that

$$\langle d\beta, \alpha \rangle = \langle \beta, \delta\alpha \rangle,$$

as required. □

1.21. REMARK. The adjoint operator d^* exists on any Riemannian manifold. But it is only for oriented manifolds that we obtain the formula relating d^* to d and the Hodge operator.

Let us now state a key theorem in analysis related to the de Rham complex:

1.22. THEOREM (Hodge). *Let M be a compact Riemannian manifold. For each p there are orthogonal direct sum decompositions*

$$\Omega^p(M) = \text{Kernel}(d) \oplus \text{Image}(d^*)$$

and

$$\Omega^p(M) = \text{Kernel}(d^*) \oplus \text{Image}(d).$$

We shall prove this in the next chapter. Note that the summands are certainly orthogonal to one another. For example if $\alpha \in \text{Kernel}(d)$ and if $\beta = d^*\gamma \in \text{Image}(d^*)$, then

$$\langle \alpha, \beta \rangle = \langle \alpha, d^*\gamma \rangle = \langle d\alpha, \gamma \rangle = 0.$$

The issue is whether or not the summands add up to all of $\Omega^p(M)$. This is a problem in PDE theory: for example, given a differential form β with $\beta \perp \text{Kernel}(d)$ we need to prove that there is a solution to the partial differential equation $d^*\gamma = \beta$.

We are going to use the Hodge theorem to exhibit the signature of M as the Fredholm index of a differential operator on M. Denote by $\Omega^*(M)$ the direct sum of all the spaces $\Omega^p(M)$. Define a differential operator

$$D\colon \Omega^*(M) \to \Omega^*(M)$$

by the formula $D\omega = d\omega + d^*\omega$.

1.23. LEMMA. $\text{Kernel}(D) = \text{Kernel}(d) \cap \text{Kernel}(d^*)$.

PROOF. Since $d^2 = 0$ and $d^{*2} = 0$, it follows that $D^2 = d^*d + dd^*$. As a result we obtain the identity

$$\|D\alpha\|^2 = \langle D^2\alpha, \alpha \rangle = \langle (d^*d + dd^*)\alpha, \alpha \rangle = \|d\alpha\|^2 + \|d^*\alpha\|^2.$$

This proves the lemma. □

1.24. DEFINITION. A differential form α on a compact Riemannian manifold is *harmonic* if $d\alpha = 0$ and $d^*\alpha = 0$.

1.25. LEMMA. *Let M be a compact Riemannian manifold. Each cohomology class for M contains exactly one harmonic form. As a result, there is a natural isomorphism* $\text{Kernel}(D) \cong \oplus_p H^p(M)$.

PROOF. Suppose that $d\alpha = 0$. From the direct sum decomposition in the Hodge theorem,

$$\Omega^p(M) = \text{Kernel}(d^*) \oplus \text{Image}(d),$$

we can write $\alpha = \alpha_1 + d\beta$, where $d^*\alpha_1 = 0$. Since $\alpha_1 = \alpha - d\beta$, we see that α_1 represents the same cohomology class as α, and is harmonic. To prove uniqueness, note that if α_1 and α_2 are harmonic, and if $\alpha_1 - \alpha_2$ is zero in cohomology, then $\alpha_1 - \alpha_2 \in \text{Image}(d)$. But in addition $\alpha_1 - \alpha_2 \in \text{Kernel}(d^*)$, and since $\text{Kernel}(d^*)$ and $\text{Image}(d)$ are orthognal, this is only possible if $\alpha_1 - \alpha_2 = 0$. □

1.26. LEMMA. *The Hodge theorem is equivalent to the formula*

$$\Omega^*(M) = \text{Kernel}(D) \oplus \text{Image}(D).$$

PROOF. If $\alpha \in \Omega^p(M)$, then we may use the above direct sum decomposition to write α as a sum

$$\alpha = \alpha_1 + d\alpha + d^*\alpha_3,$$

where $\alpha_1 \in \text{Kernel}(D)$. We can assume that $\alpha_1 \in \Omega^p(M)$, $\alpha_2 \in \Omega^{p-1}(M)$ and $\alpha_3 \in \Omega^{p+1}(M)$. The reason is that $\text{Kernel}(D) = \text{Kernel}(d) \cap \text{Kernel}(d^*)$, so that if $\alpha_1 \in \Omega^*(M)$ is a general element in $\text{Kernel}(D)$, then its various homogeneous components also belong to $\text{Kernel}(D)$. Now since $d^2 = 0$, $\text{Image}(d) \subseteq \text{Kernel}(d)$, so we immediately obtain the decompositions of the Hodge theorem. For example:

$$\alpha = \underbrace{\alpha_1 + d\alpha_2}_{\text{Kernel}(d)} + \underbrace{d^*\alpha_3}_{\text{Image}(d^*)}.$$

The converse is proved in a similar way, and is left as an exercise for the reader. □

The operator D maps $\Omega^{\text{even}}(M) = \oplus_p \Omega^{2p}(M)$ into $\Omega^{\text{odd}}(M) = \oplus_p \Omega^{2p+1}(M)$. It follows immediately from Lemma 1.26 and the Hodge theorem, together with Lemma 1.25 that

$$\text{Index}(D\colon \Omega^{\text{even}}(M) \to \Omega^{2p+1}(M)) = \chi(M),$$

where $\chi(M)$ is the *Euler characteristic* of M:

$$\chi(M) = \sum_p (-1)^p \dim H^p(M).$$

We have therefore obtained a representation of an interesting topological invariant of M as the index of a differential operator. To do the same for the signature of M we need to divide $\Omega^*(M)$ into two pieces in a more sophisticated way than separating the even and odd-degree differential forms.

1.27. DEFINITION. Assume that $\dim(M) = n = 2k$. Define an operator $\varepsilon\colon \Omega^*(M) \to \Omega^*(M)$ by the formula

$$\varepsilon\alpha = i^{k+p(p-1)} \star \alpha,$$

where $\alpha \in \Omega^p(M)$.

The operator ε is a grading operator in the sense of the following definition.

1.28. DEFINITION. A *graded inner product space* is an inner product space equipped with a *grading operator* — a self-adjoint operator ε whose square is equal to 1. An operator on a graded inner product space is *even* if it commutes with the grading operator ε and *odd* if it anticommutes.

A grading is an orthogonal decomposition of the inner product space into the ± 1 eigenspaces of the grading operator ε, and an operator is even if it preserves these subspaces, whereas it is odd if it exchanges them.

Now let D be an odd, self-adjoint operator on a graded inner product space $H = H_+ \oplus H_-$. Assume that $H = \text{Kernel}(D) \oplus \text{Image}(D)$ and that $\text{Kernel}(D)$ is finite-dimensional.

1.29. DEFINITION. In the above situation the *index* of D is the Fredholm index of the component of D which maps H_0 to H_1. In other words, it is the difference

$$\text{Index}(D, \varepsilon) = \dim(\text{Kernel}(D) \cap H_+) - \dim(\text{Kernel}(D) \cap H_-).$$

Now let us return to the consideration of the signature operator.

1.30. PROPOSITION. *If M is a closed, oriented Riemannian manifold of dimension $4k$ then the signature of M is equal to the index of $D = d + d^*$ relative to the grading ε defined above.*

PROOF. The kernel of D, that is the space of harmonic forms, decomposes into a sum of subspaces $\mathcal{H}^p$ of harmonic p-forms. If we arrange this sum as

$$\left(\mathcal{H}^0 \oplus \mathcal{H}^{4k}\right) \oplus \left(\mathcal{H}^1 \oplus \mathcal{H}^{4k-1}\right) \oplus \cdots \oplus (\mathcal{H}^{2k-1} \oplus \mathcal{H}^{2k+1}) \oplus \mathcal{H}^{2k},$$

the $2k + 1$ summands shown here are ε-invariant and so make independent contributions to the index. Think first about the middle dimension, $2k$. Here the Hodge theorem identifies $\mathcal{H}^{2k}$ with the middle-dimensional cohomology. Moreover, the intersection form $Q(\alpha, \beta) = \int \alpha \wedge \beta = \langle \alpha, \star\beta \rangle$, when considered on $\mathcal{H}^{2k}$, is positive on the positive eigenspace of $\varepsilon = \star$ and negative on the negative eigenspace. Thus, the contribution to the index from $\mathcal{H}^{2k}$ is precisely the signature.

To see that the other summands above make a zero contribution, we just need to observe that to each harmonic pair $(\alpha, \varepsilon(\alpha)) \in \mathcal{H}^\ell \oplus \mathcal{H}^{4k-\ell}$ which is in the positive eigenspace of ε, there corresponds a harmonic pair $(\alpha, -\varepsilon(\alpha))$ in the negative eigenspace. $\square$

We conclude this chapter with some exercises on Poincaré duality. As another application of the Hodge theorem, let us prove the Poincaré duality theorem, to which we referred once or twice earlier on.

1.31. EXERCISE. Let M be a compact, oriented n-dimensional manifold. Show that for every p the bilinear pairing

$$H^p(M) \times H^{n-p}(M) \to \mathbb{R}$$
$$[\alpha], [\beta] \mapsto \int_M \alpha \wedge \beta$$

is non-degenerate. (Use Lemma 1.25 to reduce to the case where α is harmonic, then take $\beta = \star\alpha$.) Show that the pairing induces an isomorphism

$$H^p(M) \xrightarrow[\cong]{} \mathrm{Hom}(H^{n-p}(M), \mathbb{R}).$$

1.32. EXERCISE. Check that the results which we earlier said 'follow from Poincaré duality' really *do* follow from this theorem. (This is easy!)

5. Notes

Aside from introducing the signature operator, this chapter should indicate to the reader that we are assuming a certain degree of familiarity with basic topics in topology, mostly organized around de Rham cohomology theory. The first two chapters of the book by Bott and Tu [] should provide adequate backgound; later on we shall cover several of the topics from Chapter 4 of that book.

CHAPTER 2

Elliptic Operators and Fredholm Theory

In this chapter we shall lay the foundations for index theory by developing the theory of elliptic linear partial differential operators on manifolds. We shall also prove the Hodge theorem (Theorem 1.22) that we discussed in the last chapter.

1. Spectral Theory

We are going to approach index theory through the spectral theory of self-adjoint linear operators on Hilbert space.

2.1. DEFINITION. A (densely defined) *unbounded operator* on a Hilbert space H is a linear operator T from a dense linear subspace of H, called the *domain* of T and denoted $\operatorname{dom} T$, into H. An unbounded operator T is *symmetric* if

$$\langle Tu, v\rangle = \langle u, Tv\rangle$$

for all $u, v \in \operatorname{dom} T$. An unbounded, symmetric operator T is *essentially self-adjoint* if the operators $T \pm iI$ map $\operatorname{dom} T$ onto dense subspaces of H. An unbounded, symmetric operator T is *self-adjoint* if the operators $T \pm iI$ map $\operatorname{dom} T$ onto the whole of H.

2.2. EXERCISE. If $\dim H < \infty$ then every symmetric operator on H is automatically self-adjoint. Thus the distinctions between symmetric, essentially self-adjoint, and self-adjoint operators arise only in infinite dimensions.

If T is symmetric then the operators $T \pm iI$ are automatically bounded below. This is because

$$\begin{aligned} \|(T \pm iI)u\|^2 &= \langle (T \pm iI)u, (T \pm iI)u\rangle \\ &= \langle (T \mp iI)(T \pm iI)u, u\rangle \\ &= \langle (T^2 + I)u, u\rangle \\ &= \|Tu\|^2 + \|u\|^2. \end{aligned}$$

As a result, the operators $(T \pm iI)^{-1}$ are well-defined and bounded linear operators from the ranges of $T \pm iI$ into H (or indeed into $\operatorname{dom} T$). If T is self-adjoint, then since the operators $(T \pm iI)$ map $\operatorname{dom} T$ bijectively onto H, we may form the inverse operators $(T \pm iI)^{-1}$, which are bounded linear operators from H to itself. Similarly, if T is essentially self-adjoint then the operators $(T \pm iI)$ map $\operatorname{dom} T$ bijectively onto dense subspaces of H, and since the inverse operators $(T \pm iI)^{-1}$ (which are defined on these dense subspaces) are bounded linear operators, they may be extended by continuity so as to obtain bounded operators from H to itself. The *spectral theorem* asserts that we can then form more general functions of the operator T:

2.3. THEOREM. *Let* T *be an essentially self-adjoint operator on a Hilbert space* H. *There is a unique homomorphism of* C^**-algebras from the algebra of continuous, bounded functions on* $\mathbb{R}$ *into the algebra of bounded operators on* H *which maps the functions* $(x \pm i)^{-1}$ *to the operators* $(T \pm iI)^{-1}$. □

The spectral theorem is proved by observing that the operators $(T \pm iI)^{-1}$ generate a commutative C^*-algebra of operators. According to the Gelfand-Naimark theorem, every commutative C^*-algebra is isomorphic to $C_0(X)$, for some locally compact space X. In this case the space X may be identified with a closed subset of $\mathbb{R}$ (the spectrum of T) in such a way that the operators $(T \pm iI)^{-1}$ correspond to the functions $(x \pm i)^{-1}$. The reader is referred to [] or [] for further details.

2.4. DEFINITION. If f is a continuous, bounded function on $\mathbb{R}$ and T is an essentially self-adjoint operator, then we shall denote by $f(T)$ the the operator associated the the pair f and T by the spectral theorem.

Suppose now that D is a differential operator acting on the sections of a smooth, complex vector bundle S over a smooth manifold M. Suppose also that S is provided with a hermitian structure and that M is provided with a smooth measure. In this case we can form the Hilbert space $L^2(M, S)$ of square-integrable sections of S by completing the space of smooth, compactly supported sections in the norm induced from the inner product

$$\langle u, v \rangle = \int_M \langle u(m), v(m) \rangle \, dm.$$

We can then regard D as an unbounded Hilbert space operator with domain the smooth, compactly supported sections of S.

2.5. PROPOSITION. *If* D *is a symmetric, order one, linear partial differential operator on a closed manifold then* D *is essentially self-adjoint.*

Before proving this, let us review some terminology concerning solutions of linear partial differential equations. If D is any symmetric linear partial differential operator on a manifold, and if $u, v \in L^2(M, S)$, then we say that $Du = v$ in the *strong* sense if there is a sequence $\{u_n\}$ of smooth, compactly supported sections such that $u_n \to u$ in $L^2(M, S)$ and $Du_n \to v$ in $L^2(M, S)$. We say that $Du = v$ in the *weak* sense (or in the sense of *distributions* if, for every smooth, compactly supported section w, $\langle D^* w, u\rangle = \langle w, v\rangle$, where D^* is the formal adjoint[1] of D. (The idea here is to think of $\langle D^* w, u\rangle$ as a substitute for $\langle w, Du\rangle$, which is not necessarily well-defined since u may not belong to $\operatorname{dom} D$.)

2.6. EXERCISE. Every strong solution is a weak solution.

2.7. LEMMA. *If D is an order one, linear partial differential operator on a manifold, and if $v \in L^2(M, S)$ is compactly supported, then every compactly supported weak solution of the equation $Du = v$ is a strong solution.*

PROOF. Let us suppose first that u and v are supported within a coordinate neighborhood U of M, over which the bundle S is trivialized. By shrinking U slightly, we may identify U with an open set in $\mathbb{R}^n$ in such a way that the restriction of D to U identifies with the restriction to U of some compactly supported order one operator D' on $\mathbb{R}^n$, acting on vector-valued functions. We will show that there are smooth, vector-valued functions u_n, compactly supported in U, such that $u_n \to u$ and $D'u_n \to v$. Let f be a compactly supported function on $\mathbb{R}^n$ with total integral 1, and for $\varepsilon > 0$ let K_ε be the operator of convolution with $\varepsilon^{-n} f(\varepsilon^{-1} x)$. The following facts may be shown about K_ε:

(i) $K_\varepsilon v \to v$ as $\varepsilon \to 0$, for every L^2-function v;
(ii) The commutator $[D', K_\varepsilon]$ extends to a bounded operator on $L^2(\mathbb{R}^n)$, for every ε, and $[D', K_\varepsilon]v \to 0$, for every L^2-function v.

If $D'u = v$ in the weak sense then $D'K_\varepsilon u = K_\varepsilon v + [D', K_\varepsilon]u$ in the honest sense (note that $K_\varepsilon u$ is a smooth function). We see that $D'K_\varepsilon u \to v$, while $K_\varepsilon u \to u$, so we can set $U_n = K_{1/n} u$.

In the general case, if $Du = v$ in the weak sense, choose a partition of unity $\{\sigma_j\}$ on M subordinate to coordinate charts. Then $D\sigma_j u = \sigma_j v + [D, \sigma_j]u$ in the weak sense (note that the commutator $[D, \sigma_j]$ is a bounded operator), and hence also in the strong sense. Summing over j, and using the fact that $\sum [D, \sigma_j] = 0$, we see that $Du = v$ in the strong sense, as required. □

PROOF OF PROPOSITION 2.5. If u is orthogonal to the range of one of $D \pm iI$, then u is a weak solution of one of the equations $(D \mp iI)u = 0$.

[1] If D is any linear partial differential operator then the formal adjoint is the unique linear partial differential operator D^* such that $\langle Du, v\rangle = \langle u, D^* v\rangle$, for all smooth, compactly supported sections u and v. We already encountered this notion in the last chapter, where we showed that $d^* = \pm \star d \star$.

It follows from the previous lemma that u is then a strong solution. If $\{u_n\}$ is a sequence of smooth sections such that $u_n \to u$ and $(D \mp iI)u_n \to 0$, then we can compute:

$$0 = \lim_{n\to\infty} \|(D \mp iI)u_n\|^2 \geq \lim_{n\to\infty} \|u_n\|^2 = \|u\|^2.$$

Hence $u = 0$, which proves the proposition. □

2.8. EXERCISE. Assume that a linear partial differential operator D is essentially self-adjoint. Show that if $Du \in L^2(M, S)$ in the weak sense, then $Du \in L^2(M, S)$ in the strong sense. Show that the range of $(D \pm iI)^{-1}$ (which we consider, after extending by continuity, to be defined on all of $L^2(M, S)$) consists of all $u \in L^2(M, S)$ such that $Du \in L^2(M, S)$ in the strong sense.

We conclude with a useful piece of terminology:

2.9. DEFINITION. If D is essentially self-adjoint then there is a unique self-adjoint operator $\overline{D}$ such that $\operatorname{dom} D \subseteq \operatorname{dom} \overline{D}$ and $Du = \overline{D}u$, for every $u \in \operatorname{dom} D$. This operator is called the *self-adjoint extension* of D. Its domain consists of all u such that $Du \in L^2(M, S)$ in the weak or strong sense.

2. Compact Resolvent and Fredholm Theory

2.10. DEFINITION. Let D be a self-adjoint, or essentially self-adjoint operator on a Hilbert space H. We shall say that D has *compact resolvent* if, for every continuous function f on $\mathbb{R}$ which vanishes at infinity, the operator $f(D)$, defined by means of the spectral theorem, is a compact operator.

If D has compact resolvent then the set $\{\, f(D) : f \in C_0(\mathbb{R}) \,\}$ is a commuting algebra of normal, compact operators. It follows from the spectral theorem for compact operators, proved in elementary Hilbert space theory, that there is an orthornomal basis $\{u_j\}$ for H consisting of simultaneous eigenvectors for all the operators $f(D)$.

2.11. PROPOSITION. *With D and $\{u_j\}$ as above, there are real scalars λ_j such that $f(D)u_j = f(\lambda_j)u_j$, for all j. The vectors u_j belong to $\operatorname{dom} D$ (or $\operatorname{dom} \overline{D}$, if D is essentially self-adjoint), and $Du_j = \lambda_j u_j$, for all j. Moreover $\lim_{j\to\infty} |\lambda_j| = \infty$.*

PROOF. Exercise. □

This allows us to prove a Hilbert space version of the Hodge theorem:

2.12. COROLLARY. *If D is a self-adjoint operator on a Hilbert space H, and if D has compact resolvent, then the kernel of D is finite-dimensional, the range of D is closed, and $\operatorname{Kernel} D \oplus \operatorname{Range} D = H$.* □

Let us suppose now that D is a self-adjoint operator on H, that H is $\mathbb{Z}/2$-graded (so that H is decomposed as a direct sum $H = H_0 \oplus H_1$), and that D is odd-graded. By the latter we mean that the grading operator ε maps $\operatorname{dom} D$ into itself, and $\varepsilon D + D\varepsilon = 0$. In matrix form, we can write

$$D = \begin{pmatrix} 0 & D_- \\ D_+ & 0 \end{pmatrix}.$$

The kernel of D decomposes as a direct sum of the kernels of D_+ and D_-, and if D has compact resolvent then both of these summands are finite-dimensional. As a result we can essentially repeat the definition of index from the last chapter (although the context here is just a bit different):

2.13. DEFINITION. The *index* of the odd, self-adjoint operator D with compact resolvent is

$$\operatorname{Index}(D, \varepsilon) = \dim \operatorname{Kernel}(D_+) - \dim \operatorname{Kernel}(D_-).$$

It follows from our "Hilbert space Hodge theorem" that this is the same as the Fredholm index of the operator D_+, considered as a linear operator from its domain into H_-. We shall show in the next chapter that the index we have defined has stability properties which are reminiscent of those familiar from the theory of bounded Fredholm operators. For example, if B is a bounded, odd-graded self-adjoint operator, then $\operatorname{Index}(D + B, \varepsilon) = \operatorname{Index}(D, \varepsilon)$. In addition, if $\{D_t\}$ is a one-parameter familty of odd-graded, self-adjoint operators with compact resolvent, and if, for every f the operators $f(D_t)$ are norm-continuous in t, then $\operatorname{Index}(D_t, \varepsilon)$ is constant in t. (However the reader might enjoy trying to prove these directly now.)

3. Sobolev Spaces and Fourier Theory

Our objective is to show that various operators such as the signature operator $D = d + d^*$ which we considered in the last chapter have compact resolvent, so that the notion of Fredholm index introduced above applies to them. Eventually we shall also prove the "C^∞-Hodge theorem" which was stated and proved in Chapter 1. One important tool which we shall need to meet these objectives is the theory of Sobolev spaces.

2.14. DEFINITION. Let u be a smooth, compactly supported function on $\mathbb{R}^n$. Let s be a non-negative integer[2]. The *Sobolev* s*-norm* of u is the quantity $\|u\|_s$ defined by

$$\|u\|_s^2 = \int_{\widehat{\mathbb{R}}^n} (1 + |\xi|^2)^s |\hat{u}(\xi)|^2 \, d\xi,$$

[2] The definition works for all real s, but the restriction to non-negative integers simplifies some arguments and is enough for our purposes.

where $\hat{u}$ denotes the Fourier transform

$$\hat{u}(\xi) = \int_{\mathbb{R}^n} e^{-i\xi\cdot x} u(x)\, dx.$$

If U is an open subset of $\mathbb{R}^n$ then the *Sobolev space* $H^s(U)$ is the completion in the Sobolev s-norm of the space of smooth functions on $\mathbb{R}^n$ which are compactly supported in U.

The Plancherel formula from Fourier theory asserts that

$$\int_{\mathbb{R}^n} |u(x)|^2\, dx = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} |\hat{u}(\xi)|^2\, d\xi.$$

Thus, up to a multiplicative constant (which will be irrelevant to us), the Sobolev 0-norm is the same thing as the ordinary L^2-norm. Observe that if $s_1 > s_2$ then $\|u\|_{s_1} > \|u\|_{s_2}$. It follows that $H^{s_1}(U)$ may be regarded as a (dense) subspace of $H^{s_2}(U)$. In particular all of the Sobolev spaces $H^s(U)$ can be regarded as subspaces of the Hilbert space $L^2(U)$.

If u is any smooth, compactly supported function on $\mathbb{R}^n$, then the Fourier transform of the function $\partial^\alpha u$ is the function $(i\xi)^\alpha \hat{u}(\xi)$. As a result of this, there is a close relation between the Sobolev spaces $H^s(U)$ and spaces of differentiable functions:

2.15. LEMMA. *If $s \geq 0$ then the Sobolev s-norm is equivalent to the norm*

$$\sum_{\alpha \leq s} \|\partial^\alpha u\|^2_{L^2(\mathbb{R}^n)}.$$

PROOF. It follows from Plancherel's theorem that

$$\sum_{\alpha \leq s} \|\partial^\alpha u\|^2_{L^2(\mathbb{R}^n)} = \frac{1}{(2\pi)^n} \sum_{\alpha \leq s} \int_{\widehat{\mathbb{R}}^n} \xi^{2\alpha} |\hat{u}(\xi)|^2\, d\xi.$$

The lemma follows from the fact that the functions $\sum_{\alpha \leq s} \xi^{2\alpha}$ and $(1+|\xi|^2)^s$ are bounded multiples of one another. □

Thus roughly speaking the Sobolev space $H^s(U)$ consists of functions supported in U all of whose derivatives of order s or less belong to $L^2(U)$.

In order to globalize the Sobolev norms to manifolds we shall need the following result:

2.16. LEMMA. *If σ is a smooth function on an open set $U \subseteq \mathbb{R}^n$ whose derivatives of all orders are bounded functions on U, then pointwise multiplication by σ extends to a bounded linear operator on $H^s(U)$, for every s. In addition, if $\phi\colon U' \to U$ in a diffeomorphism from one open set in $\mathbb{R}^n$ to another whose derivatives of all orders are bounded functions, then the operation of composition with ϕ extends to a bounded linear operator from $H^s(U')$ to $H^s(U)$.*

PROOF. Both of these facts follow easily from the alternate characterization of the s-norms given in the last lemma. □

Suppose now that M is a closed manifold. Choose a finite coordinate cover $\{U_j\}$ for M and a partition of unity $\{\sigma_j\}$ subordinate to this cover. Using this structure any function on M can be broken up into a list of compactly supported functions on $\mathbb{R}^n$; we construct a *Sobolev s-norm* of the function on u M by combining the s-norms of the constituent pieces $\sigma_j u$, which we regard as compactly supported functions on $\mathbb{R}^n$. Thus:

$$\|u\|_s^2 = \sum_j \|\sigma_j u\|_s^2.$$

The norm depends on the choices we have made, but it is not hard to check, using the lemma above, that different sets of choices give equivalent norms. Thus the following makes sense.

2.17. DEFINITION. Let M be a closed manifold. The *Sobolev space* $H^s(M)$ is the completion of $C^\infty(M)$ in the above Sobolev s-norm.

The following result is known as the *Rellich Lemma*.

2.18. PROPOSITION. *If M is a closed manifold, then the inclusion of $H^1(M)$ into $L^2(M)$ is a compact operator.*

PROOF. In view of the way the space $H^1(M)$ is constructed using partitions of unity, it suffices to show that if σ is a smooth, compactly supported function on $\mathbb{R}^n$ then the composition

$$H^1(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n) \xrightarrow{\sigma} L^2(\mathbb{R}^n)\ ,$$

where the first map is inclusion and the second is pointwise multiplication by σ, is compact. There is a commutative diagram

$$\begin{array}{ccccc} H^1(\mathbb{R}^n) & \longrightarrow & L^2(\mathbb{R}^n) & \xrightarrow{\sigma} & L^2(\mathbb{R}^n) \\ \cong\downarrow & & \cong\downarrow & & \cong\downarrow \\ L^2(\mathbb{R}^n) & \longrightarrow & L^2(\mathbb{R}^n) & \longrightarrow & L^2(\mathbb{R}^n) \end{array}$$

in which the leftmost vertical map is Fourier transform, followed by pointwise multiplication by $(1+\xi^2)^{\frac{1}{2}}$; the remaining two vertical maps are Fourier transforms; and the left bottom map is pointwise multiplication with $(1+\xi^2)^{-\frac{1}{2}}$, and the right bottom map is convolution with the Fourier transform of σ. Since the vertical maps are unitary isomorphisms it suffices to prove that the bottom composition is compact. It is left to the reader to check that the composition of pointwise multiplication by any C_0-function with convolution against any L^1-function is compact. □

2.19. EXERCISE. Prove the claim at the end of the proof of the Rellich lemma by approximating the two functions with continuous, compactly supported functions, for which the corresponding composition is then represented by a continuous, compactly supported kernel function $k(x, y)$.

2.20. REMARK. By a refinement of this argument it may be shown that the inclusion of $H^{s_1}(M)$ into $H^{s_2}(M)$ is compact, whenever $s_1 > s_2$.

The Rellich Lemma is very relevant to the problem of showing that suitable operators D have compact resolvent, as the following argument shows.

2.21. LEMMA. *Let M be a smooth, closed manifold. If D is an essentially self-adjoint operator on $L^2(M)$, and if the domain of the self-adjoint extension of D is $H^1(M)$, then D has compact resolvent.*

PROOF. Assume that the domain of $\bar{D}$ is $H^1(M)$. The range of the operators $(D \pm iI)^{-1}$ is then $H^1(M)$. Since it is easy to verify that the graph of the operator $(D \pm iI)^{-1} \colon L^2(M) \to H^1(M)$ is a closed subspace of $L^2(M) \times H^1(M)$, it follows from the closed graph theorem that the operators $(D \pm iI)^{-1}$ are bounded, viewed as operators from $L^2(M)$ into $H^1(M)$. Since the Rellich Lemma asserts that the inclusion of $H^1(M)$ into $L^2(M)$ is compact, it follows that the resolvents $(D \pm iI)^{-1}$, viewed as operators from $L^2(M)$ to itself, are compact. Since the functions $(x \pm i)^{-1}$ generate $C_0(\mathbb{R})$, it now follows from an easy approximation argument that $f(D)$ is a compact operator, for every $f \in C_0(\mathbb{R})$. □

2.22. REMARK. It follows easily from Lemma 2.15 that if D is an order one operator on a compact manifold then D extends to a bounded linear operator from $H^1(M)$ into $L^2(M)$. It follows from this that the domain of always $\bar{D}$ contains $H^1(M)$. The reverse containment is more difficult to establish, and indeed it does not hold in general.

4. Estimates for Elliptic Operators

The purpose of this section and the next is to obtain a condition on a order one operator D on a closed manifold which is sufficient to imply that the domain of $\bar{D}$ is $H^1(M)$.

We shall consider first *constant coefficient, homogeneous* differential operators on $\mathbb{R}^n$. The underlying manifold here is of course not compact, but it will turn out that the analysis of constant coefficient operators is the key to understanding operators on compact manifolds.

A constant coefficient homogenenous order one operator D on $\mathbb{R}^n$ must have the form

$$D = \sum_{j=1}^{n} a_j \frac{\partial}{\partial x^j}.$$

If we are to allow D to operate on *vector*-valued functions (corresponding to our intention to consider operators on bundles in the variable-coefficient case) then the constants a_j may be *matrices*. If D is symmetric, the matrices a_j will be skew-Hermitian.

Let us rewrite D in terms of the Fourier transform, as follows:

$$(\widehat{Du})(\xi) = i \sum a_j \xi^j \hat{u}(\xi).$$

Here u is a smooth, compactly supported function on $\mathbb{R}^n$.

2.23. DEFINITION. The constant coefficient operator D above is *elliptic* if $\sum a_j \xi^j$ is an *invertible* matrix for all nonzero ξ.

2.24. REMARK. The condition of ellipticity is invariant under linear changes of coordinates, and so we can — and later on will — speak of elliptic, translation-invariant, partial differential operators on vector spaces.

2.25. EXAMPLE. Let D be the operator

$$D = \begin{pmatrix} 0 & \overline{\partial}^* \\ \overline{\partial} & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2} & 0 \end{pmatrix}$$

on $\mathbb{C} = \mathbb{R}^2$. Then the matrix in Definition 2.23 is

$$\begin{pmatrix} 0 & -\xi_1 + i\xi_2 \\ \xi_1 + i\xi_2 & 0 \end{pmatrix},$$

and as a result, D is elliptic.

2.26. LEMMA. *If D is a symmetric, order one, constant coefficient operator on $\mathbb{R}^n$ then there is a constant $\delta > 0$ such that*

$$\|u\|_0 + \|Du\|_0 \geq \delta \|u\|_1,$$

for every $u \in C_c^\infty(\mathbb{R}^n)$.

PROOF. At the expense of altering δ, is suffices to prove the related inequality

$$\|u\|_0^2 + \|Du\|_0^2 \geq \delta \|u\|_1^2.$$

Using the Plancherel formula, this is equivalent to the inequality

$$\|\hat{u}\|_0^2 + \|\widehat{Du}\|_0^2 \geq \delta \|\hat{u}\|_1^2.$$

Bearing in mind the formula for $\widehat{Du}$, this follows from the matrix inequality

$$1 + \left(\sum a_j \xi^j\right)^* \left(\sum a_j \xi^j\right) \geq \delta(1 + \sum \xi_j^2),$$

which is implied by the invertibility of $\sum a_j \xi_j$. $\square$

2.27. COROLLARY. *If* D *is a symmetric, order one, constant coefficient operator on* $\mathbb{R}^n$*, and if* $Du \in L^2(\mathbb{R}^n)$ *in either the weak or strong sense, then* $u \in H^1(\mathbb{R}^n)$.

PROOF. Since D is essentially self-adjoint, the notions of weak and strong solution agree. If $Du = v$, then there is a sequence $\{u_n\}$ of smooth, compactly supported functions such that $u_n \to u$ and such that $Du_n \to v$ (convergence in the norm of the Hilbert space $L^2(\mathbb{R}^n)$). Thanks to Lemma 2.26, the convergence of $\{u_n\}$ and of $\{Du_n\}$ in $L^2(\mathbb{R}^n)$ implies the convergence of $\{u_n\}$ in $H^1(\mathbb{R}^n)$. Thus $u \in H^1(\mathbb{R}^n)$, as required. $\square$

In the remainder of this section we shall extend the crucial estimate in Lemma 2.26, known as *Garding's inequality*, to operators on $\mathbb{R}^n$ with variable coefficients. In the next section we shall extend the estimate still further to operators on manifolds.

To deal with a variable coefficient operator D we shall "freeze the coefficients" of D at a single point and try to approximate D by resulting constant coefficient operator in a neighborhood of that point.

2.28. DEFINITION. Let $D = \sum a_j \partial_j + b$ be a order one, partial differential operator on an open subset U of $\mathbb{R}^n$. We shall say that D is *elliptic* if, for every $m \in U$, the constant coefficient operator $D_m = \sum a_j(m)\partial_j$ is elliptic.

2.29. PROPOSITION. *Let* D *be an order one, linear elliptic partial differential operator on an open set* U *of* $\mathbb{R}^n$*. For every compact set* $K \subseteq U$ *there is a constant* $\delta > 0$ *such that*

$$\|u\|_0 + \|Du\|_0 > \delta \|u\|_1,$$

for every smooth function u *supported in* K.

PROOF. If $p \in K$, then there is a constant $\delta_p > 0$ such that

$$\|u\|_0 + \|D_p u\|_0 > \delta_p \|u\|_1,$$

for all $u \in C_c^\infty(\mathbb{R}^n)$. There is also a neighborhood U_p of p in U such that

$$\|D_p u - Du\|_0 < \frac{1}{2}\delta_p \|u\|_1,$$

for all u supported in U_p. It follows that

$$\|u\|_0 + \|D_p u\|_0 > \frac{1}{2}\delta_p \|u\|_1,$$

for all u supported in U_p. Choose a finite cover of K by neighborhoods $U_{p_1}, \ldots, U_{p_n}$ for which there exist estimates of this type. Let $\{\sigma_j\}$ be a partition of unity subordinate to $\{U_{p_j}\}$. Write

$$\begin{aligned}\|u\|_1 \le \sum_j \|\sigma_j u\|_1 &\le \text{constant} \sum_j \|\sigma_j u\|_0 + \|D\sigma_j u\|_0 \\ &\le \text{constant} \sum_j \|\sigma_j u\|_0 + \|\phi_j D u\|_0 + \|[D, \sigma_j]u\|_0.\end{aligned}$$

In the last display the first term is bounded by a multiple of $\|u\|_0$; the second term is bounded by a multiple of $\|Du\|_0$; the third term is also bounded by a multiple of $\|u\|_0$ since $[D, \sigma_j]$ is an operator of order zero. We obtain the inequality

$$\|u\|_1 \le \text{constant}\left(\|u\|_0 + \|Du\|_0\right),$$

which completes the proof. □

5. The Symbol

To extend Garding's inequality to operators on manifolds we shall use a coordinate-free way to describe the operators D_p which we obtained in the last section by freezing coefficients. This leads us to the notion of the *symbol* of a differential operator, which will be of central importance throughout the rest of these notes.

2.30. DEFINITION. Let D be an order one, linear partial differential operator on a smooth manifold M, acting on sections of a smooth, complex vector bundle S. The *symbol* of D at a point $m \in M$ is the linear map

$$\sigma\colon T^*_m M \to \text{End}(S_m)$$

given by the formula

$$\sigma\colon df \mapsto i[D, f]_m$$

2.31. REMARKS. If f is a smooth function on M, then the commutator $[D, f]$ is an endomorphism of the bundle S, and so its value $[D, f]_m$ at the point $m \in M$ is an endomorphism of the fiber S_m, as required. The value of this endomorphism of S_m depends only on the value of df at m, and so we obtain a map from T^*M into $\text{End}(S_m)$, as required. The appearance of i in our formula for the symbol is purely conventional. It has the effect that if D is a symmetric operator (with respect to some measure on M and inner product on S), then σ_D is symmetric too.

2.32. EXAMPLE. If $D = \sum a_j \partial_j + b$ in local coordinates then the symbol σ maps the cotangent vector ξ to the endomorphism $\sum a_j(m)\xi_j$ at the point $m \in M$.

Now, the space of linear maps from $T^*_m M$ into $\mathrm{End}(S_m)$ identifies with the tensor product $\mathrm{End}(S_m) \otimes T_m M$, and an element $\sum a_j \otimes X_j$ of this tensor product determines a translation-invariant, first oder linear partial differential operator D_m on the vector space $T_m M$, acting on sections of the trivial bundle with fiber S_m, by the formula

$$D_m u = \sum a_j X_j(u).$$

Here we regard the tangent vectors X_j as translation-invariant vector fields on $T_m M$ (or in other words as directional derivatives). In summary, the symbol σ of an order one operator D on M determines a translation-invariant, order one partial differential operator D_m on the the tangent space $T_m M$.

2.33. DEFINITION. We shall refer to the translation invariant operator D_m obtained from the symbol in this way as the *model operator* for D at m.

2.34. EXAMPLE. If $D = \sum a_j \partial_j + b$ in local coordinates then $D_m = \sum a_j(m)\partial_j$. Thus the model operator D_m is precisely the operator we obtained in the last section by the process of freezing coefficients at the point $m \in M$.

2.35. DEFINITION. Let D be an order one, linear partial differential operator on a smooth manifold M. The operator D is *elliptic* if all its model operators D_m ($m \in M$) are elliptic.

With these definitions in hand, we can generalize the results of the previous section to operators on manifolds.

2.36. THEOREM. *Let D be an elliptic, order one, linear partial differential operator on a smooth manifold M, acting on sections of a smooth vector bundle S. For every compact set $K \subseteq M$ there is a constant $\delta > 0$ such that*

$$\|u\|_0 + \|Du\|_0 > \delta \|u\|_1,$$

for every smooth section u of the bundle S which is supported in K.

PROOF. This is a consequence of the local result proved in Proposition 2.29, together with the partition of unity argument introduced in the proof of Proposition 2.29. □

By repeating the argument used to prove Corollary 2.27 we reach one of our main objectives for this chapter:

2.37. COROLLARY. *Let D be an order one, elliptic linear partial differential operator on a closed manifold M, acting on sections of a smooth vector bundle S. The domain of the self-adjoint extension of D is $H^1(M, S)$, and D has compact resolvent.* □

2.38. EXAMPLE. The $D = d + d^*$ which we introduced in the last chapter is elliptic. To see this, let us begin by computing the symbol of the de Rham differential d. According to the definition, if $df = \eta$ then

$$\sigma_d(\eta)\omega = i[d, f]\omega = i\eta \wedge \omega.$$

So the symbol of d is given in a very simple way by wedge product of forms. Since the symbol of the adjoint d^* is the adjoint of the symbol d, we find that the symbol of the operator $D = d + d^*$ is given by the formula

$$\sigma_D(\eta)\omega = i\eta \wedge \omega - i\eta \lrcorner \omega,$$

where the operator $\omega \mapsto \eta \lrcorner \omega$ is the adjoint of the map $\omega \mapsto \eta \wedge \omega$.

2.39. LEMMA. *Let V be a finite-dimensional inner product space and let $S = \wedge^* V$. If $v \in V$ then the operator $c\colon S \to S$ given by the formula $c(x) = v \wedge x - v \lrcorner x$ has the property that $c^2 = -\|v\|^2 I$.*

PROOF. We can assume that v is a unit vector and the first vector in an orthonormal basis $v_1, \ldots, v_k$ for V. The products $v_{i_1} \wedge \cdots \wedge v_{i_p}$, where $i_1 < \cdots < i_p$, form an orthonormal basis for $S = \wedge * V$, and in this orthonormal basis the operator $x \mapsto v \wedge x$ acts as

$$v_{i_1} \wedge \cdots \wedge v_{i_p} \mapsto \begin{cases} v \wedge v_{1_i} \wedge \cdots \wedge v_{i_p} & \text{if } i_1 \neq 1 \\ 0 & \text{if } i_1 = 1 \end{cases}$$

The operator is therefore a partial isometry, and its adjoint is therefore given by the formula

$$v_{i_1} \wedge \cdots \wedge v_{i_p} \mapsto \begin{cases} v_{1_2} \wedge \cdots \wedge v_{i_p} & \text{if } i_1 = 1 \\ 0 & \text{if } i_1 \neq 1 \end{cases}$$

The lemma follows immediately from these formulas. □

As a result of this computation, the square of the symbol of $D = d + d^*$ is $\|\xi\|^2 I$. Thus, the symbol is invertible — up to a scalar multiple it is its own inverse — for all $\xi \neq 0$. This kind of algebra will be developed further in our discussion of Dirac-type operators in Chapter 6.

6. Elliptic Regularity and the Hodge Theorem

We shall now refine the results obtained so far so as to prove the Hodge Theorem from the last chapter. We shall also prove a result about representing the operators $f(D)$ in the functional calculus by kernels which strengthens the assertion that an elliptic operator on a closed manifold has compact resolvent.

The following *Sobolev embedding lemma* is a crucial feature of Sobolev space theory. It relates the norms $\| \; \|_s$ to the ordinary notion of differentiability of functions.

2.40. LEMMA. *If* $s > \frac{n}{2} + k$ *then* $H^s(\mathbb{R}^n)$ *is included within* $C^k_0(\mathbb{R}^n)$, *the Banach space of* k*-times continuously differentiable functions on* $\mathbb{R}^n$, *whose derivatives up to order* k *vanish at infinity.*

PROOF. We need to show that the C^k-norm of a smooth, compactly supported function is bounded by a multiple of the Sobolev s-norm, whenever $s > \frac{n}{2} + k$. This will imply that the identity map on $C^\infty_c(\mathbb{R}^n)$ extends to a continuous map of $H^s(\mathbb{R}^n)$ into $C^k(\mathbb{R}^n)$, as required. If $|\alpha| \leq k$ we compute, using the Fourier inversion formula, that

$$\partial^\alpha u(x) = \int e^{ix\xi}(i\xi)^\alpha \hat{u}(\xi)\, d\xi.$$

Therefore, by the Cauchy-Schwarz inequality,

$$|\partial^\alpha u(x)|^2 \leq \int (1+\xi^2)^{-s}\xi^{2\alpha}\, d\xi \cdot \int (1+\xi^2)^s |\hat{u}(\xi)|^2\, d\xi.$$

If $s > \frac{n}{2} + k$ and $k \geq |\alpha|$ then the first integral is finite. Taking square roots we get the required estimate $\sup_x |\partial^\alpha u(x)| \leq \text{constant}\|u\|_s$. □

2.41. LEMMA. *Let* $u \in L^2(\mathbb{R}^n)$. *If* $u \in H^s(\mathbb{R}^n)$, *for some* $s \geq 1$, *and if* $\partial_j u \in H^s(\mathbb{R}^n)$ *for all* $j = 1, \ldots, n$ *(in the weak sense), then* $u \in H^{s+1}(\mathbb{R}^n)$.

PROOF. Thanks to the Plancherel formula, the Fourier transform extends from smooth compactly supported functions to L^2-functions. The hypothesis that $\partial_j u \in H^s(\mathbb{R}^n)$ implies that

$$\int_{\mathbb{R}^n} (1+\xi^2)^s \xi_j^2 |\hat{u}(\xi)|^2\, d\xi < \infty.$$

This being true for all j, it follows that

$$\int_{\mathbb{R}^n} (1+\xi^2)^{s+1} |\hat{u}(\xi)|^2\, d\xi < \infty,$$

and it is then easy to prove by an approximation argument that $u \in H^{s+1}(\mathbb{R}^n)$. □

2.42. LEMMA. *Let* D *be an order one, linear elliptic partial differential operator on an open set* $U \subseteq \mathbb{R}^n$. *Let* $u \in L^2(\mathbb{R}^n)$ *and assume that* u *has compact support within* U. *If* $u \in H^s(\mathbb{R}^n)$ *for some* $s \geq 0$, *and if* $Du \in H^s(\mathbb{R}^n)$, *then in fact* $u \in H^{s+1}(\mathbb{R}^n)$.

PROOF. Suppose first that $s = 0$, that $u \in L^2(\mathbb{R}^n)$ has compact support in U, and that $Du = v \in L^2(\mathbb{R}^n)$, in the weak sense. The proof of Proposition 2.5 shows that there is a sequence $\{u_n\}$ of smooth functions, compactly supported in U, such that $u_n \to u$ and $Du_n \to v$. It follows from Garding's inequality that $\{u_n\}$ converges to u in $H^1(\mathbb{R}^n)$. To prove the lemma for higher s we use induction. If $u \in H^s(U)$ and $Du \in H^s(U)$, then from the formula

$$D\partial_j u = \partial_j D u + [D, \partial_j]u,$$

together with the fact that $[D, \partial_j]$ is an order zero operator, we see that $D\partial_j u \in H^{s-1}(U)$, for all j. It follows from the induction hypothesis that $\partial_j u \in H^s(U)$, for all j. It now follows from Lemma 2.41 that $u \in H^{s+1}(U)$, as required. □

2.43. THEOREM. *Let* D *be an order one, essentially self-adjoint, linear elliptic partial differential operator on a smooth manifold* M. *If* $u \in L^2(M)$ *and if* Du *is smooth then* u *is smooth.*

PROOF. Assume that $u \in L^2(M)$ and that Du is smooth. We will show that if σ is any smooth function on M which is compactly supported in a coordinate neighbourhood U, then $\sigma u \in H^s(U)$, for all s. In view of Lemma 2.40 this will suffice. We shall use induction, noting that the case $s = 0$ is trivial. Assume then that $\sigma u \in H^s(U)$, for all smooth σ which are compactly supported in U. It follows that if A is any smooth, order zero operator, which is compactly supported in U, then $Au \in H^s(U)$. We want to show that $\sigma u \in H^{s+1}(U)$. From the equation

$$D\sigma u = \sigma D u + [D, \sigma]u,$$

together with the fact that $[D, \sigma]$ is compactly supported and of order zero, we see that $D\sigma u \in H^s(U)$, and hence, by the previous lemma, that $\sigma u \in H^{s+1}(U)$, as required. □

2.44. REMARK. If we were to introduce the language of distribution theory we could prove without much difficulty the following stronger version of the theorem: if D is elliptic, and if u is a distribution such that Du is smooth, then u itself is smooth. This property of D is called *hypoellipticity*.

We close this section by considering the problem of representing operators $f(D)$ by continuous kernel functions $k(x, y)$. Since the operators we are interested in act on $L^2(M, S)$, we shall need to consider kernels which are not scalar-valued but *S-valued*, in the sense that $k(x, y) \in \mathrm{Hom}(S_y, S_x)$ (thus k is a continuous section of a bundle over $M \times M$).

2.45. PROPOSITION. *Let* D *be an essentially self-adjoint, elliptic order one operator on a smooth manifold* M *and let* $f \in \mathcal{S}(\mathbb{R})$ *be a rapidly decaying function. There is a continuous, S-valued kernel* k *such that*

$$f(D)u(m_2) = \int_M k(m_2, m_1)u(m_1)\, dm_1,$$

for every compactly supported section u.

PROOF. We shall assume that M is closed, which simplifies things a little bit, and leave the general case to the reader. Fix $\ell \geq 0$ and write the rapidly decreasing function f as a product

$$f(x) = (x^2+1)^{-\ell} g(x)(x^2+1)^{-\ell},$$

where g is also rapidly decreasing, and in particular bounded. Using the functional calculus we see that

$$f(D) = (D^2+I)^{-\ell} g(D)(D^2+I)^{-\ell}.$$

We shall prove the proposition by analyzing the operator $(I+D^2)^{-\ell}$. Since $(D^2+I)^{-1} = (D+iI)^{-1}(D-iI)^{-1}$, it follows from Theorem 2.43 that the range of the operator $(I+D^2)^{-\ell}$ is the Sobolev space $H^{2\ell}(M)$. So if $k \geq 0$ and if $\ell > \frac{n}{2}$ then $(I+D^2)^{-\ell}$ maps $L^2(M)$ continuously into $C(M)$. Taking Banach space adjoints, and using the fact that

$$\langle (I+D^2)^{-\ell} u, v\rangle = \langle u, (I+D^2)^{-\ell} v\rangle,$$

for all $u, v \in L^2(M)$, it follows that $(I+D^2)^{-\ell}$ extends to a continuous map of the dual space $C(M)^*$ into $L^2(M)$. Returning to our product decomposition of $f(D)$, we see that $f(D)$ extends to a continuous map of $C(M)^*$ into $C(M)$. Now, each element $m \in M$ determines an element $\delta_m \in C(M)^*$ by the formula $\delta_m(\phi) = \phi(m)$. We can therefore define a kernel function on $M \times M$ by the formula

$$k(m_1, m_2) = (f(D)\delta_{m_2})(m_1).$$

It may be verified that this is a continuous kernel which represents $f(D)$ in the required fashion. □

2.46. REMARK. The same sort of argument shows that k is in fact a smooth function, so that $f(D)$ is a *smoothing operator*.

7. A More General Version of the Elliptic Package

In later chapters we shall need to consider not single elliptic operators but families of elliptic operators on smooth families of smooth manifolds. In this section, which can be omitted on a first reading, we set up the necessary details.

Let us begin by recalling the following basic concept.

2.47. DEFINITION. A *submersion* is a surjective map $\pi\colon E \to B$ between smooth Riemannian manifolds with the property that for every $p \in E$ there are local coordinates $x_1, \ldots, x_{n+m}$ near p and local coordinates $y_1, \ldots, y_n$ near $\pi(p)$ in B such that π has the form

$$\pi(x_1, \ldots, x_{n+m}) = (y_1, \ldots, y_n).$$

2.48. REMARK. According to the implicit function theorem, the existence of these local coordinates is equivalent to the fact that the differential $D\pi\colon T_pE \to T_{\pi(p)}B$ is a surjective linear map.

If $\pi\colon E \to B$ is a submersion, then for every $b \in B$ the fiber $E_b = \pi^{-1}[b]$ is a smooth submanifold of E (the fibers of π need not be diffeomorphic to one another). Let D be an order one, linear partial differential operator on E (acting on sections of some smooth bundle S) which acts fiberwise, so that D restricts to a family of linear partial differential operators D_b on the fibers E_b.

2.49. EXERCISE. To say that D acts fiberwise is the same thing as to say that if f is any smooth function on B which is pulled back to E via π, then the commutator $[D, f]$ is zero. Show that this implies that if u is any section, then the restriction $(Du)_b$ of Du to a fiber E_b depends only on the restriction u_b of u to E_b, and that there is a unique operator D_b on E_b such that $D_b u_b = (Du)_b$.

Assume now that the bundle S is equipped with an inner product, and that the manifolds E_b are equipped with smooth measures μ_b which vary smoothly with b in the sense that if u is a smooth, compactly supported function on E then the formula

$$b \mapsto \int_{E_b} u_b(m)\, dm$$

defines a smooth function on B. The inner product and measures determine L^2-spaces $L^2(E_b, S_b)$, and these assemble to form a *continuous field* of Hilbert spaces, or *Hilbert* $C_0(B)$*-module*, in the following way. On the space of continuous, compactly supported sections of the bundle $H(E, S)$ over B define a $C_0(B)$-valued inner product by the formula

$$\langle u, v\rangle_b = \langle u_b, v_b\rangle_{L^2(E_b, S_b)},$$

and denote by $H(E, S)$ the completion in the associated norm

$$\|u\| = \max_{b\in B} \langle u_b, u_b\rangle_b^{\frac{1}{2}}.$$

The $C_0(B)$-valued inner product extends to $H(E, S)$, and we obtain a Hilbert module (see [] for an introduction to Hilbert modules). An element of $H(E, S)$ can be viewed as a continuous family of sections $u_b \in L^2(E_b, S_b)$.

Let us return to the operator D. We want to consider it as an unbounded operator on the Hilbert module $H(E, S)$. For this purpose we extend the definition of the Sobolev spaces that we introduced in Section 3. For simplicity, we shall consider only submersions for which E is compact.

2.50. DEFINITION. Let $\pi\colon E \to B$ be a submersion for which E is a compact manifold, and let s be a non-negative integer. Denote by $H^s(E, S)$ the space of all families $\{u_b\}$ in $H(E, S)$ such that $u_b \in H^s(E_b, S_b)$, for all b, and such that if X is any fiberwise differential operator of order s or less, then the family $\{X_b u_b\}$ belongs to $H(E, S)$.

Thus $H^s(E, S)$ consists of families $u_b \in H^s(E_b, S_b)$ which—in a suitable sense—vary continuously in the Sobolev s-norm.

2.51. PROPOSITION. *If E is compact then the operators $(D \pm iI)$ map $H^1(E, S)$ bijectively to $H(E, S)$.*

PROOF. Let $\{v_b\}$ be any family in $H(E, S)$. We need to show that the families $(D_b \pm iI)^{-1} v_b$, defined fiberwise, belong to $H^1(E, S)$. Since $(D_b \pm iI)^{-1} v_b \in H^1(E_b, S_b)$, for all b, the problem is to show continuity, in the appropriate sense, in the Sobolev 1-norm. Since E is compact the (compact) fibers of E are locally diffeomorphic to one another, and indeed the submersion is locally a product $M \times U \to U$, in such a way that S may be viewed as pulled back to the product $M \times U$ from some bundle on M. The proposition amounts to the assertion that if v_b ($b \in U$) is a norm-continuous family in $L^2(M, S)$ then $(D_b \pm iI)^{-1} v_b$ is norm-continuous in the Hilbert space $H^1(M, S)$. This follows from the formula

$$(D_{b_1} \pm iI)^{-1} v_{b_1} - (D_{b_2} \pm iI)^{-1} v_{b_2} = (D_{b_1} \pm iI)^{-1}(v_{b_1} - v_{b_2}) \\ + (D_{b_1} \pm iI)^{-1}(D_{b_2} - D_{b_1})(D_{b_2} \pm iI)^{-1} v_{b_2}$$

and Garding's inequality. (We need the fact that the constant in Garding's inequality can be chosen to be the same for all b; this follows from a simple compactness argument.) $\square$

2.52. PROPOSITION. *If $u \in H^1(E, S)$ and $Du \in H^s(E, S)$, then in fact $u \in H^{s+1}(E, S)$.*

PROOF. This follows by a similar argument, using the fact that in the product situation the operators $(D_b \pm iI)^{-1}$ map $H^s(M, S)$ continuously (and in fact equicontinuously) into $H^{s+1}(E, S)$ $\square$

2.53. DEFINITION. If $\pi\colon E \to B$ is a submersion, then denote by $E \times_B E$ the submanifold $\{(x, y) \in E \times E : \pi(x) = \pi(y)\}$ of $E \times E$. A *continuous family $\{k_b\}$ of S-valued continuous kernels on the fibers of* E is a continuous section of the vector bundle over $E \times_B E$ whose fiber at (x, y) is $\mathrm{Hom}(S_y, S_x)$.

2.54. PROPOSITION. *Let $\pi\colon E \to B$ be a submersion, with E compact, and let D be an order one, fiberwise elliptic operator on E, acting on sections of a bundle S. If $f\colon \mathbb{R} \to \mathbb{R}$ is a rapidly decreasing function* *then there is a continuous family* $\{k_b\}$ *of* S*-valued continuous kernels on the fibers of* E *such that*

$$f(D_b)u_b(m_2) = \int_{E_b} k_b(m_2, m_1)u_b(m_1)\, dm_1$$

for every $b \in B$ *and every section* u *of* S*.*

PROOF. This is proved following the argument used to prove Proposition 2.45. ☐

We shall also need a version of this proposition which applies to noncompact submersions. The following lemma allows us to reduce its proof to the compact case.

2.55. LEMMA. *Let* D *be an essentially self-adjoint differential operator on* M*. Let* K *be a compact subset of* M *and let* U *be an open neighborhood of* K*. There exists* $\varepsilon > 0$ *such that if* u *is supported within* K*, and if* $|s| < \varepsilon$*, then* $e^{isD}u$ *is supported within* U*.*

2.56. EXAMPLE. Let $M = \mathbb{R}$ and let $D = i\frac{d}{dx}$. Then e^{isD} is the operator on $L^2(\mathbb{R})$ of translation by s, and therefore has the “finite propagation” property of the lemma.

PROOF. By treating treating separately the two operators $\pm D$, it suffices to consider nonnegative s. Let $g\colon M \to \mathbb{R}$ be a smooth function which is equal to 1 on K, and which is compactly supported within U. Let $f\colon \mathbb{R} \to [0,1]$ be a smooth, *non-decreasing* function such that

$$\begin{cases} f(t) < 1 & \text{if } t < 1 \\ f(t) = 1 & \text{if } t \geqslant 1. \end{cases}$$

Finally, let $h_s(m) = f(g(m) + cs)$, where c is a positive constant which we will specify in a moment. Note that h_s is bounded by 1. Moreover if $s < 1/c$, then the set where h_s actually attains the value 1 is contained within the support of g, and hence within U. Suppose now that u is a smooth section which is supported within K, and let $u_s = e^{isD}u$. We are going to show that

$$s \geq 0 \quad \Rightarrow \quad \langle h_s u_s, u_s\rangle \geqslant \langle h_0 u_0, u_0\rangle\ .$$

Since u is supported within K, $\langle h_0 u_0, u_0\rangle = \langle u_0, u_0\rangle$; since e^{isD} is a unitary operator, $\langle u_0, u_0\rangle = \langle u_s, u_s\rangle$. By incorporating these identities we obtain the relation

$$s \geq 0 \quad \Rightarrow \quad \langle h_s u_s, u_s\rangle \geqslant \langle u_s, u_s\rangle.$$

This implies that $h_s u_s = u_s$, so that u_s is supported within the set where $h_s = 1$, and hence within U, so long as $s < 1/c$.

To prove that $\langle h_s u_s, u_s\rangle \geqslant \langle h_0 u_0, u_0\rangle$ for $s \geq 0$, let us first note that

$$\frac{dh_s}{ds}(m) = cf'(g(m) + cs) \geq 0.$$

We may also calculate the differential of the function $h_s\colon M \to \mathbb{R}$: we see that $dh_s(m) = f'(g(m) + cs)\, dg(m)$, and as a result

$$dh_s = \frac{1}{c}\frac{dh_s}{ds}dg,$$

and hence $[D, h_s] = \frac{1}{c}\frac{dh_s}{ds}[D, g]$. Now let us choose the constant c in such a way that $c > \|[D, g]\|$. Having done so, we see that

$$\frac{dh_s}{ds} - i[D, h_s] = \frac{1}{c}\frac{dh_s}{ds}\big(c - i[D, g]\big) \geqslant 0\,.$$

Returning now to the smooth section u which is supported within K, we conclude that

$$\frac{\partial}{\partial s}\big\langle h_s u_s, u_s\big\rangle = \big\langle \frac{dh_s}{ds}u_s, u_s\big\rangle - \big\langle i[D, h_s]u_s u_s\big\rangle \geq 0,$$

as required. □

2.57. THEOREM. *Let D be an order one, fiberwise elliptic, linear partial differential operator on a submersion $\pi\colon E \to B$. Assume that each fiber operator D_b is essentially self-adjoint. Let $f \in C_0(\mathbb{R})$ be a smooth function with compactly supported Fourier transform. There is a continuous family $\{k_b\}$ of continuous kernels such that*

$$f(D_b)u_b(m_2) = \int_{E_b} k_b(m_2, m_1)u_b(m_1)\, dm_1$$

for every $b \in B$ and every compactly supported section u of S. □

PROOF. The existence of a family of kernels k_b representing the operators $f(D_b)$ follows from Proposition 2.45; the problem is to show continuity of the family in b. For this purpose we shall use that following geometric fact: for every compact set L of E and every $b \in B$ there is a neighbourhood of $L \cap E_b$ in E and a compact submersion $E' \to B'$ and which agrees with $E \to B$ in that neighbourhood, along with a fiberwise elliptic operator D' on E' which agrees with D on the neighbourhood. This will allow us to reduce the proposition to the compact case already proved in Proposition 2.54.

Let $f\colon \mathbb{R} \to \mathbb{R}$ be a rapidly decreasing function and assume that the Fourier transform $\hat{f}$ is supported in some bounded interval $I \subseteq \mathbb{R}$. It follows from Lemma 2.55 that for every compact subset K of E there is a larger compact subset L of E such that if u is supported in K and if $s \in I$ then

$e^{isD}u$ is supported in L. If u is supported within K it now follows from finite propagation speed that

$$e^{isD}u = e^{isD'}u$$

for $s \in I$. To see this, observe that in this range

$$\frac{\partial}{\partial s}(e^{isD}e^{-isD'}u) = e^{isD}(D - D')e^{-isD'}u = 0.$$

Now by the Fourier inversion formula one has

$$f(D) = \frac{1}{2\pi}\int_I \hat{f}(s)e^{isD}\,ds,$$

where the integral converges in the strong topology on $\mathcal{B}(H)$. It follows that if u is supported in K then $f(D)u = f(D')u$. Because of this, we obtain a representation for $f(D')$ by a continuous family of kernels using Proposition 2.54, which provides a representation for $f(D')$ using a continuous family of kernels. $\square$

CHAPTER 3

C*-Algebras and K-theory

We assume that the reader has some prior acquaintance with K-theory, including the topological K-theory of compact spaces and K-theory for C^*-algebras, up to, but not necessarily including, the Bott periodicity theorem. But to fix notation we shall rapidly review some of the essential ideas in the first two sections of the chapter, before going on to present an alternate description of K-theory which is particularly well-suited to the study of index theory.

1. Review of K-Theory

If A is a unital ring then $K(A)$ (also called $K_0(A)$) is the abelian group with one generator for each isomorphism class of finitely generated projective modules over A, and with relations

$$[M_1] + [M_2] = [M_1 \oplus M_2]$$

for finitely generated projective modules M_1 and M_2. We can equivalently describe $K(A)$ as the group generated by equivalence classes $[p]$ of idempotents in matrix rings over A, subject to the relation

$$[p] + [q] = [p \oplus q]$$

(the notation $p \oplus q$ represents a block diagonal sum of matrices). Two idempotent matrices p and q are equivalent if there are matrices u and v (not necessarily square matrices) such that $p = uv$ and $q = vu$.

There is a standard device (adjoining a unit) by means of which the definition of $K(A)$ can be extended to non-unital rings. There are more elaborate ways of approaching the same problem, which show that if p and q are idempotent matrices of the same size over a ring B which contains A as an ideal, and if $p - q$ is a matrix with entries in A, then there is an associated class $[p \ominus q] \in K(A)$. The following exercise is designed to remind the reader of the details.

3.1. EXERCISE. If A is an ideal in B, construct the ring

$$C = \{(b_1, b_2) \in B \times B \,:\, b_1 - b_2 \in A\}.$$

Show (using the elementary properties of K-theory you know) that there is a short exact sequence

$$0 \longrightarrow K(A) \longrightarrow K(C) \longrightarrow K(B) \longrightarrow 0$$

in which the map $K(A) \to K(C)$ is induced from the ring homomorphism $a \mapsto (a, 0)$. The sequence is split by the homomorphism $b \mapsto (b, b)$ from B into C, and as a result, $K(A)$ can be viewed as a direct summand of $K(C)$. If p and q are idempotent matrices of the same size over B, and if $p - q$ is a matrix with entries in A, then the projection of the class $[(p, q)] \in K(C)$ into $K(A)$ defines a class in $[p \ominus q] \in K(A)$. Show that if p and q are actually matrices over A then $[p \ominus q] = [p] - [q]$.

3.2. REMARK. If A is an ideal in B, and if $\phi_0, \phi_1 \colon D \to B$ are ring homomorphisms which are equal, modulo A, then the pair (ϕ_0, ϕ_1) determines a ring homomorphism ϕ from D into the ring C of the previous exercise. By composing the induced map $\phi_* \colon K(D) \to K(C)$ with the projection from $K(C)$ to $K(A)$ we obtain a homomorphism $\phi_* \colon K(D) \to K(A)$.

The K-theory functor has a multiplicative property: if A_1 and A_2 are any two rings then there is a functorial pairing

$$K(A_1) \otimes K(A_2) \to K(A_1 \otimes A_2).$$

If we restrict from rings to algebras, say over $\mathbb{C}$, then we can take the tensor product over $\mathbb{C}$ (rather than over $\mathbb{Z}$, as is implicit here). The formula for the product is very simple in the unital case: if p_1 and p_2 are idempotent matrices over A_1 and A_2 respectively, then we can think of $p \otimes q$ as an idempotent matrix over $A_1 \otimes A_2$. The K-theory pairing maps $[p_1] \otimes [p_2] \to [p_1 \otimes p_2]$, and is characterized by this property.

If A is a commutative ring then the multiplication map $A \otimes A \to A$ is a ring homomorphism, and the composition of the K-theory product

$$K(A) \otimes K(A) \to K(A \otimes A)$$

with the K-theory map

$$K(A \otimes A) \to K(A)$$

induced from multiplication gives $K(A)$ the structure of a commutative ring.

More generally, if $A \to B$ is a homomorphism from a commutative ring into the center of a ring B then $K(B)$ is a module over the ring $K(A)$.

2. C*-Algebra K-Theory

If X is a locally compact space, then we shall write $K(X)$ for $K(C_0(X))$, where $C_0(X)$ is the ring of complex-valued, continuous functions on X

which vanish at infinity. If X is a *compact* space we can interpret $K(X)$ in terms of complex vector bundles over X, via the correspondence between vector bundles and projection-valued functions which was mentioned in Chapter 1. This *topological* K-*theory* functor has several properties which are not enjoyed by the more general algebraic K-theory functor of the previous section, but which are however shared by the restriction of algebraic K-theory to the category of C^*-algebras.

The special features of K-theory for C^*-algebras stem from the interaction between algebra and analysis in C^*-algebra theory, often ultimately boiling down to the statement that if x is small enough then $1+x$ is invertible. It follows from this, for instance, that K-theory for C^*-algebras is a *homotopy functor*: if two morphisms

$$\phi_0, \phi_1 \colon A \to B$$

of C^*-algebras are linked by a continuous path $\{\phi_t\}$ of morphisms, then

$$\phi_{0,*} = \phi_{1,*} \colon K(A) \to K(B).$$

Also, in the unital case $K(A)$ can be represented by *homotopy classes* of *projections* (self-adjoint idempotents) in matrix algebras over A.

3.3. EXERCISE. A C^*-algebra J is called *contractible* if the identity homomorphism $J \to J$ is homotopic to the zero homomorphism. Show that if J is a contractible ideal in a C^*-algebra A, then the induced map $K(A) \to K(A/J)$ is an isomorphism. (There is an obvious argument using the 'six term exact sequence' of K-theory, and therefore implicitly involving Bott periodicity. If you know a bit more, you can prove this with less.)

In the C^*-algebra theory a special role is played by the C^*-algebra $\mathcal{K} = \mathcal{K}(H)$ of compact operators on a Hilbert space H (we usually require H to be separable, although this detail will be of no concern to us in these notes). By considering operators which are zero on the orthogonal complements of finite-dimensional subspaces it is easy to see that $\mathcal{K}$ contains an increasing family of matrix algebras whose union is dense in $\mathcal{K}$. So, roughly speaking, $\mathcal{K}$ plays the role of an algebra of matrices of arbitrarily large but finite size.

In algebraic K-theory the map $M_n(A) \to M_{n+k}(A)$ which includes an $n \times n$ matrix as the upper left corner of an $(n+k) \times (n+k)$ matrix (whose entries are otherwise zero) induces an isomorphism $K(M_n(A)) \cong K(M_{n+k}(A))$. This fact can be strengthened in C^*-algebra K-theory, as follows:

3.4. LEMMA. *If* H_1 *and* H_2 *are Hilbert spaces, then the natural inclusion* $\mathcal{K}(H_1) \subseteq \mathcal{K}(H_1 \oplus H_2)$ K-*theory isomorphisms*

$$K(A \otimes \mathcal{K}(H_1)) \cong K(A \otimes \mathcal{K}(H_1 \oplus H_2)),$$

for every C^**-algebra* A. □

3.5. COROLLARY. *The inclusion of* $\mathbb{C}$ *as a one-dimensional subspace of any Hilbert space* H *induces an isomorphism* $K(A) \cong K(A \otimes \mathcal{K})$. □

3.6. REMARK. In C^*-algebra theory there are, in general, various different tensor products $A \otimes B$ (that is, different C^*-algebra completions of the algebraic tensor product of A and B over $\mathbb{C}$). In these notes it will not matter which C^*-algebraic completion of the tensor product over $\mathbb{C}$ we choose, since in fact in all the cases we shall consider, all the tensor products will agree. But for our purposes the most natural tensor product to consider would be the *maximal* product.

At one or two points we shall use a small extension of the lemma. Let $\mathcal{E}$ be a Hilbert module over a C^*-algebra A. Recall that this means that $\mathcal{E}$ is simultaneously:

(a) A Banach space
(b) A right A-module
(c) A space equipped with a sesquilinear, positive definite form

$$\langle\ ,\ \rangle \colon \mathcal{E} \times \mathcal{E} \to A.$$

These structures must be compatible with one another in the way that the various structures on a Hilbert space are compatible with one another (indeed, a Hilbert module over $A = \mathbb{C}$ is exactly the same thing as a Hilbert space).

3.7. DEFINITION. The C^*-algebra of A-*compact operators* on $\mathcal{E}$ is the C^*-algebra $\mathcal{K}_A(\mathcal{E})$ of A-linear operators on $\mathcal{E}$ generated by the "rank-one" operators $K_{v_1,v_2} \colon v \mapsto v_1\langle v_2, v\rangle$, where $v, v_1, v_2 \in \mathcal{E}$.

3.8. PROPOSITION. *If* $\mathcal{E}_1$ *and* $\mathcal{E}_2$ *are Hilbert* A*-modules, and if* $\langle \mathcal{E}_1, \mathcal{E}_1\rangle = A$ *then the natural inclusion*

$$\mathcal{K}_A(\mathcal{E}_1) \to \mathcal{K}_A(\mathcal{E}_1 \oplus \mathcal{E}_2)$$

induces an isomorphism in K*-theory.* □

3.9. EXAMPLE. If $\ell^2(\mathbb{N}, A)$ is the completion of the algebraic direct sum $\oplus_{j=1}^{\infty} A$ in the norm

$$\|\{a_j\}\|^2 = \|\sum_{j=1}^{\infty} a_j^* a_j\|$$

then $\ell^2(\mathbb{N}, A)$ is a Hilbert module and $\mathcal{K}_A(\ell^2(\mathbb{N}, A)) \cong A \otimes \mathcal{K}(\ell^2(\mathbb{N}))$. In this way Proposition 3.8 generalizes Lemma 3.4.

The feature of K-theory exhibited in Lemmas 3.4 and 3.8 is called *Morita invariance*.

3. Graded Algebras

3.10. DEFINITION. A *graded algebra* is an (associative, complex) algebra A equipped with an automorphism α whose square is the identity. An element $a \in A$ such that $\alpha(a) = a$ is called *even*, and one such that $\alpha(a) = -a$ is called *odd*. If the algebra A is a C^*-algebra then we require the automorphism α to be a C^*-algebra automorphism (that is, we require it to be compatible with the $*$-operation). Elements in a graded algebra which are either even or odd are called *homogeneous*.

The product of two even elements, or two odd elements, is even; the product of an even and an odd element is odd; a general element of A can be written uniquely as the sum of an even part and an odd part.

3.11. REMARK. We might more properly say that A is $\mathbb{Z}/2$*-graded*, since the grading is in effect a decomposition of A into a direct sum of even and odd subspaces, or in other words a decomposition into a direct sum whose summands are labeled by elements of the group $\mathbb{Z}/2$, in such a way that multiplication in the algebra is compatible with addition in the group $\mathbb{Z}/2$. It is possible to consider more elaborate gradings, in which $\mathbb{Z}/2$ is replaced by another abelian group. But we shall not need to consider these.

3.12. EXAMPLE. The definition is designed to be compatible with our previous discussion of graded Hilbert spaces. Let $H = H_0 \oplus H_1$ be a graded Hilbert space, or more generally a graded Hilbert module . Then the C^*-algebra of compact operators on H is a graded C^*-algebra. The even elements are those which preserve the grading of H; the odd elements are those that reverse it. In other words the even elements preserve each of the subspaces H_0 and H_1, whereas the odd elements exchange the subspaces H_0 and H_1. In particular the algebra $M_2(A)$ can be graded by declaring that the diagonal matrices are even and the off-diagonal matrices are odd.

The following definition provides a second very important example of a graded C^*-algebra.

3.13. DEFINITION. We denote by $\mathcal{S}$ the algebra $C_0(\mathbb{R})$ with the grading automorphism $\alpha(f(x)) = f(-x)$. Thus the even and odd elements of $\mathcal{S}$ are those functions which are even and odd in the usual sense of elementary calculus.

We shall encounter one further source of examples later one, in Chapter 6. But in truth our recourse to graded algebras will be rather minor, and is done for notational convenience more than anything else. For our purposes, the great advantage of using graded algebras is that they allow for a considerable simplification of K-theory formulas involving *differences* (for example, differences of projections). Since the index can be regarded as just such a difference (of two integers, or of two kernel projections in $\mathcal{K}$), this feature of graded algebras is very useful in index theory.

3.14. DEFINITION. A *graded homomorphism* between graded algebras is a homomorphism which sends even elements to even, and odd elements to odd. In the C^*-algebra case we shall obviously also require that the map be compatible with the $*$-operation.

There are some quite surprising differences between the categories of graded and ungraded C^*-algebras. For example the map $\varepsilon\colon \mathcal{S} \to \mathbb{C}$ defined by $f \mapsto f(0)$ is a graded $*$-homomorphism which, when one forgets the grading, is in a simple way homotopic to the zero $*$-homomorphism. But as a graded $*$-homomorphism it is not at all trivial at the level of homotopy.

3.15. EXERCISE. Show that the map $\mathcal{S} \to M_2(\mathbb{C})$ defined by the formula

$$f \mapsto \begin{pmatrix} f(0) & 0 \\ 0 & f(0) \end{pmatrix}$$

is a graded $*$-homomorphism which is homotopic (through graded homomorphisms) to the 0-homomorphism, whereas the $*$-homomorphism

$$f \mapsto \begin{pmatrix} f(0) & 0 \\ 0 & 0 \end{pmatrix}$$

is *not* null-homotopic.

The "cancellation" phenomenon which we see in the previous exercise suggests the following construction. Suppose that A is a unital (ungraded) C^*-algebra. Let p, q be projections in A, whose formal difference defines a K-theory class $[q] - [p] \in K_0(A)$. Define a graded $*$-homomorphism

$$\phi_{p,q}\colon \mathcal{S} \to M_2(A),$$

where $\mathcal{S}$ and $M_2(A)$ are graded as above, by

$$\phi_{p,q}(f) = \begin{pmatrix} pf(0) & 0 \\ 0 & qf(0) \end{pmatrix}.$$

Clearly, homotopic projections give rise to homotopic homomorphisms; and (by Exercise 3.15) if $p = q$, then $\phi_{p,q}$ is homotopic to the 0-homomorphism. Replacing A by $A \otimes \mathcal{K}$ allows us to handle projections in matrix algebras over A, and we obtain a canonical map

$$\Phi\colon K(A) \to [\mathcal{S}, A \otimes \mathcal{K}],$$

where the $[\ldots]$ notation denotes homotopy classes of graded $*$-homomorphisms, and where $\mathcal{K}$ is the C^*-algebra of operators on a graded Hilbert space $H = H_0 \oplus H_1$, whose homogeneous subspaces are both infinite-dimensional.

3.16. EXERCISE. The collection of homotopy classes $[\mathcal{S}, A \otimes \mathcal{K}]$ can be given a group structure: addition is by direct sum followed by the map

$$\mathcal{K}(H) \oplus \mathcal{K}(H) \to \mathcal{K}(H \oplus H) \cong \mathcal{K}(H)$$

associated to a graded unitary isomorphism $H \oplus H \cong H$. The inverse of a $*$-homomorphism f is obtained by reflecting the domain ($x \mapsto -x$) and reversing the grading on the range. With this group structure, the map Φ above is a homomorphism of groups.

3.17. PROPOSITION. *For any (ungraded)* C^**-algebra* A*, the map*

$$\Phi\colon K(A) \to [\mathcal{S}, A \otimes \mathcal{K}]$$

defined above is an isomorphism.

PROOF. Using the Cayley transform

$$x \mapsto \frac{x+i}{x-i}$$

let us identify $\mathcal{S} = C_0(\mathbb{R})$ with the algebra of continuous functions on the unit circle $\mathbb{T}$ which vanish at $1 \in \mathbb{T}$. In this way a homomorphism from $\mathcal{S}$ into $A \otimes \mathcal{K}$ corresponds to a unital homomorphism from the algebra $C(\mathbb{T})$ into the algebra obtained by adjoining a unit to $A \otimes \mathcal{K}$. By spectral theory, such a homomorphism corresponds to a unitary u in the unitalization of $A \otimes \mathcal{K}$. If we begin with a graded homomorphism then the unitary u we obtain has the property that $\alpha(u) = u^*$, where α is the grading automorphism. Now, for the purposes of this proof, let us use the term *skew-unitary* for any unitary in a graded C^*-algebra B which has this property. If the grading is *internal*, which is to say that $\alpha(x) = \varepsilon x \varepsilon$ for some self-adjoint unitary $\varepsilon \in B$, then there is a bijective correspondence between skew-unitaries in B and projections in B, given by $u \mapsto \frac{1}{2}(1 + u\varepsilon)$. In view of all this, if B is any unital, graded C^*-algebra containing both $A \otimes \mathcal{K}$ and the grading operator $\varepsilon = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, then we obtain from a graded $*$-homomorphism $\phi\colon \mathcal{S} \to A \otimes \mathcal{K}$ a projection $p_\phi \in B$ which is equal to the projection $p_\varepsilon = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ modulo $A \otimes \mathcal{K}$. The correspondence

$$[\phi] \mapsto [p_\phi \ominus p_\varepsilon]$$

defines a map from $[\mathcal{S}, A \otimes \mathcal{K}]$ into $K(A) \cong K(A \otimes \mathcal{K})$ which is inverse to Φ. ☐

3.18. EXERCISE. Provide the remaining details in the proof of the proposition.

3.19. EXERCISE. If you are familiar with Kasparov's KK-theory, show that the inverse map $[\mathcal{S}, A \otimes \mathcal{K}]$ can also be constructed in the following way. Identify $A \otimes \mathcal{K}$ with the compact operators on the universal graded Hilbert

A-module $\mathcal{E} = \ell^2(\mathbb{N}, A)$ (see Example 3.9). Given a $*$-homomorphism $\phi\colon \mathcal{S} \to A \otimes \mathcal{K}$, let $\mathcal{E}_\phi$ be the Hilbert submodule $\overline{\phi[\mathcal{S}]\mathcal{E}}$. Then ϕ extends to a homomorphism from the bounded, continuous functions on $(-\infty, \infty)$ to the bounded operators on $\mathcal{E}_\phi$. Let $F \in \mathcal{B}(\mathcal{E}_\phi)$ be the operator corresponding to the odd function $x \mapsto x(1+x^2)^{-1/2}$. Verify that F describes a Kasparov cycle for $KK(\mathbb{C}, A) = K_0(A)$.

4. K-Theory and Index Theory

Having quickly reviewed some topics in K-theory, and introduced a new picture of K-theory using the graded C^*-algebra $\mathcal{S}$, let us show how to associate two fundamental K-theory classes to an elliptic operator D.

Let D be a symmetric, odd-graded, elliptic operator on a graded vector bundle S over a compact manifold. Thus we can write

$$D = \begin{pmatrix} 0 & D_- \\ D_+ & 0 \end{pmatrix}$$

where $D_- = D_+^*$. To keep within the framework developed in the previous chapter, let us assume that D has order 1, although nothing we do in this chapter will depend on this additional assumption. The space of L^2 sections of S is a graded Hilbert space H. According to the elliptic package developed in Chapter 2 (Proposition 2.45), the functional calculus homomorphism $\phi_D\colon f \mapsto f(D)$ gives a graded $*$-homomorphism $\phi\colon \mathcal{S} \to \mathcal{K}(H)$, which according to our discussion above defines an element of $K(\mathbb{C})$.

3.20. PROPOSITION. *The element* $[\phi_D] \in K(\mathbb{C}) \cong \mathbb{Z}$ *is the Fredholm index of the operator* D.

3.21. REMARK. Recall that by the index of D we mean the integer $\dim(\text{Kernel}(D) \cap H_+) - \dim(\text{Kernel}(D) \cap H_-)$.

PROOF. One can use the homotopy of $*$-homomorphisms $\phi_{s^{-1}D}(f) = f(s^{-1}D)$. At $s = 1$ we have ϕ_D and at $s = 0$ we have the homomorphism $f \mapsto f(0)P$, where P is the projection onto the kernel of D. This corresponds to the integer $\dim(\text{Kernel}(D) \cap H_+) - \dim(\text{Kernel}(D) \cap H_-)$, which is the index of D in our sense. □

We see that by means of the algebra $\mathcal{S}$ we can encode the integer Fredholm index in K-theory. The great advantage of using $\mathcal{S}$ for this purpose is that the following very similar construction also places the symbol of an elliptic operator into K-theory.

3.22. DEFINITION. Let Z be a locally compact space and let S be a graded Hermitian vector bundle over Z. Let $c\colon S \to S$ be a self-adjoint, odd endomorphism of S. Then for each $z \in Z$, the fiber operator $c(z)\colon S_z \to S_z$

is a positive operator. We say that $c\colon S \to S$ is *elliptic* if the norm of the operator $(I + c(z)^2)^{-1}\colon S_z \to S_z$ tends to zero, as $z \in Z$ tends to infinity.

3.23. REMARK. In particular, an c must be invertible outside a compact set.

3.24. EXAMPLE. Recall that the ellipticity of the operator D can be expressed in terms of the *symbol* $\sigma_D\colon \pi^*S \to \pi^*S$, as in Remark **??**. The symbol of an elliptic operator is an elliptic endomorphism of the bundle $\pi^*(S)$ over T^*M. This both explains our terminology and provides the most fundamental instances of Definition 3.22.

Given a elliptic endomorphism $c\colon S \to S$ over a locally compact space Z, the map $f \mapsto f(c)$ is a graded $*$-homomorphism

$$\phi_c\colon \mathcal{S} \to C_0(Z; \mathrm{End}(S)),$$

where the target algebra is the algebra of continuous sections, vanishing at infinity, of the endomorphism bundle $\mathrm{End}(S)$. But this target algebra is precisely the algebra of $C_0(Z)$ compact operators on the Hilbert $C_0(Z)$-module $C_0(Z, S)$ of continuous sections of S which vanish at infinity. So by Morita invariance we obtain a K-theory class $c \in K(Z)$.

3.25. DEFINITION. We shall call $c \in K(Z)$ the *difference class* of the elliptic endomorphism $c\colon S \to S$.

3.26. REMARK. The idea is that $s \in K(Z)$ represents the difference "$S_0 - S_1$" where the endomorphism c is used to given meaning to this difference. This is similar to the way that the index element in $K(\mathbb{C})$ associated to an elliptic operator represents "$H_0 - H_1$," where D is used to give meaning to the difference (by cancelling out all but $\mathrm{Kernel}(D)$ in H_0 and H_1).

3.27. DEFINITION. Let D be an elliptic operator acting on sections of a graded Hermitian vector bundle S. Assume that D is odd-graded and symmetric. The *symbol class* of D is the difference class $\sigma_D \in K(T^*M)$ associated to the symbol $\sigma_D\colon \pi^*S \to \pi^*S$.

5. A Functorial Property

Proposition 3.20 and Definition 3.27 provide two different K-theory classes associated to the same elliptic operator D. The proof of the Atiyah-Singer Index Theorem hinges on devising a procedure to relate them. To conclude this chapter we shall take the first steps towards doing this by enlarging the class of morphisms between K-groups beyond those simply induced by $*$-homomorphisms.

We saw above how a $*$-homomorphism $\phi\colon \mathcal{S} \to A \otimes \mathcal{K}$ gives rise to an element of $K(A)$. In this section we are going to generalize this statement as follows.

3.28. PROPOSITION. *A graded $*$-homomorphism $\phi\colon \mathcal{S} \otimes A \to B \otimes \mathcal{K}$ determines a K-theory map $\phi_*\colon K(A) \to K(B)$, with the following properties:*

(i) *The correspondence $\phi \mapsto \phi_*$ is functorial with respect to composition with $*$-homomorphisms $A_1 \to A$ and $B \to B_1$.*
(ii) *The map ϕ_* depends only on the homotopy class of ϕ.*
(iii) *If $\phi\colon \mathcal{S} \otimes A \to A \otimes \mathcal{K}(H)$ is of the form*

$$\phi(f \otimes a) = f(0)a \otimes p \in A \otimes \mathcal{K}(H)$$

where p is a rank-one projection operator whose range is a even-graded subspace of H, then $\phi_*\colon K(A) \to K(A)$ is the identity.

PROOF. Let C be the image of $\mathcal{S} \otimes A$ under ϕ; it is a C^*-subalgebra of $B \otimes \mathcal{K}$. From ϕ we obtain homomorphisms $\phi_{\mathcal{S}}$ and ϕ_A of $\mathcal{S}$ and A separately into the multiplier algebra of C. From $\phi_{\mathcal{S}}$ we obtain two projections p_0 and p_1 in the multiplier algebra of C, using exactly the same formulas we used earlier in our proof of Proposition 3.17. Thus if we view $\phi_{\mathcal{S}}$ as a homomorphism $f \mapsto f(D)$, then we can form the Cayley transform $U_D = (D + i)(D - i)^{-1}$ in the multiplier algebra of C, and if ε is the grading operator then the projections p_0 and p_1 are the positive projections of the involutions $U\varepsilon$ and ε. Now these projections *commute* with the image of the $*$-homomorphism ϕ_A. So the maps

$$a \mapsto p_0\phi_A(a) \quad \text{and} \quad a \mapsto p_1\phi_A(a)$$

are both $*$-homomorphisms. The difference of these two homomorphisms maps A into C, so as we noted earlier in Remark 3.2, the two homomorphisms determine a homomorphism from $K(A)$ to $K(C)$. If we follow with the inclusion of C into $B \otimes \mathcal{K}$, and then use the isomorphism $K(B) \cong K(B \otimes \mathcal{K})$, we obtain a map

$$\phi_*\colon K(A) \to K(B).$$

The verification of the properties listed is left to the reader. □

6. A More General Functorial Property

The construction of the previous section permits us to build many interesting homomorphisms between K-theory groups. We will, however, need to make use of a still more general kind of functoriality. This arises on replacing the notion of *morphism* of graded C^*-algebras by that of *asymptotic morphism*.

3.29. DEFINITION. Let A and B be C^*-algebras. An *asymptotic morphism* from A to B is a family of functions $\phi_t\colon A \to B$, where $t \in [1, \infty)$, such that

(i) For each $a \in A$ the map $t \mapsto \phi_t(a) \in B$ is continuous and bounded.
(ii) For all $a, a_1, a_2 \in A$ and $\lambda_1, \lambda_2 \in \mathbb{C}$,

$$\lim_{t\to\infty} \left\{ \begin{array}{c} \phi_t(a_1a_2) - \phi_t(a_1)\phi_t(a_2) \\ \phi_t(\lambda_1 a_1 + \lambda_2 a_2) - \lambda_1\phi_t(a_1) + \lambda_2\phi_t(a_2) \\ \phi_t(a^*) - \phi_t(a)^* \end{array} \right\} = 0.$$

3.30. DEFINITION. An *homotopy* of asymptotic morphisms from A to B is an asymptotic morphism from A to $C([0, 1]; B)$.

Ordinary $*$-homomorphisms of course give rise asymptotic morphisms (if ϕ is a homomorphism we can set $\phi_t = \phi$, for all t), and ordinary homotopies of $*$-homomorphisms give rise homotopies of asymptotic morphisms. Less trivial examples will have to wait until Chapter 7; in fact, the key construction in the index theorem is a certain asymptotic morphism.

3.31. DEFINITION. We will denote the collection of asymptotic homotopy classes of asymptotic morphisms from A to B by $[\![A, B]\!]$.

3.32. PROPOSITION. *An asymptotic morphism $\phi_t\colon A \to B$ determines a K-theory map $\phi_*\colon K(A) \to K(B)$, with the following properties:*

(i) *The correspondence $\phi \mapsto \phi_*$ is functorial with respect to composition with $*$-homomorphisms $A_1 \to A$ and $B \to B_1$.*
(ii) *The map ϕ_* depends only on the homotopy class of ϕ.*
(iii) *If each ϕ_t is actually a $*$-homomorphism, then $\phi_*\colon K(A) \to K(B)$ is the map induced by ϕ_1.*

PROOF. Denote by $\mathcal{C}(B)$ the algebra of bounded, continuous functions from $[1, \infty)$ into B and denote by $\mathcal{J}(B)$ the ideal of functions which vanish at infinity. Denote by $\mathcal{Q}(B)$ the quotient C^*-algebra. There is a short exact sequence

$$0 \longrightarrow \mathcal{J}(B) \longrightarrow \mathcal{A}(B) \longrightarrow \mathcal{Q}(B) \longrightarrow 0.$$

Since the ideal $\mathcal{J}(B)$ is contractible, the quotient map in the short exact sequence induces an isomorphism in K-theory. An asymptotic morphism from $\mathcal{S} \otimes A$ into B induces a $*$-homomorphism

$$\widetilde{\phi}\colon A \to \mathcal{Q}(B).$$

The required map $\phi\colon K(A) \to K(B)$ is then the one which fits into the following diagram:

$$\begin{array}{ccc} K(A) & \xrightarrow{\phi_*} & K(B) \\ {\scriptstyle \widetilde{\phi}_*}\big\downarrow & & \big\uparrow \\ K(\mathcal{Q}(B)) & \xleftarrow[\cong]{} & K(\mathcal{A}(B)). \end{array}$$

The right upwards arrow is induced from evaluation at $1 \in [1, \infty)$. □

This construction can be combined with Proposition 3.28. Let us say that an asymptotic morphism $\phi_t\colon A \to B$ between graded C^*-algebras is graded if the maps ϕ_t commute asymptotically with the grading operators on A and B.

3.33. PROPOSITION. *A graded asymptotic morphism* $\phi_t\colon \mathcal{S} \otimes A \to B \otimes \mathcal{K}$ *determines a* K*-theory map* $\phi_*\colon K(A) \to K(B)$*, with the following properties:*

(i) *The correspondence* $\phi \mapsto \phi_*$ *is functorial with respect to composition with* $*$*-homomorphisms* $A_1 \to A$ *and* $B \to B_1$.
(ii) *The map* ϕ_* *depends only on the homotopy class of* ϕ.
(iii) *If each* ϕ_t *is actually a* $*$*-homomorphism, then* $\phi_*\colon K(A) \to K(B)$ *is the map induced by* ϕ_1*, as in the proof of Proposition 3.28.* □

7. A Remark on Graded Tensor Products

In this optional section we shall show how the constructions of the previous two sections can be made a bit more cleanly by making a further investment in the technology of graded algebras.

To begin with, let us discuss the *graded tensor product* of graded algebras[1]. This is the algebraic tensor product over $\mathbb{C}$, as a vector space, but we introduce a "twist" into the multiplication in the following way. Let A and B be graded algebras and let $a_1 \widehat{\otimes} b_1$ and $a_2 \widehat{\otimes} b_2$ be elementary tensors in their graded tensor product. Assume that the a's and b's are homogeneous (either even or odd) and use the symbol ∂ to denote degrees (0 for even, 1 for odd). Then we decree that

$$(a_1 \widehat{\otimes} b_1) \cdot (a_2 \widehat{\otimes} b_2) = (-1)^{\partial(b_1)\partial(a_2)} (a_1 a_2) \widehat{\otimes} (b_1 b_2).$$

This extends by linearity to define the multiplication on $A \widehat{\otimes} B$.

[1] As with the usual tensor product, when we pass to C^*-algebras we shall take the C^*-completion of the algebraic graded tensor product. There are in general various possible choices for the product C^*-norm. However, it is fortunately the case that in the situations we consider the product norm is unique, so we will not need to worry about this complication.

3.34. EXERCISE. Show that if A and B are graded $*$-algebras then the formula $(a\widehat{\otimes}b)^* = (-1)^{\partial a \partial b} a^*\widehat{\otimes}b^*$ makes $A\widehat{\otimes}B$ into a graded $*$-algebra.

3.35. EXERCISE. Let H and H' be graded Hilbert spaces. Check that the algebra $\mathcal{K}(H)\widehat{\otimes}\mathcal{K}(H')$ is isomorphic to $\mathcal{K}(H \otimes H')$, where $H \otimes H'$ is graded by the product of the grading operators, $\varepsilon \otimes \varepsilon'$. Moreover, any two such isomorphisms are homotopic.

3.36. EXERCISE. Show that the algebra $\mathcal{S}\widehat{\otimes}\mathcal{S}$ is isomorphic to the algebra of matrix-valued functions on the quarter-plane, $f\colon (\mathbb{R}^+)^2 \to M_2(\mathbb{C})$, having the properties that for each x the value $f(x,0)$ belongs to the 2-dimensional subalgebra of matrices of the form $\left(\begin{smallmatrix} a & b \\ b & a\end{smallmatrix}\right)$, and for each y the value $f(0,y)$ belongs to the 2-dimensional subalgebra of matrices of the form $\left(\begin{smallmatrix} a & b \\ -b & a\end{smallmatrix}\right)$. (Hint: First give a similar description of $\mathcal{S}$ itself as functions on the half-line with values in a certain graded algebra.)

Notice that in the previous exercise we have described $\mathcal{S}\widehat{\otimes}\mathcal{S}$ as a subalgebra of the endomorphisms of a certain graded vector bundle (namely $\mathbb{C} \oplus \mathbb{C}$) over $(\mathbb{R}^+)^2$. The endomorphism

$$Z = \begin{pmatrix} 0 & x+iy \\ x-iy & 0 \end{pmatrix},$$

is odd, self-adjoint and elliptic. The difference construction described in Definition 3.25 gives us a "comultiplication" map

$$\Delta\colon \mathcal{S} \to \mathcal{S}\widehat{\otimes}\mathcal{S}$$

defined by $f \mapsto f(Z)$.

The map Δ can be described very concisely using the notion of "unbounded multiplier," or in other words using unbounded operators on C^*-algebras, considered as Hilbert modules over themselves (see for example Lance's book [] on Hilbert modules). There is a natural unbounded, self-adjoint multiplier X on $\mathcal{S}$, namely the function $x \mapsto x$. If we apply the functional calculus to X and $f \in \mathcal{S}$ we obtain the element $f(X)$ in $\mathcal{S}$ which is, of course, just f itself. The operator $X\widehat{\otimes}1 + 1\widehat{\otimes}X$ is an unbounded multiplier of $\mathcal{S}\widehat{\otimes}\mathcal{S}$, and the map Δ is given by the attractive formula

$$\Delta\colon f(X) \mapsto f(X\widehat{\otimes}1 + 1\widehat{\otimes}X).$$

3.37. PROPOSITION. *The diagrams*

$$\begin{array}{ccc} \mathcal{S} & \xrightarrow{\ \Delta\ } & \mathcal{S}\widehat{\otimes}\mathcal{S} \\ {\scriptstyle\Delta}\downarrow & & \downarrow{\scriptstyle 1\widehat{\otimes}\Delta} \\ \mathcal{S}\widehat{\otimes}\mathcal{S} & \xrightarrow[\ \Delta\widehat{\otimes}1\ ]{} & \mathcal{S}\widehat{\otimes}\mathcal{S}\widehat{\otimes}\mathcal{S} \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathcal{S}\widehat{\otimes}\mathcal{S} & \xrightarrow{\ 1\widehat{\otimes}\varepsilon\ } & \mathcal{S} \\ {\scriptstyle\varepsilon\widehat{\otimes}1}\downarrow & \overset{\Delta}{\nwarrow} & \uparrow{\scriptstyle =} \\ \mathcal{S} & \xleftarrow[\ =\ ]{} & \mathcal{S} \end{array}$$

commute. □

Thus $\mathcal{S}$ is a sort of "graded C^*-bialgebra." Because of this we can form a category whose objects are graded C^*-algebras and for which the morphisms from A to B are the graded $*$-homomorphisms $\phi\colon \mathcal{S}\widehat{\otimes}A \to B$. The composition of two morphisms ϕ and ψ is given by the prescription

$$\mathcal{S}\widehat{\otimes}A \xrightarrow{\Delta\widehat{\otimes}1} \mathcal{S}\widehat{\otimes}\mathcal{S}\widehat{\otimes}A \xrightarrow{1\widehat{\otimes}\phi} \mathcal{S}\widehat{\otimes}B \xrightarrow{\psi} C.$$

The identity morphism is the $*$-homomorphism $\varepsilon\widehat{\otimes}1\colon \mathcal{S}\widehat{\otimes}A \to A$. The category of graded C^*-algebras is included within this "$\mathcal{S}$-category" by the map

$$\left[\, A \xrightarrow{\phi} B \,\right] \mapsto \left[\, \mathcal{S}\widehat{\otimes}A \xrightarrow{\varepsilon\widehat{\otimes}\phi} B \,\right].$$

If A is a graded C^*-algebra, let us define $K(A)$ to be the set of homotopy classes of graded $*$-homorphisms from $\mathcal{S}$ to $A\widehat{\otimes}\mathcal{K}$. If the grading on A is trivial, this agrees with our previous definition. In general, $K(A)$ is just the set of homotopy classes of morphisms from $\mathbb{C}$ to $A\widehat{\otimes}\mathcal{K}$ in the $\mathcal{S}$-category.

3.38. PROPOSITION. *Let* A *and* B *be* C^**-algebras, possibly graded. A graded* $*$*-homomorphism from* $\mathcal{S}\widehat{\otimes}A \to B\widehat{\otimes}\mathcal{K}$ *gives rise to an induced homomorphism* $K(A) \to K(B)$ *of* K*-theory groups.*

PROOF. The induced homomorphism is just composition in the $\mathcal{S}$-category. □

The case of asymptotic morphisms is taken care of by the following simple result.

3.39. PROPOSITION. *Let* D *be any graded* C^**-algebra. The forgetful map*

$$[\mathcal{S}, D] \to [\![\mathcal{S}, D]\!]$$

is an isomorphism.

3.40. EXERCISE. Prove this.

3.41. COROLLARY. *Let* A *be a* C^**-algebra. One has an identification* $K(A) = [\![\mathcal{S}, A\widehat{\otimes}\mathcal{K}]\!]$. □

3.42. PROPOSITION. *Let* A *and* B *be* C^**-algebras, possibly graded. An asymptotic morphism from* $\mathcal{S}\widehat{\otimes}A \to B\widehat{\otimes}\mathcal{K}$ *gives rise to an induced homomorphism* $K(A) \to K(B)$ *of* K*-theory groups.*

PROOF. Composition in the $\mathcal{S}$-category gives a map

$$[\mathcal{S}, A\widehat{\otimes}\mathcal{K}] \to [\![\mathcal{S}, A\widehat{\otimes}\mathcal{K}]\!].$$

□

Notice that in the above proof we did not attempt the (rather technical) feat of composing two asymptotic morphisms (for which the reader can refer to [], for example). We only composed an asymptotic morphism with a regular one, and this is easy.

CHAPTER 4

Characteristic Classes

In this chapter we are going to study in detail the *characteristic classes* of vector bundles. We mentioned these briefly in Chapter 1.

4.1. DEFINITION. A *characteristic class* for vector bundles (of a certain kind, for instance complex k-dimensional vector bundles) is a natural map c which assigns, to each vector bundle V of that kind over a base M, a cohomology class $c(V) \in H^*(M)$. Here *natural* means that c commutes with pull-backs: if $f\colon M' \to M$ is a map and V is a vector bundle over M, then $f^*(c(V)) = c(f^*(V))$.

For the purposes of these notes, there will be no loss of generality if we think of M as a compact manifold, and the cohomology as de Rham cohomology. But sometimes it is very important to understand that certain characteristic classes are *integral*, that is, they are elements of the integral cohomology groups $H^*(M;\mathbb{Z})$. We will touch on this at the end of the chapter.

There is an extensive theory of characteristic classes; the canonical reference is the book by Milnor and Stasheff. Our motivation in studying the theory is the following. As we have seen, an elliptic operator D on a compact manifold M gives rise to a *symbol class* $[\sigma_D] \in K^0(T^*M)$, and the problem to which the Index Theorem gives an answer is that of computing, in some explicit way, how the index of D depends on its symbol class. Since K-theory is constructed out of vector bundles, characteristic classes will give rise to maps from K-theory to cohomology. Moreover the resulting cohomology classes are explicitly computable (we will not have time to give many examples of this, but again we refer to Milnor and Stasheff's book for techniques of calculation with characteristic classes). We are therefore going to regard the index problem as solved if we can find an e explicit formula for the index of D in terms of characteristic classes of σ_D.

1. Classifying Spaces and Cohomology

Recall from Chapter 1 that the *Grassmannian* $G_k(\mathbb{C}^n)$ is the space of k-dimensional subspaces of V. It is a compact manifold. There is a *canonical bundle* of k-dimensional vector spaces over $G_k(\mathbb{C}^n)$, and if M is any compact manifold then for n sufficiently large, the isomorphism classes of complex vector bundles on M of rank k correspond to the homotopy classes of maps from M to $G_k(\mathbb{C}^n)$ via the operation which assigns to any map the pullback of the canonical bundle.

In order to obviate the need to continually make n "sufficiently large" it is convenient to speak of the space

$$G_k(\mathbb{C}^\infty) = \lim_{n\to\infty} G_k(\mathbb{C}^n).$$

This is a legitimate topological space in its own right (when given the direct limit topology). But for our purposes we can think of a map from a compact manifold into $G_k(\mathbb{C}^\infty)$ as a compatible family of maps into the $G_k(\mathbb{C}^n)$, for all large enough n, while by a cohomology class on $G_k(\mathbb{C}^\infty)$ we shall mean a family of cohomology classes, one on each $G_k(\mathbb{C}^n)$ which are compatible with one another under the maps in cohomology induced from the inclusions $G_k(\mathbb{C}^m) \subseteq G_k(\mathbb{C}^n)$. Notice that the canonical bundles on the $G_k(\mathbb{C}^n)$, for different n, are compatible with one another under these inclusion maps. With these conventions, we shall speak of the cohomology ring of $G_k(\mathbb{C}^\infty)$, the canonical bundle over $G_k(\mathbb{C}^\infty)$, and so on.

4.2. PROPOSITION. *Let k be a positive integer.*

(i) *There is a bijection between isomorphism classes of complex, rank k vector bundles over a compact manifold M and homotopy classes of maps from M to $G_k(\mathbb{C}^\infty)$.*

(ii) *There is a bijection between characteristic classes of rank k complex vector bundles on compact manifolds and classes in the cohomology ring*

$$H^*(G_k(\mathbb{C}^\infty)) = \prod_p H^p(G_k(\mathbb{C}^\infty)).$$

The first bijection associates to a map f the pullback f^*E of the canonical bundle. The second bijection associates to a map f the pullback of the class in $H^*(G_k(\mathbb{C}^\infty))$ along the map. □

4.3. DEFINITION. The map f is called a *classifying map* for the (isomorphism class of) bundle f^*V.

2. Characteristic Classes for Complex Line Bundles

The space $G_1(\mathbb{C}^n)$ is none other than the projective space $\mathbb{C}P^{n-1}$ of lines in $\mathbb{C}^n$. So in order to determine the characteristic classes of one-dimensional complex vector bundles — in other words complex line bundles — we need to compute the cohomology rings of complex projective space.

To do so, recall from Chapter 1 that associated to any oriented, rank-d real vector bundle V over a compact manifold M there is a *Thom class* u_V in the compactly supported de Rham cohomology group $H^d(V)$. In our case, the real bundle E underlying the canonical line bundle on $\mathbb{C}P^{n-1}$ is 2-dimensional and oriented as follows: if e is any non-zero local section then we deem the pair e, ie to be an oriented local frame of the underlying real bundle (the orientation so-defined does not depend on e).

4.4. DEFINITION. If V is an oriented, rank d, real vector bundle over a compact manifold M, the *Euler class* of E is the image $e_V \in H^d(M)$ of the Thom class $u_V \in H^d(V)$ under the map induced from including M into V as the zero section.

4.5. REMARK. This is a characteristic class of real, oriented vector bundles. The name is derived from the following beautiful theorem (which we shall not need, except to compute examples): if e_{TM} is the Euler class of an oriented, closed manifold, then $\int_M e_{TM}$ is equal to the Euler characteristic of M.

4.6. PROPOSITION. *The cohomology ring* $H^*(\mathbb{C}P^{n-1})$ *is the unital algebra freely generated by the Euler class* $e_E \in H^2(\mathbb{C}P^{n-1})$, *subject to the relation* $e_v^n = 0$.

To prove this we shall use the following important result, which will also figure in later computations.

4.7. THEOREM (Thom Isomorphism Theorem). *If V is an oriented, rank d, real vector bundle over a compact manifold M, then $H^*(V)$ is freely generated as a module over $H^*(M)$ by the Thom class.*

PROOF (SKETCH). If M is a point then the result follows from the characteristic property of the Thom class, that its restriction to each fiber of V generates the cohomology of the fiber. If M is a contractible open manifold then the result follows from the homotopy invariance on cohomology.[1] The general result follows by choosing a suitable open cover by contractible sets, and applying a Mayer-Vietoris argument. □

PROOF OF PROPOSITION 4.6. Associated to any vector bundle over a compact M there is a long exact cohomology sequence

$$\cdots \longrightarrow H^p(V) \longrightarrow H^p(\overline{V}) \longrightarrow H^p(S(V)) \longrightarrow H_{p+1}(V) \longrightarrow \cdots$$

where $\overline{V}$ is the compactification of V obtained by adding a sphere at infinity to each fiber, and $S(V)$ is the bundle of spheres. (These spaces are smooth

[1] When dealing with non-compact manifolds we should use de Rham cohomology with compact supports only in the fiber direction of V. See [].

manifolds in a natural way: if we put an inner product on V then $\overline{V}$ is diffeomorphic to the closed unit ball bundle and $S(V)$ is diffeomorphic to the unit sphere bundle.) If V is oriented and if we incorporate the Thom isomorphism, we obtain the *Gysin* long exact sequence

$$\cdots \longrightarrow H^{p-d}(M) \xrightarrow{e_V} H^p(M) \longrightarrow H^p(S(V)) \longrightarrow H^{p-d+1}(M) \longrightarrow \cdots$$

in which the map labeled e_V is multiplication by the Euler class. When M is the complex projective space $\mathbb{C}P^n$ and V is the canonical line bundle, the space $S(V)$ may be identified with the unit sphere in $\mathbb{C}^n$. Knowing the cohomology of the unit sphere, it is now easy to deduce the result. □

Hence:

4.8. THEOREM. *The ring* $H^*(G_1(\mathbb{C}^\infty))$ *is isomorphic to ring of formal power series over the Euler class* $x = e_E$ *of the canonical line bundle. As a result, the characteristic classes of complex line bundles are in one-to-one correspondence with formal power series.* □

To put it another way, the only characteristic classes of line bundles are the Euler class of the underlying real, oriented plane bundle, and the other classes obtained from it by simple algebraic operations (squaring, cubing, etc, and linear combinations of these).

While this may seem disappointingly simple,[2] there are nonetheless some interesting questions to be answered. For instance, the set of isomorphism classes of complex line bundles over M has the structure of an abelian group. The group operation is tensor product of line bundles, and the inverse of L is the class of the conjugate line bundle $\overline{L}$ (this is the same real plane bundle, but with the complex conjugate complex structure $i \cdot_{\text{conjugate}} v = -iv$; it is isomorphic to the dual bundle L^*). How is this group structure reflected in the theory of characteristic classes?

4.9. PROPOSITION. *If* L *and* L *are line bundles over* M *then* $e_{L\otimes L'} = e_L + e_{L'}$. *Moreover if* L *is any line bundle on* M *then* $e_{\overline{L}} = -e_L$.

PROOF. We'll prove the first relation; the second follows from the easily verified fact that the Euler class of the trivial bundle is zero. Let us consider first the universal situation in which $M = G_1(\mathbb{C}^n)$ and L and L' are both the canonical line bundle. Construct over $M \times M$ the line bundle L'' whose fiber over a pair (m, m') is $L_m \otimes L'_{m'}$. What is its Euler class? The

[2] Or reassuringly simple, depending on your perspective.

Kunneth formula in cohomology says that wedge product of forms sets up an isomorphism

$$H^r(M \times M) \cong \oplus_{p+q=r} H^p(M) \otimes H^q(M).$$

In our case we are interested in the formula

$$H^2(M \times M) \cong H^2(M) \otimes H^0(M) \oplus H^0(M) \otimes H^2(M)$$

(there are no $H^1(M)$ terms since H^1 is zero for $M = G_1(\mathbb{C}^n)$). By restricting L'' to $M \times \{pt\}$ and $\{pt\} \times M$ we see that

$$e_{L''} = e_L \otimes 1 + 1 \otimes e_{L'}.$$

If we now restrict to the diagonal $M \subseteq M \times M$, over which L'' becomes $L \otimes L'$, then by functoriality of the Euler class we obtain the formula $e_{L \otimes L'} = e_L + e_{L'}$, as required. In the case of general M and general line bundles, pull back this formula via the product of classifying maps $M \times M \to G_1(\mathbb{C}^n) \times G_1(\mathbb{C}^n)$. □

4.10. EXERCISE. Let L be a complex line bundle over the base space M. We can assume without loss of generality that L is provided with a Hermitian metric in each fiber. Now since L is locally trivial, we can cover M by open sets[3] $\{U_j\}$ such that each restriction $L_{|U_j}$ is trivial and so admits a unit section s_j. When U_j and U_k intersect, we can find transition functions $\phi_{jk}\colon U_j \cap U_k \to \mathbb{R}$ measuring the difference between these sections, so that

$$s_j = e^{2\pi i \phi_{jk}} \cdot s_k.$$

On triple intersections $U_j \cap U_k \cap U_l$, we must have

$$c_{jkl} := \phi_{jk} + \phi_{kl} - \phi_{jl} \in \mathbb{Z}.$$

This means that c is a 2-cocycle for the Cech cohomology $H^2(M; Z)$. Denote by $c_1(L)$ the associated cohomology class. Verify that the construction of $c_1(L)$ given above depends only on the isomorphism class of L. If L and L' are line bundles, we can form their *tensor product* $L \otimes L'$. Prove that $c_1(L \otimes L') = c_1(L) + c_1(L')$. For extra points, identify de Rham and Cech cohomology, and thereby identify $c_1(L)$ with e_L.

4.11. REMARK. In fact one can show that c_1 gives an *isomorphism* between the abelian group of isomorphism classes of line bundles (under tensor product) and $H^2(M; \mathbb{Z})$; but we will not need this.

[3]We may assume that these sets and all their intersections are contractible.

3. Characteristic Classes of Higher Rank Bundles

There is a natural map

$$f_k\colon \underbrace{G_1(\mathbb{C}^n) \times G_1(\mathbb{C}^n) \times \cdots \times G_1(\mathbb{C}^n)}_{k \text{ times}} \to G_k(\mathbb{C}^{kn})$$

(there is an obvious direct formula, but in homotopy theoretic terms, the left had space has over it a k-fold direct sum of canonical line bundles, and there is therefore a classifying map to $G_k(\mathbb{C}^N)$, for some N, which classifies this rank k vector bundle). Passing to cohomology, and to the limit as $n \to \infty$, we obtain a canonical homomorphism

$$H^*(G_k(\mathbb{C}^\infty)) \to H^*(G_1(\mathbb{C}^\infty) \times \cdots \times G_1(\mathbb{C}^\infty)).$$

Now the symmetric group Σ_k acts on the right-hand cohomology ring by permuting the factors $G_1(\mathbb{C}^\infty)$. The fundamental fact about $H^*(G_k(\mathbb{C}^\infty))$ is this:

4.12. THEOREM. *The above ring homomorphism identifies* $H^*(G_k(\mathbb{C}^\infty))$ *with the permutation-invariant elements in* $H^*(G_1(\mathbb{C}^\infty) \times \cdots \times G_1(\mathbb{C}^\infty))$. *Thus the ring* $H^*(G_k(\mathbb{C}^\infty))$ *is isomorphic to the ring of formal power series in degree 2 indeterminates* $x_1, \ldots, x_k$ *which are symmetric under permutation of the* x_j. *Under the map on homology induced from the map*

$$f_k\colon G_1(\mathbb{C}^\infty) \times G_1(\mathbb{C}^\infty) \times \cdots \times G_1(\mathbb{C}^\infty) \to G_k(\mathbb{C}^\infty)$$

which classifies the k*-fold direct sum of canonical line bundles, the generator* x_j *maps to the Euler class of the canonical line bundle over the* j*th factor.* □

We shall not prove this result here (but see Remark 4.28 in Section 6 for some comments on the proof). The theorem is sometimes called the 'splitting principle', because it effectively tells us that, in calculations with characteristic classes, we can behave as though every (complex) vector bundle is a direct sum of line bundles. See the next section for an example of this.

4. The Chern Character

What the theorem in the previous section definitely tells us is that to specify a characteristic class of rank k vector bundles it is enough to specify a symmetric formal power series in k degree 2 variables, $x_1, \ldots, x_k$. Let us illustrate how this is done in what is perhaps the most important case: that of the *Chern character*.

4.13. DEFINITION. The *Chern character* is the characteristic class of rank k vector bundles which corresponds to the symmetric formal power series

$$e^{x_1} + e^{x_2} + \cdots + e^{x_k}$$

This definition applies to any positive integer k, and if V is any vector bundle over M we shall denote by $\mathrm{ch}(V) \in H^*(M)$ its Chern character, obtained by applying the formula in the definition for the appropriate k.

The Chern character is important because it is a sort of 'ring-homomorphism' from vector bundles to cohomology:

4.14. PROPOSITION. *Let V and W be complex vector bundles over M. Then*

$$\mathrm{ch}(V \oplus W) = \mathrm{ch}(V) + \mathrm{ch}(W)$$

and

$$\mathrm{ch}(V \otimes W) = \mathrm{ch}(V)\,\mathrm{ch}(W).$$

PROOF. It suffices to prove these identities in the case where V and W are the pullbacks to the product $G_k(\mathbb{C}\infty) \times G_\ell(\mathbb{C}^\infty)$ of the universal rank k and ℓ bundles on the two factors (compare the proof of Proposition 4.9). By the Kunneth formula and the theorem in the last section, the classifying map

$$f_{k,\ell}\colon \underbrace{G_1(\mathbb{C}^\infty) \times G_1(\mathbb{C}^\infty) \times \cdots \times G_1(\mathbb{C}^\infty)}_{k+\ell \text{ times}} \to G_k(\mathbb{C}^\infty) \times G_\ell(\mathbb{C}^\infty)$$

is injective on cohomology. It therefore suffices to verify the identity in the cohomology of the product of the $G_1(\mathbb{C}^\infty)$. In particular, it suffices to prove the formula when V and W are complex vector bundles over some space which are direct sums of line bundles. But for a direct sum of line bundles $L_1 \oplus \cdots \oplus L_p$ the meaning of Definition 4.13 is that

$$\mathrm{ch}(L_1 \oplus \cdots \oplus L_p) = e^{L_1} + \cdots + e^{L_p}.$$

Additivity is therefore obvious, while multiplicativity follows from the case of individual line bundles, which is handled by Proposition 4.9. $\square$

It follows easily that:

4.15. THEOREM. *The Chern character gives a homomorphism of rings*

$$\mathrm{ch}\colon K(M) \to H^{\mathrm{even}}(M),$$

for any compact space M. $\square$

4.16. REMARK. Atiyah and Hirzebruch showed that this homomorphism passes to an *isomorphism* $K^0(M) \otimes \mathbb{C} \to H^{\mathrm{even}}(M)$ (the factor $\mathbb{C}$ is appropriate if we are using de Rham theory with complex coefficients). However, we shall not need this fact.

5. Multiplicative Characteristic Classes

4.17. DEFINITION. A characteristic class for complex vector bundles, say $\mathcal{C}$, is *multiplicative* if

$$\mathcal{C}(V \oplus V') = \mathcal{C}(V) \cdot \mathcal{C}(V'),$$

for any vector bundles V and V'.

4.18. REMARK. Strictly speaking, a multiplicative characteristic class is, like the Chern character, a whole family of characteristic classes, one for each dimension of complex vector bundles.

We could equally well have defined *additive* classes, of which the Chern character would be an example. However multiplicative classes arise more frequently in the sequel. The great virtue of multiplicative (or additive) classes, is that they may be determined by computation of a very limited set of examples, as the following proposition shows.

4.19. PROPOSITION. *Two multiplicative characteristic classes are equal if they are equal on the canonical line bundles over all the* $G_1(\mathbb{C}^n)$.

PROOF. To show that two classes are equal, it suffices to show that they are equal on the universal bundles over $G_k(\mathbb{C}^\infty)$. But by the splitting principle, as illustrated in the last section, it then suffices to show they are equal for direct sums of line bundles. By multiplicativity we can then reduce to single line bundles; and by universality it finally suffices to consider the canonical line bundle over $G_1(\mathbb{C}^\infty)$. □

4.20. PROPOSITION. *Let* $F(x)$ *be a formal power series in* x. *There is a unique multiplicative class* $\mathcal{C}_F$ *such that, on line bundles,*

$$\mathcal{C}_F(L) = F(e_L) \in H^*(M),$$

where e_L *is the Euler class.*

PROOF. On rank k bundles, let $\mathbb{C}_F$ be the characteristic class associated to the formal power series

$$F(x_1) \times \cdots \times F(x_k) \in H^*(G_k(\mathbb{C}^\infty)).$$

By the splitting principle, as illustrated in the previous section, this defines a multiplicative characteristic class. □

4.21. REMARK. A multiplicative class $\mathcal{C}$ such that $\mathcal{C}(1) = 1$, where the first 1 denotes the trivial line bundle, is called a *genus*. Genera correspond to formal power series $F(x)$ whose order zero term is 1.

4.22. EXAMPLE. Later on we shall consider the formal power series

$$\frac{x}{1-e^{-x}} = 1 + \frac{1}{2}x + \frac{1}{12}x^2 + \frac{1}{720}x^4 + \cdots .$$

(As an easy exercise, show that no odd powers of x higher than the first appear in this expansion. The coefficient of x^n for even n is $B_n/n!$, where B_n is the n'th Bernoulli number.) The associated genus is called the *Todd genus*, denoted $\mathrm{Todd}(V)$.

4.23. EXERCISE. The inverse (with respect to the product in cohomology) of the multiplicative class associated to the formal power series F is the multiplicative class associated to the formal power series $1/F$.

6. Chern Classes

It is a theorem of algebra that the algebra of symmetric formal power series in $x_1, \ldots, x_k$ is isomorphic to the algebra of all formal power series in the indeterminates $c_1, \cdots, c_k$, via the map which sends c_j to the jth *elementary symmetric function* in $x_1, \ldots, x_k$. This is the degree j coefficient in the polynomial $\prod_{i=1}^{k}(1+x_i)$.

4.24. DEFINITION. The jth *Chern class* (not to be confused with the Chern character) is the characteristic class $c_j(V)$ associated to the jth elementary symmetric function in the ring of symmetric formal power series in $x_1, \ldots, x_k$.

Thus, $H^*(G_k(\mathbb{C}^\infty))$ is an algebra of formal power series in the Chern classes.

4.25. EXERCISE. The *total Chern class* is, by definition, the characteristic class

$$c(V) = 1 + c_1(V) + c_2(V) + \cdots .$$

Show that $c(V)$ is the genus associated to the power series (in fact polynomial) $F(x) = 1 + x$.

It is traditional to expand characteristic classes in terms of the Chern classes; the following exercises give some examples, which also hint at the one useful reason for doing this.

4.26. EXERCISE. On 2-dimensional bundles the formula

$$\frac{x_1}{1-e^{-x_1}} \cdot \frac{x_2}{1-e^{-x_2}} = 1 + \frac{1}{2}(x_1 + x_2) + \frac{1}{12}(3x_1x_2 + x_1^2 + x_2^2) + \ldots,$$

gives

$$\mathrm{Todd}(V) = 1 + \frac{1}{2}c_1(V) + \frac{1}{12}(c_1(V)^2 + c_2(V)) + \cdots .$$

Show that this is valid for a bundle of any rank, and compute the next term in the Todd genus, again for any rank.

4.27. EXERCISE. By expanding the power series for the exponential function, show that

$$\operatorname{ch}(V) = k + c_1(V) + \tfrac{1}{2}(c_1^2 - 2c_2) + \cdots .$$

An even more important reason for focusing on the Chern classes is that they are *integral*, which is to say that in fact they may be defined in integral cohomology:

$$c_j(V) \in H^{2j}(M, \mathbb{Z}).$$

This stems ultimately from the fact that the Thom class of an oriented vector bundle is integral too. As we shall see very briefly at the end of this chapter, this integrality may be played of very effectively against other integrality phenomena to introduce interesting arithmetic constraints into manifold theory.

4.28. REMARK. The Chern classes also arise naturally in the computation of the cohomology ring $H^*(G_k(\mathbb{C}^\infty)$. By geometric arguments similar to those used in the proof of Proposition 4.6 it is possible to construct a long exact sequence

???

from which the rings $H^*(G_k(\mathbb{C}^\infty))$ can be computed by induction on k. The map labelled c_k is multiplication by the kth Chern class of the canonical bundle on $G_k(\mathbb{C}^\infty)$.

7. Real Characteristic Classes

We now want to consider characteristic classes for *real* vector bundles. By working with de Rham cohomology we can avoid complicated issues involving torsion in cohomology, and in so doing we can reduce, by the process of *complexification*, characteristic class theory in the real case to the complex case already considered.

Complexification (that is, the process of tensoring real vector spaces or bundles by $\mathbb{C}$) gives rise to an inclusion map

$$G_k(\mathbb{R}^\infty) \to G_k(\mathbb{C}^\infty),$$

and therefore we get an induced map on cohomology from $H^*(G_k(\mathbb{C}^\infty))$, which we have already computed, to $H^*(G_k(\mathbb{C}^\infty))$. This map is *not* an isomorphism. However, we do have the following fact (which, once again, we shall not prove):

4.29. PROPOSITION. *Suppose that* $k = 2m$ *is even. Then the induced map*

$$H^*(G_k(\mathbb{C}^\infty)) = \mathbb{C}_{sym}[[x_1, y_1, \dots, x_m, y_m]] \to H^*(G_k(\mathbb{R}^\infty))$$

is surjective, and its kernel is the ideal generated by $x_1 + y_1, \dots, x_m + y_m$. □

4.30. REMARK. There is a similar proposition for $k = 2m + 1$, but we won't need it.

What underlies the proposition is that the composition of operations

$$\{\text{Real Bundles}\} \longrightarrow \{\text{Complex Bundles}\} \longrightarrow \{\text{Real Bundles}\}$$

where the first arrow is complexification, whereas the second is "realification," the passage to the real bundle underlying a complex bundle, satisfies: $V \mapsto V \oplus V$, whereas the composition

$$\{\text{Complex Bundles}\} \longrightarrow \{\text{Real Bundles}\} \longrightarrow \{\text{Complex Bundles}\}$$

satisifies $V \mapsto V \oplus \overline{V}$, and the operation $V \mapsto \overline{V}$ corresponds to the operation $x_j \mapsto -x_j$ in $H^*(G_k(\mathbb{C}^\infty))$.

Since the quotient $\mathbb{C}_{sym}[[x_1, \dots, y_m]]/\langle x_1 + y_1, \dots, x_m + y_m\rangle$ can be identified with $\mathbb{C}_{sym}[[x_1^2, \dots, x_m^2]]$, follows that

$$H^*(G_{2m}(\mathbb{R}^{2m})) = \mathbb{C}_{sym}[[x_1^2, \dots x_m^2]] \tag{1}$$

4.31. DEFINITION. The characteristic classes corresponding to the elementary symmetric functions of $x_1^2, \dots, x_m^2$ are called the *Pontrjagin classes* p_j.

4.32. REMARK. Tracing through the identifications we have made above one sees that

$$p_j(V) = (-1)^j c_{2j}(V \otimes \mathbb{C}).$$

The Pontrjagin classes are therefore *integral*.

Let us now study multiplicative characteristic classes for even-dimensional for real vector bundles (these are defined analogously to their complex counterparts). For simplicity we shall consider even-dimensional bundles only. One can argue just as in the complex case to see that:

4.33. LEMMA. *There is a bijective correspondence between mutliplicative classes* $\mathcal{C}$ *for real vector bundles and formal power series in* x^2, $F(x^2)$. □

The correspondence goes like this: given a power series $F(x^2)$, find a symmetric power series $\tilde{F}(x_1, y_1, \ldots, x_m, y_m)$ such that

$$\tilde{F}(x_1, -x_1, x_2, -x_2, \ldots, x_m, -x_m) = F(x_1^2)F(x_2^2)\cdots F(x_m^2).$$

The power series $\tilde{F}$ lies in $H^*(G_k(\mathbb{C}^\infty))$ and hence defines a characteristic class $\tilde{\mathcal{C}}$ of complex vector bundles. If V is a real vector bundle one defines $\mathcal{C}(V) = \tilde{\mathcal{C}}(V_{\mathbb{C}})$.

4.34. EXAMPLE. The simplest such class is the one corresponding to $F(x^2) = 1 + x^2$; it is the *total Pontrjagin class*

$$p(V) = 1 + p_1(V) + p_2(V) + \cdots .$$

4.35. EXAMPLE. The $\widehat{A}$ *genus* and the L *genus*, which appear prominently in index theory, are the real genera associated to the formal power series

$$F(x^2) = \frac{x/2}{\sinh(x/2)}, \quad \text{and} \quad F(x^2) = \frac{x}{\tanh x}$$

respectively. (The L genus is the characteristic class that appeared in Hirzebruch's signature theorem in Chapter 1).

4.36. EXERCISE. Let V be a real vector bundle. Prove that $\widehat{A}(V)^2 = \mathrm{Todd}(V \otimes \mathbb{C})$.

CHAPTER 5

The Index Problem

In Chapter 2 we saw that a linear elliptic partial differential operator D on a smooth closed manifold has a Fredholm index,

$$\mathrm{Index}(D) \in \mathbb{Z}.$$

In Chapter 3 we saw that associated to D there is a symbol class

$$\sigma_D \in K(T^*M).$$

In Chapter 4 we discussed the Chern character and characteristic classes of vector bundles. In this chapter we shall give in outline form the solution of the following *index problem:* to compute $\mathrm{Index}(D)$ in terms of $\mathrm{ch}(\sigma_D)$.

The answer to the problem is the famous Atiyah-Singer Index Theorem.

5.1. THEOREM (Atiyah and Singer). *Let D be a linear elliptic partial differential operator*[1] *on a smooth, closed even-dimensional*[2] *manifold M, and denote by $[\sigma_D] \in K(T^*M)$ its symbol class. Then*

$$\mathrm{Index}(D) = \int_{T^*M} \mathrm{ch}[\sigma_D]\, \mathrm{Todd}(TM \otimes \mathbb{C}).$$

Recall that the Todd class $\mathrm{Todd}(V)$ of a complex vector bundle V is the genus associated to the formal power series $x/(1-e^{-x})$.

5.2. REMARK. In many of the applications of the index theorem, the integral over T^*M is evaluated in two stages: first integrate over the fibers of T^*M and then integrate the result over the base space M. For example, we shall see that this is how the expression $\int_M L(TM)$ arises in the Hirzebruch signature theorem.

[1]We are only considering first order operators in these notes, but the result applies more generally.

[2]There is a version of the index theorem for operators on odd-dimensional manifolds too, but to obtain interesting examples one must move outside the world of differential operators to the larger class of *pseudodifferential* operators. That is the reason for the restriction to even-dimensional manifolds here.

1. The Analytic Index Map

The first and major step in solving the index problem is to recast it in K-theoretic terms, using the following result, which we will prove in Chapter 7.

5.3. THEOREM. *For each smooth manifold* M *(compact or not), there is a homomorphism*

$$\alpha_M \colon K(T^*M) \to K(\mathrm{pt})$$

that has the following property: if M is *compact, and if* $\sigma_D \in K(T^*M)$ *is the symbol class of an elliptic operator* D *on* M*, then* $\alpha(\sigma_D) = \mathrm{Index}(D)$ *in* $K(\mathrm{pt}) \cong \mathbb{Z}$.

5.4. REMARK. This theorem does not yet tell us everything we need to know about the *analytic index map* α. Further necessary properties of the construction will be described in Theorem 5.20.

Once we have the map α in hand, the index problem will amount to filling in the blank in the following diagram:

$$\begin{array}{ccc} K(T^*M) & \xrightarrow{\ \alpha\ } & K(\mathrm{pt}) \\ {\scriptstyle \mathrm{ch}}\downarrow & & \downarrow{\scriptstyle \mathrm{ch}} \\ H^{\mathrm{even}}(T^*M) & \xrightarrow[?]{} & H^{\mathrm{even}}(\mathrm{pt}) \end{array}$$

We shall do so by reducing from M to the simpler manifold $\mathbb{R}^k$, and in order to successfully carry out this program we shall need to understand the correspondence between constructions in K-theory and their counterparts in cohomology theory. The major part of the present chapter will be devoted to this.

2. The Thom Homomorphism in K-Theory

Let V be a complex Hermitian vector bundle over a locally compact base space X. We are going to construct a *Thom homomorphism*

$$\phi \colon K(X) \to K(V)$$

which is in many ways analogous to the Thom homomorphism in cohomology described in the previous chapter.

Let $\bigwedge^* V$ be the exterior algebra bundle of V, which is a $\mathbb{Z}/2$-graded hermitian vector bundle over X (it is graded by its decomposition into forms of even and odd degree). Let $\pi \colon V \to X$ be the projection, and form the pullback $\pi^* \bigwedge^* V$. Define an endomorphism

$$b \colon \pi^* \textstyle\bigwedge^* V \to \pi^* \bigwedge^* V$$

by the formula

$$b(v)s = v \wedge s + v \lrcorner s,$$

where the map $s \mapsto v \lrcorner s$ is the adjoint of the operator $s \mapsto v \wedge s$ of exterior multiplication by v. Obviously b is a self-adjoint, odd endomorphism. Its most important property is this:

5.5. LEMMA. $b(v)^2 = \|v\|^2 \cdot I$.

PROOF. We may assume that $v \in V_b$ has norm one. Choose an orthonormal basis $\{v_1, \ldots, v_k\}$ for V in which $v = v_1$. The exterior algebra $\bigwedge^* V_b$ has then the orthonormal basis made up of all products $v_{i_1} \wedge \cdots \wedge v_{i_p}$, where $i_1 < \cdots < i_p$. The operator $v \wedge \_$ satisfies

$$v \wedge v_{i_1} \wedge \cdots \wedge v_{i_p} = \begin{cases} v_1 \wedge v_{i_1} \wedge \cdots \wedge v_{i_p} & \text{if } i_1 \neq 1 \\ 0 & \text{if } i_1 = 1. \end{cases}$$

From this it follows that the exterior product operator is a partial isometry, and that

$$v \lrcorner v_{i_1} \wedge \cdots \wedge v_{i_p} = \begin{cases} 0 & \text{if } i_1 \neq 1 \\ v_{i_2} \wedge \cdots \wedge v_{i_p} & \text{if } i_1 = 1. \end{cases}$$

The lemma follows easily from this. □

It follows from the lemma that the endomorphism b behaves just like the symbol of an elliptic operator. Indeed it is an elliptic element in the sense of Definition 3.22. Thus b determines a K-theory class $[b]$ in $K(V)$ by the difference bundle construction.

5.6. DEFINITION. Let V be a complex Hermitian vector bundle over a compact base space X and form the pullback $\pi^* \bigwedge^* V$ of the exterior algebra bundle of V to V. The *Thom element* is the class $b_V \in K(V)$ determined by the elliptic endomorphism $b\colon \pi^* \bigwedge^* V \to \pi^* \bigwedge^* V$ constructed above. The *Thom homomorphism* is the homomorphism

$$\phi\colon K(X) \to K(V)$$

determined by the formula $\phi(x) = x \cdot b_V$ (recall that $K(V)$ is a module over the ring $K(X)$).

If V is a complex vector bundle over a non-compact (but still locally compact) base X, then the Thom *element* $b_V \in K(V)$ is no longer defined. However the Thom *homomorphism* $\phi\colon K(X) \to K(V)$ may still be constructed. One way to do this is to use C^*-algebra homomorphisms, as follows.

Suppose that V is a complex vector bundle over the locally compact space X, and denote by $C_0(V, \mathrm{End}(\bigwedge^* V))$ the C^*-algebra of continuous sections, vanishing at infinity, the bundle $\operatorname{End}(\pi^* \wedge^* V)$ over V. This is a $\mathbb{Z}/2$-graded C^*-algebra, Morita equivalent to $C_0(V)$.[3] Define a graded $*$-homomorphism

$$\phi\colon \mathcal{S} \otimes C_0(X) \to C_0(V, \operatorname{End}(\wedge^* V))$$

by the formula

$$\phi(f \otimes h)(v) = f(b(v))h(\pi(v)),$$

where $\pi\colon V \to X$ is the projection. According to the observations we made in Chapter 3, this induces a homomorphism $K(X) \to K(V)$ of K-theory groups, which we will again denote by ϕ.

5.7. DEFINITION. The homomorphism $\phi\colon K(X) \to K(V)$ constructed in the preceding paragraph is called the *Thom homomorphism* for V.

5.8. EXERCISE. Show that if X is compact then this is the same Thom homomorphism as in Definition 5.6.

5.9. REMARK. Another way to construct the Thom homomorphism in the non-compact case is to reduce to the compact case as follows. Write X as a union of an increasing sequence (or net) of open subsets X_j with compact closure in X. Let Y_j be the compact space obtained by joining together two copies of $\overline{X_j}$ along ∂X_j. Then X_j may be viewed as an open subset of Y_j (embed X_j in the first copy of Y_j) and the bundle V over X may be extended in the obvious way to Y_j. There is a commuting diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K(X_j) & \longrightarrow & K(Y_j) & \longrightarrow & K(\overline{X_j}) & \longrightarrow & 0 \\ & & & & \downarrow \phi & & \downarrow \phi & & \\ 0 & \longrightarrow & K(V|_{X_j}) & \longrightarrow & K(V|_{Y_j}) & \longrightarrow & K(V|_{\overline{X_j}}) & \longrightarrow & 0 \end{array}$$

using which we may define $\phi\colon K(X_j) \to K(V|_{X_j})$. Since $K(X) = \varinjlim K(X_j)$ and $K(V) = \varinjlim K(V|_{X_j})$, and since the maps just constructed are compatible with direct limits, we obtain a map $\phi\colon K(X) \to K(V)$, as required.

5.10. EXERCISE. Check that this is consistent with our previous construction of the Thom homomorphism in the non-compact case.

In ordinary cohomology, the Thom class had the crucial property that its restriction to each fiber of an oriented vector bundle was a generator for the cohomology (with compact supports) of that fiber. The corresponding property in K-theory is just as important: it is the famous Bott periodicity theorem.

[3]It is the algebra of compact operators on the Hilbert module $C_0(V, \wedge^* V)$.

Let V be a finite-dimensional complex vector space. We may regard it as a vector bundle over a point, and so associate to it a Thom class, which we shall call in this special case the *Bott element*. Thus the Bott element is the class in $K(V)$ associated to the endomorphism $b\colon \bigwedge^* V \to \bigwedge^* V$ of the trivial bundle over V with fiber $\bigwedge^* V$.

5.11. THEOREM (Bott Periodicity). *Let V be a finite-dimensional, complex Hermitian vector space. The abelian group $K(V)$ is freely generated by the Bott element.*

This theorem is at the center of topological K-theory. It allows us to promote the functor $K(X)$ to a fully fledged cohomology theory, with long exact sequences, excision and so one. Using the Mayer-Vietoris argument hinted at during our discussion of the Thom isomorphism in cohomology, one can generalize Bott Periodicity to a Thom isomorphism theorem in K-theory:

5.12. THEOREM (Thom Isomorphism). *Let V be a complex Hermitian vector bundle over a locally compact space X. The Thom homomorphism*

$$\phi\colon K(X) \to K(V)$$

is an isomorphism.

We shall not need the K-theory Thom Isomorphism Theorem in these notes, but the Bott Periodicity Theorem is a central component of the K-theory proof of the index theorem. We shall prove it in Chapter 8.

3. Comparison of Thom Homomorphisms

The underlying real vector space of any complex vector space is canonically oriented: pick a complex basis $v_1, \ldots, v_k$ then decree that the real basis

$$v_1, iv_1, \ldots, v_k, iv_k$$

is oriented. Thus the real bundle underlying every complex vector bundle V is oriented, and we can therefore consider the Thom homomorphism in cohomology, $\psi_V\colon H^{\text{even}}(M) \to H_c^{*+2k}(V)$. In this section we shall compare the Thom homomorphisms in K-theory and in cohomology.

Let V be a k-dimensional, smooth, complex vector bundle over a smooth manifold M. Does the diagram

$$\begin{array}{ccc} K(M) & \xrightarrow{\ \phi\ } & K(V) \\ {\scriptstyle \text{ch}}\downarrow & & \downarrow{\scriptstyle \text{ch}} \\ H^{\text{even}}(M) & \xrightarrow[\ \psi\ ]{} & H_c^{\text{even}}(V) \end{array}$$

which relates the Thom homomorphisms in K-theory and cohomology commute? Assume for a moment that M is compact. Then since both Thom homomorphisms are module maps (over $K(M)$ and $H^{\text{even}}(M)$), and the Chern character is a module homomorphism, the question is equivalent to asking if the Chern character of the K-theory Thom class b_V is the cohomology class u_V.

5.13. REMARK. Here we have used the Chern character for a non-compact space, namely V. But it is easy to extend the definition of the Chern character from compact to non-compact (but locally compact) spaces, by way of the following diagram which relates such a space X to its one-point compactification X^+:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K(X) & \longrightarrow & K(X^+) & \longrightarrow & K(pt) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle \text{ch}} & & \downarrow{\scriptstyle \text{ch}} & & \\ 0 & \longrightarrow & H_c^{\text{even}}(X) & \longrightarrow & H^{\text{even}}(X^+) & \longrightarrow & H^{\text{even}}(pt) & \longrightarrow & 0 \end{array}$$

The fact that ch is a module homomorphism in this context follows from the fact that it is a ring homomorphism for X^+.

In general, *the answer to our question is no*. According to the cohomology Thom isomorphism theorem, $H_c^{\text{even}}(V)$ is a free module over $H^{\text{even}}(X)$, generated by the cohomology Thom class u_V. We can therefore write

$$\text{ch}(b_V) = \tau(V) \cdot u_V,$$

for some unique class $\tau(V) \in H^{\text{even}}(X)$. The class $\tau(V)$ is not in general 1. But it is functorial in V, in the sense that $f^*\tau(V) = \tau(f^*V)$, for every map $f\colon M_1 \to M_2$ (because the Thom classes and the Chern character are functorial). Moreover the following proposition shows that $\tau(V)$ is multiplicative, and therefore, according to the previous chapter, quite computable.

5.14. PROPOSITION. *Let V be a k-dimensional, smooth, complex vector bundle over a smooth, closed manifold M. Define a cohomology class $\tau(V) \in H^{\text{even}}(M)$ by the formula*

$$\text{ch}(b_V)) = \tau(V) \cdot u_V,$$

where b_V is the K-theory Thom class and $u_V \in H^{\text{even}}(V)$ is the cohomology Thom class. Then $\tau(V)$ is a multiplicative characteristic class of complex vector bundles.

This is proved using the following multiplicative property of Thom classes. Suppose that V_1 and V_2 are two complex vector bundles over X. We can view V_2 as a vector bundle over the total space of V_1 by pulling back to

V_1. The total space of this pullback bundle is equal to the total space of the bundle $V_1 \oplus V_2$ over X. We can then compose Thom homomorphisms:

$$K(X) \xrightarrow{\phi} K(V_1) \xrightarrow{\phi} K(V_1 \oplus V_2),$$

and of course we can do the same thing in cohomology.

5.15. LEMMA. *Suppose that V_1 and V_2 are two complex vector bundles over M. We can view V_2 as a vector bundle over the total space of V_1 by pulling back to V_1. The compositions of Thom homomorphisms*

$$K(M) \xrightarrow{\phi_V} K(V_1) \xrightarrow{\phi_{\pi^* V_2}} K(V_1 \oplus V_2),$$

and

$$H^{\text{even}}(M) \xrightarrow{\psi_V} H^{\text{even}}(V_1) \xrightarrow{\psi_{\pi^* V_2}} H^{\text{even}}(V_1 \oplus V_2),$$

are equal to the Thom homomorphism, in K-theory and cohomology respectively, for the complex vector bundle $V_1 \oplus V_2$ over M.

We shall postpone the proof for a moment, and proceed with a proof of multiplicativity of $\tau(V)$, followed by a computation of this class.

PROOF OF PROPOSITION 5.14. The basic idea is very simple: we want to show that

$$\text{ch}(\phi_{V_1 \oplus V_2}(x)) = \tau(V_1)\tau(V_2)\psi_{V_1 \oplus V_2}(x),$$

for every $X \in K(X)$. We obtain this formula by factoring the Thom homomorphisms $\phi_{V_1 \oplus V_2}$ and $\psi_{V_1 \oplus V_2}$ using Lemma 5.15. In the following computation we shall omit the pullback symbol π^* from V_2. Here we go:

$$\begin{aligned}\text{ch}(\phi_{V_1 \oplus V_2}(x)) &= \text{ch}(\phi_{V_2}(\phi_{V_1}(x))) \\ &= \tau(V_2)\psi_{V_2}(\text{ch}(\phi_{V_1}(x))) = \tau(V_2)\psi_{V_2}(\tau(V_1)\psi_{V_1}(\text{ch}(x))).\end{aligned}$$

Using the fact that ψ_{V_1} is an $H^{\text{even}}(M)$ module map, and using the lemma again, we get

$$\begin{aligned}&\tau(V_2)\psi_{V_2}(\tau(V_1)\psi_{V_1}(\text{ch}(x))) \\ &\quad = \tau(V_2)\tau(V_1)\psi_{V_2}(\psi_{V_1}(\text{ch}(x))) = \tau(V_2)\tau(V_1)\psi_{V_2 \oplus V_1}(\text{ch}(x)),\end{aligned}$$

as required. There is a small detail here: the Thom homomorphism ψ_{V_2} is really $\psi_{\pi * V_2} \colon K(V_1) \to K(V_1 \oplus V_2$, and we need to know the formula

$$\text{ch}(\phi_{\pi^* V_2}(y)) = \tau(V_2)\psi_{\pi * V_2}(\text{ch}(y)),$$

for every $y \in H^{\text{even}}(V_1)$. (The class $\tau(V_2)$ is fits, by definition, into the analogous formula for the bundle V_2 over M, rather than the pullback of V_2 over V_1.) We leave this small issue to the reader as an exercise. □

As a result of this calculation we now know that the Thom homomorphisms in K-theory and cohomology are related by

$$\mathrm{ch}(\phi(x)) = \tau(V) \cdot \psi(\mathrm{ch}\, x) \tag{2}$$

where τ is a certain multiplicative characteristic class. But from Proposition 4.20 we know that such classes are determined simply by formal power series in one variable.

5.16. THEOREM. *The multiplicative class* τ *above is associated to the power series* $(1 - e^x)/x$.

PROOF. The formal power series associated to a multiplicative characteristic class just tells us what that class does to the canonical line bundle L over $BU(1) = \mathbb{C}P^\infty$. Let us calculate in $\mathbb{C}P^N$ for large N, let $x = c_1(L)$ be the generator of the cohomology ring, and let $y \in H^{\mathrm{even}}(\mathbb{C}P^N)$ be the cohomology class $(1 - e^x)/x$.

We want to compare the two elements $\mathrm{ch}(b_L)$ and $y \cdot u_L$ in $H_c^{\mathrm{even}}(L)$. Consider the map $\iota \colon M \to L$ which includes M as the zero section of L. Restricting the endomorphism representing the Thom class in K-theory to the zero section, we see that $\iota^*(b_L) = 1 - [L] \in K(M)$, and therefore

$$\iota^*(\mathrm{ch}(b_L)) = \mathrm{ch}(1 - [L]) = 1 - e^x.$$

On the other hand, we pointed out in Chapter 4 that

$$\iota^*(u_L) = x;$$

restricting the Thom class of L in cohomology gives the generator of the cohomology ring of $\mathbb{C}P^n$. Therefore

$$\iota^*(\mathrm{ch}(b_L)) = \iota^*(y \cdot u_L).$$

But under the Thom isomorphism $H^{\mathrm{even}}(L) \cong H^{\mathrm{even}}(M)$ the restriction map corresponds to multiplication by x, which is injective except in the highest non-zero degree of cohomology. Thus

$$\mathrm{ch}(b_L) = y \cdot u_L$$

in all degrees except possibly in $H^{2N}(\mathbb{C}P^N)$. Letting $N \to \infty$ we complete the proof. □

5.17. REMARK. As a very special case, of the above calculation, we see that for any complex vector space W of dimension k,

$$\int_W \mathrm{ch}(b) = (-1)^k,$$

where b denotes the Bott generator. (Consider W as a trivial vector bundle over a point.)

5.18. EXERCISE. It is clear that the multiplicative characteristic class τ (associated to $(e^x - 1)/x$) is closely related to the Todd genus (associated to $x/(1 - e^{-x})$). The actual relation, for a k-dimensional complex vector bundle, is

$$\tau(V) = \frac{(-1)^k}{\operatorname{Todd} \overline{V}},$$

where the bar denotes complex conjugation. Prove this.

PROOF OF LEMMA 5.15. Let $S_1 = \bigwedge^* V_1$ and $S_2 = \bigwedge^* V_2$, and observe that

$$S_2 \widehat{\otimes} S_1 \;\cong \bigwedge\nolimits^*(V_2 \oplus V_1).$$

The Thom homomorphism $\phi\colon K(V_1) \to K(V_2 \oplus V_1)$ is induced from the $*$-homomorphism $\phi\colon \mathcal{S} \otimes C_0(V_1) \to C_0(V_2, \operatorname{End}(S_1 \widehat{\otimes} S_2))$, but it is also induced from the $*$-homomorphism

$$\widehat{\phi}\colon \mathcal{S} \widehat{\otimes} C_0(V_1, \operatorname{End}(S_1)) \to C_0(V_2 \oplus V_1, \operatorname{End}(S_2 \widehat{\otimes} S_1).$$

given by the formula

$$\widehat{\sigma}_2(f \widehat{\otimes} h)(v_2, v_1) = f(c_2(v_2)) \widehat{\otimes} h(v_1).$$

The composition

$$\mathcal{S} \otimes C_0(X) \xrightarrow{\Delta\phi_{V_1}} \mathcal{S} \widehat{\otimes} C_0(V_1, \operatorname{End}(S_1)) \xrightarrow{\widehat{\phi_{V_2}}} C_0(V_2 \oplus V_1, \operatorname{End}(S_2 \widehat{\otimes} S_1)$$

is precisely $\phi_{V_1 \oplus V_2}$. □

5.19. REMARK. Topological K-theory is a 'generalized cohomology' theory, and in particular it is *contravariantly* functorial: if $f\colon X \to Y$ is a map, there is an induced homomorphism $f^*\colon K(Y) \to K(X)$. If we are working with non-compact spaces we need to add an extra condition whose purpose is to ensure that f^* maps $C_0(Y)$ to $C_0(X)$; the most convenient such condition is to require that f should be *proper* (the inverse image of a compact set is compact).

Nevertheless it is the case that certain *non*-proper maps g also induce homomorphisms on K-theory, not contravariantly but *covariantly*, so that $g\colon X \to Y$ induces $g_!\colon K(X) \to K(Y)$. A simple example of such a 'wrong way' map occurs when X is an open subset of Y, and g is the inclusion. Then any element of $C_0(X)$ extends by zero to an element of $C_0(Y)$, so we get a homomorphism $C_0(X) \to C_0(Y)$ and g is the induced map on K-theory.

It is convenient to consider the Thom homomorphism to be a 'wrong way' map also, induced by the inclusion of the zero-section $M \to V$. Lemma 5.15 then shows that these 'wrong way maps' are functorial.

There are functorial 'wrong way' maps in (compactly supported) cohomology also. However — and this is the main point of this section — the

Chern character is *not* a natural transformation for this wrong way functoriality. We have explicitly computed the 'naturality defect' in terms of the class $\tau(V)$.

4. Axioms for the Analytic Index Map

Let us now return to the index problem, as formulated in Section 1. There we explained that we are going construct, for each manifold M, a homomorphism $\alpha_M\colon K(TM) \to \mathbb{Z}$ which implements the analytic index, in the sense that

$$\alpha_M[\sigma_D] = \text{Index}(D)$$

for any elliptic operator D.

Once we have the homomorphism α_M to hand, the proof of the index theorem will require us to identify it in cohomological terms. We shall do this by showing that α_M satisfies the hypotheses of the theorem below, whose conclusion is the 'right hand side' of the Index Theorem.

5.20. THEOREM. *Assume that to every manifold M there is associated a homomorphism $\alpha_M\colon K(T^*M) \to \mathbb{Z}$ with the following properties:*

(i) *If M_1 is embedded as an open subset of M_2 then the diagram*

$$\begin{array}{ccc} K(T^*M_1) & \xrightarrow{\alpha_{M_1}} & \mathbb{Z} \\ \downarrow & & \downarrow \\ K(T^*M_2) & \xrightarrow[\alpha_{M_2}]{} & \mathbb{Z} \end{array}$$

commutes.

(ii) *If V is a real vector bundle of dimension k over M, and if $\phi\colon K(T^*M) \to K(T^*V)$ denotes the Thom homomorphism*[4]*, then the following diagram commutes:*

$$\begin{array}{ccc} K(T^*M) & \xrightarrow{\alpha_M} & \mathbb{Z} \\ {\scriptstyle\phi}\downarrow & & \downarrow{\scriptstyle =} \\ K(T^*V) & \xrightarrow[\alpha_V]{} & \mathbb{Z}. \end{array}$$

[4] This requires us to give T^*V the structure of a complex vector bundle over T^*M. The way to do this is described in the remark following the statement of the theorem.

(iii) *If* $b \in K(T^*\mathbb{R}^n)$ *is the Bott element*[5], *then* $\alpha_{\mathbb{R}^n}(b) = 1$.

Then

$$\alpha_M(x) = (-1)^{\dim(M)} \int_{T^*M} \mathrm{Todd}(TM \otimes \mathbb{C}) \cdot \mathrm{ch}(x), \tag{3}$$

for every M *and every* $x \in K(T^*M)$.

5.21. REMARK. We need to make various conventions about complex structures, orientations, and so on. We need to know that T^*V is naturally a *complex* vector bundle over T^*M. By choosing Euclidean metrics one can identify TM and T^*M as real vector bundles, and similarly one can identify TV and T^*V. Thus it is enough to exhibit TV as a complex vector bundle over TM. But in fact

$$TV \cong \pi^*(V \oplus V), \quad \text{(where } \pi\colon TM \to M\text{)},$$

and we can identify $V \oplus V = V \otimes \mathbb{C}$.

The final formula 3 requires that we orient T^*M. This we do as follows. Choose local coordinates $\{x_1, \ldots, x_n\}$ on M and corresponding coordinates $\xi_1, \ldots, , \xi_n$ in the fibers of T^*M. Then we deem that the list $x_1, \xi_1, \ldots, x_n, \xi_n$ is an oriented local coordinate system on T^*M.

5.22. REMARK. Note that axioms (i) and (ii) above simply say that α is natural with respect to the two sorts of 'wrong way maps' that we identified in Remark 5.19.

PROOF OF THEOREM 5.20. We are going to approach this proof by easy stages, so consider first the case of Euclidean space $\mathbb{R}^n$, and the associated homomorphism

$$\alpha_{\mathbb{R}^n}\colon K(T^*\mathbb{R}^n) \to K(\mathrm{pt}).$$

According to the normalization axiom (iii), $\alpha_{\mathbb{R}^n}(b) = 1 \in K(\mathrm{pt})$. On the other hand, on the right-hand side of formula 3 the Todd genus is equal to 1 (because the tangent bundle to $\mathbb{R}^n$ is trivial) and so we are required simply to integrate the Chern character of the Bott class. By remark 5.17, the result is $(-1)^n$. So formula 3 is correct on $b \in K(T^*\mathbb{R}^n)$. But, according to the Bott Periodicity Theorem 5.11 (which is here used in a crucial way), the element b generates all of $K(T^*\mathbb{R}^n)$. Thus the formula is correct on every element of $K(T^*\mathbb{R}^n)$. Now using axiom (i), it is easy to show that the formula is correct on any open subset U of $\mathbb{R}^n$. (Of course, for such a U the tangent bundle is still trivial, so that the Todd genus is again 1 and the formula reads $\alpha_U(x) = (-1)^n \int_{T^*U} \mathrm{ch}(x)$.)

[5] We consider $T^*\mathbb{R}^n$ as a complex vector space via the formula $i \cdot (x, \xi) = (-\xi, x)$, where $X \in \mathbb{R}^n$ and $\xi \in T^*_x\mathbb{R}^n \cong \mathbb{R}^n$

The Tubular Neighborhood Theorem of differential topology tells us that, given any manifold M, there is a real vector bundle V of some dimension k over M such that the total space of V is diffeomorphic to an open subset of $\mathbb{R}^n$. Thus the formula 3 holds for V and we will finish the proof by deducing the formula for M from that for V, using of course the axiom (ii) which refers to the Thom isomorphism.

By that axiom we obtain, for $x \in K(T^*M)$,

$$\alpha_M(x) = \alpha_V(\phi(x)) = (-1)^{n+k} \int_{T^*V} \mathrm{ch}(\phi(x)),$$

using formula 3 for V. On the right hand side of this equation apply Proposition 5.14 to get

$$\alpha_M(x) = (-1)^{n+k} \int_{T^*M} \tau(V \otimes \mathbb{C})\, \mathrm{ch}(x),$$

where τ is the genus corresponding to the power series $(1 - e^x)/x$. (We are using here the fact that T^*V, as a complex vector bundle over T^*M, is isomorphic to $\pi^*(V \otimes \mathbb{C})$.) To finish the proof, note that the direct sum $V \oplus TM$ is isomorphic to a trivial bundle (of dimension $n + k$). Thus

$$\tau(V \otimes \mathbb{C}) = (-1)^k / \mathrm{Todd}(V \otimes \mathbb{C}) = (-1)^k \, \mathrm{Todd}(TM \otimes \mathbb{C})$$

using Exercise 5.18 and the fact that the complexification of a real vector bundle is isomorphic to its conjugate. Substituting this into the previously displayed equation we obtain the result. □

5. The Signature Operator

In this section we are going to outline an important application of the Hirzebruch signature theorem to the construction of an *exotic sphere*. This is due to Milnor (1957) and it highlighted the importance of playing off against one another two sources of 'integrality' in the signature theorem: the fact the the signature (or more generally the index of an elliptic operator) is an integer, and the fact that the Pontrjagin classes are integral cohomology classes. It is interesting that in *non*commutative geometry only the first source of integrality (index theory) is available to us.

The geometric input that is needed is a construction of manifolds with prescribed intersection form (remember that the *intersection form* is the form defined by the cup-product on the middle-dimensional cohomology). We will be considering manifolds W with boundary, whose boundary is topologically a sphere; if you don't want to work out a general theory of intersection forms for manifolds with boundary, just *define* the intersection form of such a manifold to be the intersection form of the topological manifold obtained by capping off the boundary with a disk.

A quadratic form over the integers is said to be *even* if it can be represented by a matrix all of whose diagonal entries are even, and *unimodular* if its determinant is ± 1. Milnor gave an explicit construction, sometimes called 'plumbing', which will produce a smooth W with prescribed even intersection form; unimodularity implies that the boundary is topologically a sphere. In particular

5.23. THEOREM (Milnor Plumbing). *There is a smooth 8-dimensional manifold W with boundary, such that*

- $\Sigma = \partial W$ *is homeomorphic to* S^7*;*
- W *is parallelizable (its tangent bundle is trivial);*
- *The intersection form of* W *is the* E_8 *matrix,*

$$E_8 = \begin{bmatrix} 2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{bmatrix}$$

The E_8 form is even, unimodular, and *positive definite*: it is the 'smallest' integral quadratic form with these properties.

We are going to show that Σ is *not diffeomorphic* to S^7: it is an 'exotic sphere'. For, suppose that it were. Then we could form a smooth, closed 8-manifold M by attaching an 8-disk to ∂W. Applying the Hirzebruch signature theorem we get

$$\text{Sign}(M) = \langle L(TM), [M] \rangle;$$

that is

$$8 = \frac{1}{45}(7p_2 + p_1^2)$$

where the Pontrjagin classes p_1 and p_2 are (implicitly) evaluated on the fundamental class of M. Recall, however, that the tangent bundle of W is trivial. Thus TM is obtained by 'clutching' two trivial bundles over the 7-sphere, and in such circumstances it is easy to see that all but the highest Pontrjagin classes must vanish. We conclude that $p_1 = 0$ so

$$p_2 = \frac{56}{45}$$

which contradicts the integrality of the Pontrjagin classes.

CHAPTER 6

The Dirac Operator

Before commencing the proof of the index theorem are going to work out the especially important example of the *Dirac operator*. Atiyah and Singer were developed (or rediscovered) the Dirac operator to serve as a counterpart in the realm of real manifolds, of the Dolbeault operator in complex manifold theory. Accordingly, we shall take a quick look at the Dobeault operator first.

1. The Dolbeault Operator

In this section we shall assume that the reader has some very basic familiarity with complex manifold theory. See for instance []. Let M be a compact *complex hermitian manifold* of complex dimension k, and hence real dimension $2k$ (see for example [] for an introductory account). The space of ordinary 1-forms on M (with complex coefficients) decomposes as a direct sum

$$\wedge^1 M = \wedge^{0,1} M \oplus \wedge^{1,0} M,$$

with the first summand generated locally by the $d\overline{z_i}$ and the second by the dz_j. The de Rham differential decomposes as a direct sum

$$d = \overline{\partial} + \partial \colon \Omega^0(M) \to \wedge^{0,1} M \oplus \wedge^{1,0} M.$$

There is a corresponding decomposition of differential forms and the de Rham operator in higher degrees, so that for example

$$\wedge^r M = \oplus_{p+q=r} \wedge^{p,q} M.$$

The space $\wedge^{0,q} M$ is isomorphic to the space of smooth sections of the bundle $\wedge^q TM$, where here we regard TM as a complex vector bundle to define the exterior power.

We can consider the *Dolbeault complex*

$$\Omega^0(M) \xrightarrow{\overline{\partial}} \Omega^{0,1}(M) \xrightarrow{\overline{\partial}} \cdots \xrightarrow{\overline{\partial}} \Omega^{0,k}(M)$$

and associated *Dolbeault operator* $D = \overline{\partial} + \overline{\partial}^*$. This is an elliptic operator, and in fact its symbol is a familiar object. Namely, after we use the hermitian metric to identify T^*M and TM as smooth manifolds (not as complex vector bundles) the symbol of D can be identified with the Thom element for the pullback of $\bigwedge^* TM$ to TM:

$$\sigma_D = b \colon \pi^* \textstyle\bigwedge^* TM \to \pi^* \bigwedge^* TM.$$

Because of this we can readily compute the contribution of its Chern character to the index formula. We get

$$\text{Index}(D) = \int_{T^*M} \tau(TM) u_{T^*M} \,\text{Todd}(T_{\mathbb{C}}M)$$

where the overall sign $(-1)^n$ has been dropped since n is even. Now M, being a complex manifold, is naturally oriented, and if we orient T^*M using local coordinates $x_1, \ldots, x_n, \xi_1, \ldots, \xi_n$, where $x_1, \ldots, x_n$ are oriented local coordinates on M, then we can compute the integral by first integrating along the fibers of T^*M. We get

$$\int_{T^*M} \tau(TM) u_{T^*M} \,\text{Todd}(T_{\mathbb{C}}M) = \int_M \tau(TM) \,\text{Todd}(T_{\mathbb{C}}M).$$

However this new orientation on T^*M differs from the orientation provided in Remark 5.21 by the sign $(-1)^{\frac{n(n-1)}{2}}$. Bearing this in mind, and since $(-1)^{\frac{n(n-1)}{2}} = (-1)^k$, we obtain the index formula

$$\text{Index}(D) = (-1)^k \int_M \tau(TM) \,\text{Todd}(T_{\mathbb{C}}M).$$

Finally, the bundle $T_{\mathbb{C}}M$ is isomorphic, as a complex vector bundle, to $TM \oplus \overline{TM}$ (the overline denotes the complex conjugate bundle). As a result

$$\text{Todd}(T_{\mathbb{C}}M) = \text{Todd}(TM) \cdot \text{Todd}(\overline{TM}).$$

Now using exercise 5.18 again, we obtain the *Hirzebruch Riemann-Roch formula*

$$\text{Index}(D) = \int_M \text{Todd}(TM).$$

6.1. EXERCISE. (For those who know some complex manifold theory.) Check Hirzebruch's formula for $\mathbb{C}P^1$. For extra credit, do the same for $\mathbb{C}P^n$ (in all cases you should get $1 = 1$).

Let M be an oriented, Riemannian manifold. The signature operator D on M was discussed in Lecture 1. Its square is the Laplace operator on differential forms. If we square its symbol σ we find the key property that

$$\sigma(x, \xi)^2 = \|\xi\|^2 \cdot I,$$

which we used to infer that D is elliptic.

Formulas of this type are common throughout K-theory and index theory. For example we encountered essentially the same identity in our treatment of the Bott element and the Thom homomorphism. We saw in Example **??** that the signature operator is not the only operator whose symbol has this feature that it is the square root of the function $\|\xi\|^2 \cdot I$. In this lecture we shall define and study the Dirac operator, which is in many respects the most important and most basic example of such an operator.

2. Clifford Symbols

6.2. DEFINITION. Let V be a euclidean vector bundle over some base X. A (complex) *Clifford symbol* associated to V consists of the following:

(i) A $\mathbb{Z}/2$-graded hermitian vector bundle S over X;

(ii) An $\mathbb{R}$-linear vector bundle map

$$c\colon V \to \mathrm{End}(S)$$

whose values are all odd-graded, self-adjoint endomorphisms of S, which satisfies the relation

$$c(v)^2 = \|v\|^2 \cdot I,$$

for all $v \in V$.

6.3. REMARK. We can also define a *real* Clifford symbol in the same way, by replacing the hermitian vector bundle S with a euclidean vector bundle. We will take a quick look at these at the end of the lecture.

We shall be most interested in the case where V is the cotangent bundle of a Riemannian manifold, in which case we can view a Clifford symbol as the symbol of some elliptic operator on M. Notice that a Clifford symbol defines an elliptic endomorphism of the pullback π^*S of S over V, and thus defines a K-theory class $[c] \in K(V)$ by the difference bundle construction of 3.25.

6.4. EXAMPLE. Suppose that V is a complex hermitian bundle, and let $S = \wedge^* V$. The formula

$$b(v)w = v \wedge w + v \lrcorner w$$

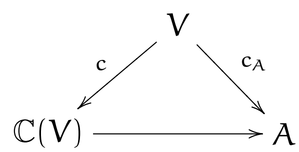
(which we used to define the Thom class in K-theory) is an example of a Clifford symbol.

To put it another way, Clifford symbols generalize the Thom element construction that we introduced in the previous lecture. The name "Clifford symbol" is borrowed from the following construction in algebra:

6.5. DEFINITION. Let V be a finite-dimensional euclidean vector space. The *complex Clifford algebra* $\mathbb{C}(V)$ is the complex, associative algebra with unit which is characterized up to canonical isomorphism by the following properties:

(i) There is a real linear map $c\colon V \to \mathbb{C}(V)$, such that $c(v^2) = \|v\|^2 I$, for all $v \in V$.

(ii) If A is any associative algebra with unit equipped with a real linear map $c_A\colon V \to A$ such that $c(v^2) = \|v\|^2 I$, for all $v \in V$, then there is a unique algebra homomorphism $\mathbb{C}(V) \to A$ such that the diagram



commutes

It is easy to check that if $v_1, \ldots, v_k$ is a basis for V then the set of products $c(v_{i_1}) \cdots c(v_{i_p})$, where $i_1 < \cdots < i_p$, is a linear basis for $\mathbb{C}(V)$. Thus $\mathbb{C}(V)$ is a finite-dimensional algebra, with

$$\dim(\mathbb{C}(V)) = 2^{\dim(V)}.$$

The algebra $\mathbb{C}(V)$ is $\mathbb{Z}/2$-graded by assigning the monomial $c(v_{i_1}) \cdots c(v_{i_p})$ even or odd degree, according as p is even or odd. A little less obvious is the following important fact:

6.6. PROPOSITION. *If V has even dimension $2k$, then $\mathbb{C}(V)$ is isomorphic to the algebra of $2^k \times 2^k$ complex matrices.*

PROOF (SKETCH/EXERCISE). We shall construct an explicit representation from $\mathbb{C}(V)$ into the matrix algebra, and proving using a linear basis for $\mathbb{C}(V)$ that it is injective (and hence surjective too, by dimension counting). To do this, observe that If $v_1, \ldots, v_{2k}$ is an orthonormal basis for V, and if matrices $E_1, \ldots, E_{2k}$ are given such that

$$E_i^2 = I \quad \text{and} \quad E_iE_j + E_jE_i = 0 \quad \text{when} \quad i \neq j$$

then the formula

$$c(a_1v_1 + \cdots + a_{2k}v_{2k}) = a_1E_1 + \cdots + a_nE_n$$

defines a representation of the Clifford algebra. For example, if $k = 1$, then we can define

$$c(v_1) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad c(v_2) = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}.$$

We leave it to the reader to work out suitable formulas for general k. (For $k = 2$ you will find them in Dirac's book on quantum mechanics.) □

6.7. EXERCISE. We can make this argument a little slicker if we are prepared to use the notion of *graded tensor product*, which we already discussed in Lecture 3 in connection with C^*-algebras. For it is not hard to see from the universal property that $\mathbb{C}(V \oplus W) \cong \mathbb{C}(V)\widehat{\otimes}\mathbb{C}(W)$. On the other hand, we have explicitly computed above that if V is 2-dimensional, $\mathbb{C}(V)$ is isomorphic to the graded algebra of endomorphisms of the graded vector space $H = \mathbb{C} \oplus \mathbb{C}$. Therefore we obtain

$$\mathbb{C}(\mathbb{C}^{2k}) = \mathbb{C}(\mathbb{C}^2)^{\widehat{\otimes}k} = \mathrm{End}(H^{\widehat{\otimes}k}) = \mathrm{End}(\mathbb{C}^{2k-1} \oplus \mathbb{C}^{2k-1}) = M_{2^k}(\mathbb{C}).$$

This argument gives us the grading and $*$-algebra structure too (see the next remark).

6.8. EXERCISE. Following up on the previous exercise, suppose that c_1 and c_2 are Clifford symbols for bundles V_1 and V_2, acting on S_1 and S_2 respectively. Show that their *sharp product*

$$c_1 \sharp c_2 = c_1 \widehat{\otimes} 1 + 1 \widehat{\otimes} c_2$$

is a Clifford symbol for $V_1 \oplus V_2$ acting on $S_1 \widehat{\otimes} S_2$. For extra credit, show that the associated K-theory classes satisfy

$$[c_1 \sharp c_2] = [c_1] \cdot [c_2],$$

and thus that $\mathrm{ch}(c_1 \sharp c_2) = \mathrm{ch}(c_1)\,\mathrm{ch}(c_2)$.

6.9. REMARK. It is easy to check that if V is any euclidean vector space, then there is a unique $*$-algebra structure on $\mathbb{C}(V)$ for which $c(v)^* = c(v)$, for all v. If $\dim(V) = 2k$ then $\mathbb{C}(V)$ is $*$-isomorphic to the matrix algebra $M_{2^k}(\mathbb{C})$, with its usual $*$-algebra structure of conjugate transpose. In addition, we can find a grading preserving $*$-isomorphism, where $M_{2^k}(\mathbb{C})$ is graded as an algebra of block 2×2 matrices. We shall use these refinements of Proposition 6.6 at one or two points below.

6.10. REMARK. Proposition 6.6 is not true for odd-dimensional V, and this is the reason that we shall restrict to even-dimensional V for the rest of this lecture. There are odd-dimensional counterparts to the proposition and to most of what follows, but they are rather more complicated and will not be discussed in these notes.

To return to our notion of Clifford symbol, from the vector bundle V we can form the bundle $\mathbb{C}(V)$ of Clifford algebras, and it is clear that a Clifford symbol is the same thing as a homomorphism of bundles from $\mathbb{C}(V)$ into $\mathrm{End}(S)$, which is fiberwise a homomorphism of $\mathbb{Z}/2$-graded algebras.

3. Dirac Symbols

6.11. DEFINITION. Let V be a euclidean vector bundle of *even* dimension $n = 2k$. A *Dirac symbol* associated to V is a Clifford symbol $c\colon V \to \operatorname{End}(S)$ such that the vector spaces S_x have (complex) dimension 2^k.

The condition on $\dim(S)$ specifies the minimal possible dimension of S, in view of the following result:

6.12. LEMMA. *If* $c\colon V \to \operatorname{End}(S)$ *is a Clifford symbol associated to a euclidean vector bundle, and if* $\dim(V) = 2k$*, then the fiber dimension of* S *is a multiple of* 2^k*.*

PROOF. The fibers of S are representation spaces of the Clifford algebra of $2k$-dimensional euclidean vector spaces. Since the Clifford algebras are all isomorphic to the matrix algebra $M_{2^k}(\mathbb{C})$, all such representations are multiples of the standard representation, of dimension 2^k. □

Why the interest in Dirac symbols? They play the same role in K-theory that orientations of vector bundles play in cohomology theory. It can be shown that if V is an even-dimensional euclidean vector bundle over X, and if $c\colon V \to \operatorname{End}(S)$ is a Dirac symbol, then the K-theory class $c \in K(V)$ freely generates $K(V)$ as a module over $K(M)$. Thus the existence of a Dirac symbol is a sufficient (and as it happens necessary) condition for the formulation of a Thom isomorphism theorem in K-theory.

Not every vector bundle V admits a Dirac symbol. At the very least, V must be orientable:

6.13. LEMMA. *If* V *is even-dimensional and if* $\sigma\colon V \to \operatorname{End}(S)$ *is a Dirac symbol then* V *is oriented by the following requirement: a local orthonormal frame* $v_1, \ldots, v_{2k}$ *is oriented if and only if the operator*

$$\gamma = i^k \sigma(v_1) \cdots \sigma(v_{2k})$$

is the grading operator of the bundle S *(in other words* γ *is* $+I$ *on the even part of* S *and* $-I$ *on the odd part).*

PROOF. The element γ has the following properties: $\gamma = \gamma^*$; $\gamma^2 = 1$; and γ anticommutes with every $c(v)$. Using the explicit basis for $\mathbb{C}(V)$ given earlier, it is not hard to check that there are precisely two elements in any Clifford algebra with these properties, which differ from one another by a sign only. Under the isomorphism of bundles $c\colon \mathbb{C}(V) \to \operatorname{End}(S)$, one of $\pm\gamma$ corresponds to the grading operator and one to its negative (since the grading operator and its negative have the same properties). We can therefore specify a family of consistent orientations in the fibers of V by requiring that in every fiber, γ corresponds to the grading operator. □

6.14. REMARK. From now on we shall assume that V is oriented and that the orientation is compatible with the grading on S, as specified in the lemma.

Not every orientable vector bundle admits a Dirac symbol; for instance, the tangent bundle to the orientable manifold $SU(3)/SO(3)$ does not[1]. There is however a simple sufficient condition: if V is (the underlying real vector bundle of) a complex hermitian bundle, it admits a Dirac symbol, namely the one given in Example 6.4.

We shall say a bit more later about conditions necessary to guarantee the existence of Dirac symbols. But let us note now that Dirac symbols are not necessarily unique. Indeed, if $c\colon V \to \mathrm{End}(S)$ is a Dirac symbol and if L is a complex line bundle, then the object

$$c \otimes \underset{L}{\mathrm{id}}\colon V \to \mathrm{End}(S \otimes L)$$

is also a Dirac symbol.

6.15. LEMMA. *Let* $c_1\colon V \to \mathrm{End}(S_1)$ *and* $c_2\colon V \to \mathrm{End}(S_2)$ *be two Dirac symbols associated to an even-dimensional, oriented euclidean vector bundle* V*. The formula*

$$L = \underset{V}{\mathrm{Hom}}(S_1, S_2)$$

defines a line bundle, for which $c_2\colon V \to \mathrm{End}(S_2)$ *is isomorphic to the tensor product*

$$c_1 \otimes \underset{L}{\mathrm{id}}\colon V \to \mathrm{End}(S_1 \otimes L).$$

PROOF. By $\mathrm{Hom}_V(S_1, S_2)$ we mean the vector bundle whose fibers are the complex linear maps from the fibers of S_1 to the fibers of S_2 which are of even $\mathbb{Z}/2$-grading degree and which commute with the action of the fibers of V. Neglecting the orientation condition, the fact that the action of the fiber V_x corresponds to an irreducible representation of the Clifford algebra $\mathbb{C}(V_x)$ proves that the fibers of $\mathrm{Hom}_V(S_1, S_2)$ are one-dimensional vector spaces (this is Schur's Lemma in representation theory). The fact that the elements of $\mathrm{Hom}_V(S_1, S_2)$ are grading-preserving follows from our orientation assumption. The isomorphism in the statement of the lemma comes from the canonical evaluation map

$$S_1 \otimes \underset{V}{\mathrm{Hom}}(S_1, S_2) \to S_2,$$

so the proof of the lemma is complete. □

[1] The proof of this result is out of reach using the techniques we have developed so far.

4. Chern Character of Dirac Symbols

If $c\colon V \to \operatorname{End}(S)$ is a Dirac symbol then so is the adjoint map $c^*\colon V \to \operatorname{End}(S^*)$. The construction in Lemma 6.15 therefore allows us to associate to a single Dirac symbol $c\colon V \to \operatorname{End}(S)$ a line bundle E_c by the formula

$$E_c = \operatorname*{Hom}_V(S^*, S).$$

Note that by the canonical evaluation map given in the proof of Lemma 6.15, $E_c \otimes S^* \cong S$.

6.16. EXERCISE. Show that if L is an auxiliary line bundle, and if $\sigma_L = c \otimes \mathrm{id}_L$, then $E_{c_L} \cong E_c \otimes L \otimes L$.

6.17. EXERCISE. Show that if V is an n-dimensional complex vector bundle and if $b\colon V \to \operatorname{End}(\wedge^* V)$ is the Thom element, viewed as a Dirac symbol, then $E_b = \wedge^n V$. (This is a little tricky.)

Our aim is to prove the following result:

6.18. PROPOSITION. *Let V be an euclidean vector bundle of rank $2k$ over a compact manifold M. If $c\colon V \to \operatorname{End}(S)$ is a Dirac symbol, then*

$$\operatorname{ch}(c) = (-1)^k \sqrt{\operatorname{ch}(E_c)}\sqrt{\tau(V \otimes \mathbb{C})}u_V \in H^*(W)$$

where $u_V \in H^*(V)$ is the cohomology Thom class of V and τ is the multiplicative characteristic class of complex vector bundles associated to the power series $(1 - e^x)/x$.

The formula requires a little bit of interpretation. Note that the classes $\tau(V \otimes \mathbb{C})$ and $\operatorname{ch}(E_c)$ both are elements of the graded ring $H^*(M)$ and have degree zero term equal to 1. Square roots of such elements (in any graded ring) may be defined by the usual binomial formula

$$(1 + x)^{\frac{1}{2}} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \cdots .$$

In fact, since E_c is a line bundle, its Chern character is simply e^x, where $x = c_1(E_c)$ is the Chern class of E_c. The square root of this Chern character is of course just $e^{x/2}$.

PROOF OF THE PROPOSITION (SKETCH). Consider the euclidean vector bundle $V \oplus V$. There are *two* natural ways to construct a Dirac symbol on this bundle:

(i) We may form the product symbol[2] $c \sharp c$ of two copies of the given Dirac symbol c;

[2] See Exercise 6.8.

(ii) Ignoring the given symbol c entirely, we may consider $V \oplus V$ as the underlying real vector bundle of the complex vector bundle $V \otimes \mathbb{C}$, and then form the associated Dirac symbol according to example 6.4.

Call these Dirac symbols c_1, c_2, acting on S_1, S_2 respectively. According to Lemma 6.15 there is a line bundle L such that $S_2 = S_1 \otimes L$ and $c_2 = c_1 \otimes 1$. What is this line bundle? We shall see that it is simply E_c^*, the dual of the canonical line bundle associated to the original Dirac symbol.

Once we have this information we may proceed as follows. Let $c \in K(V)$ be the K-theory class of the Dirac symbol $c\colon V \to \mathrm{End}(S)$, and let $\mathfrak{a}(c)$ be the cohomology class determined by the formula

$$\mathrm{ch}(c) = \mathfrak{a}(V) u_V \in H^*(V).$$

By following the same line of reasoning that we used in the last lecture, one can show that $\mathfrak{a}(c)$ is multiplicative, in the sense that $\mathfrak{a}(c'\sharp c'') = \mathfrak{a}(c') \cdot \mathfrak{a}(c'')$. In particular we have

$$\mathrm{ch}(c_1) = \mathfrak{a}(c)^2 \cdot u_{V\otimes\mathbb{C}} \in H^*(V \otimes \mathbb{C}).$$

On the other hand, c_2 is just the K-theory Thom class associated to the complex vector bundle $V \otimes C$, so according to Theorem 5.16 from the previous lecture,

$$\mathrm{ch}(c_2) = \tau(V \otimes \mathbb{C}) u_{V\otimes\mathbb{C}}.$$

Using $S_2 = S_1 \otimes E_c^*$, we obtain

$$\tau(V \otimes \mathbb{C}) = \mathfrak{a}(c)^2 \, \mathrm{ch}(E_c^*),$$

and so

$$\mathfrak{a}(c) = \pm\sqrt{\mathrm{ch}(E_c)}\sqrt{\tau(V \otimes \mathbb{C})}.$$

This is what we want, except that we have to check that the sign is correct. To do this we just need to work out the degree zero part of $\mathfrak{a}(c)$ in $H^0(M)$, and to do this we can restrict the bundle V to a single point in M. Here, over a single point, we can give V a complex structure, and since all line bundles over a point are trivial, the restriction of the Dirac symbol to our point is isomorphic to the Bott element for V over this point (considered as a complex vector space). The computations in the previous lecture now tell us that the correct sign is $(-1)^k$ (where k is the complex dimension of the restricted V).

It remains to explain why it is that $S_2 = S_1 \otimes E_c^*$. Begin by considering $\mathbb{C}(V)$, the bundle of Clifford algebras over V. Like any algebra, the Clifford algebra is of course a *bimodule* over itself, using the actions of left and right multiplication. These actions commute: but we can make them anticommute instead by introducing a small twist from the grading automorphism α:

$$L(u)\cdot x = ux, \qquad R(v)\cdot x = \alpha(x)v.$$

Now L and R define anticommuting Clifford symbols for V, so the pair (L,R) defines a Clifford symbol for $V\oplus V$, which (by dimension counting) must in fact be a Dirac symbol. In fact, this Dirac symbol is a familiar one: there is a canonical isomorphism of vector spaces from $\mathbb{C}(V)$ to the complexified exterior algebra of V, and under this isomorphism the symbol (L,R) just passes to the symbol of Example 6.4. In other words, the bundle $\mathbb{C}(V)$, considered as a Dirac bundle over $V\oplus V$ by the action (L,R), just is S_2.

But we can apply our knowledge of the representation theory of the Clifford algebra to understand $\mathbb{C}(V)$. We know that the Clifford algebra is a matrix algebra over its spin space: thus, $\mathbb{C}(V) = \mathrm{End}(S) = S^*\otimes S$. The effect of the 'twist' that we introduced above to make the left and right actions anticommute is to replace the ordinary tensor product here by a *graded* tensor product, so that

$$S_2 = S^*\widehat{\otimes} S.$$

On the other hand,

$$S_1 = S\widehat{\otimes} S,$$

so the desired result $S_2 = S_1\otimes E_c^*$ now follows from the definition of the bundle E_c. □

6.19. DEFINITION. We shall call a (symmetric, first-order) differential operator a *Dirac operator* if its symbol is a Dirac symbol associated to T^*M.

A Dirac operator is necessarily elliptic. Our calculation of the Chern character of Dirac symbols allows us to write out the index formula for Dirac operators in fairly explicit terms.

6.20. THEOREM. *Let D be a Dirac operator associated to a Dirac symbol σ on a compact oriented (even-dimensional) manifold M. Then*

$$\mathrm{Index}(D) = \int_M \sqrt{\mathrm{ch}(E_\sigma)}\,\widehat{A}(TM).$$

where the genus $\widehat{A}(TM)$ is defined in Example 4.35. □

PROOF. We will deduce this from the general form of the Index Theorem 5.1. The idea is the same as in Example **??**.

Substituting the formula of Proposition 6.18, which gives the Chern character of the symbol, into the index theorem 5.1, we get

$$\operatorname{Index}(D) = \int_{T^*M} \sqrt{\operatorname{ch}(E_\sigma)}\sqrt{\tau(TM \otimes \mathbb{C})}\operatorname{Todd}(TM \otimes \mathbb{C})u_{TM}.$$

Now the class τ is just the inverse of the Todd class (see Exercise 5.18; there are no signs because we are in the even-dimensional case). Using this, and integrating over the fiber, we get

$$\operatorname{Index}(D) = \int_M \sqrt{\operatorname{ch}(E_\sigma)}\sqrt{\operatorname{Todd}(TM \otimes \mathbb{C})}.$$

But according to exercise 4.36, the $\widehat{A}$ genus is the square root of the Todd genus of the complexification. This completes the proof. □

6.21. EXAMPLE. If M is a complex manifold then the symbol of the *Dolbeault operator* $D = \bar\partial + \bar\partial^*$ acting on $S = \wedge^{0,*}T^*_{\mathbb{C}}M \cong \wedge^* TM$ is a Dirac symbol. The line bundle L_S is the dual of the *canonical line bundle*: $E_{\sigma_D} = \wedge^n TM$ (we form the highest exterior power using the complex structure on TM).

5. Spin^c-Structures and Principal Bundles

This short section is optional, and aimed at people with some familiarity with principal bundle theory.

6.22. DEFINITION. A Spin^c-*structure* on a Riemannian manifold is an isomorphism class of Dirac symbols associated to T^*M.

Let us discuss in more detail the problem of determining whether or not an oriented Riemannian manifold admits a Spin^c structure.

Consider the complex Clifford algebra of $\mathbb{R}^{2k}$. It is isomorphic to matrix algebra $M_{2^k}(\mathbb{C})$:

$$\operatorname*{Cliff}_{\mathbb{C}}(\mathbb{R}^{2k}) \cong M_{2^k}(\mathbb{C}).$$

Let us fix such an isomorphism. The group $SO(2k)$ acts on $\mathbb{R}^{2k}$ and therefore on $\operatorname{Cliff}_{\mathbb{C}}(\mathbb{R}^{2k})$, and therefore on $M_{2^k}(\mathbb{C})$ via the given, fixed, isomorphism. Since every automorphism of $M_{2^k}(\mathbb{C})$ is induced from an automorphism of C^{2^k}, which is determined up to a scalar multiple of the identity, we obtain a group homomorphism

$$SO(2k) \to U(2^k)/Z,$$

where Z denotes the center of the unitary group $U(2^k)$ (Z is isomorphic to S^1 and consists of scalar multiples of the identity).

6.23. Definition. Denote by $\mathrm{Spin}^c(2k)$ the group which fits into the pullback diagram

$$\begin{array}{ccccc} S^1 & \longrightarrow & \mathrm{Spin}^c(2k) & \longrightarrow & SO(2k) \\ {\scriptstyle =}\downarrow & & \downarrow & & \downarrow \\ Z & \longrightarrow & U(2^k) & \longrightarrow & U(V)/Z \end{array}$$

Observe that the group $\mathrm{Spin}^c(2k)$ comes with canonical representations on the spaces $\mathbb{R}^{2k}$ and $\mathbb{C}^{2^k}$. The map $\mathbb{R}^{2k} \otimes \mathbb{C}^{2^k} \to \mathbb{C}^{2^k}$ which is induced from our fixed isomorphism $\mathbb{C}(\mathbb{R}^{2k}) \cong M_{2^k}(\mathbb{C})$ is $\mathrm{Spin}^c(2k)$-equivariant.

One can prove the following result.

6.24. Theorem. *An oriented Riemannian* $2k$*-manifold admits a* Spin^c*-structure if and only if the principal* $SO(2k)$*-bundle* F *of oriented frames admits a reduction* $\widetilde{F}$ *to the group* $\mathrm{Spin}^c(2k)$*. In this case, the cotangent bundle* TM *is given by*

$$T^*M = \widetilde{F} \times_{\mathrm{Spin}^c(2k)} \mathbb{R}^{2k},$$

and the formula

$$S = \widetilde{F} \times_{\mathrm{Spin}^c(2k)} \mathbb{C}^{2^k}$$

defines a hermitian bundle equipped with an action $T^*M \otimes S \to S$ *which is a Dirac symbol. In this way,* Spin^c*-structures correspond bijectively to reductions of the oriented frame bundle to* $\mathrm{Spin}^c(2k)$. □

6. Spin-Structures

Let us finish by making some remarks about Dirac operators associated to real (as opposed to complex) Dirac symbols. In the real case, we shall restrict to manifolds of dimension $8n$ (for the definitions we have given here). This is to accommodate the following result:

6.25. Proposition. *The real Clifford algebra* $\mathbb{R}(\mathbb{R}^{8m})$ *is isomorphic to the matrix algebra* $M_{2^{4k}}(\mathbb{R})$. □

We define real Dirac symbols in the $8k$-dimensional case by putting a minimal dimensionality requirement on the bundle S (we shall orient M compatibly with the symbol). We define a Spin-structure to be an isomorphism class of real Dirac symbols.

By following a similar reasoning to that used in the previous section one can prove:

6.26. Theorem. *An oriented Riemannian* $8k$*-manifold admits a Spin-structure if and only if the principal* $SO(8k)$*-bundle* F *of oriented frames* *admits a reduction $\widetilde{F}$ to the group $\mathrm{Spin}(8k)$. Here $\mathrm{Spin}(8k)$ is the double cover of $SO(8k)$ which fits into the diagram*

$$\begin{array}{ccccc} \mathbb{Z}/2 & \longrightarrow & \mathrm{Spin}(8k) & \longrightarrow & SO(8k) \\ \downarrow{=} & & \downarrow & & \downarrow \\ Z & \longrightarrow & SO(2^{4k}) & \longrightarrow & SO(2^{4k})/Z \end{array}$$

(one can show that $\mathrm{Spin}(8n)$ *is the simply connected double cover of $SO(8n)$). In this case, the bundle S is given by*

$$S = \widetilde{F} \times_{\mathrm{Spin}(8n)} \mathbb{R}^{2^{4k}},$$

and the action of T^*M on it is induced from the action of $\mathbb{R}^{8n}$ on $\mathbb{R}^{2^{4k}}$.

If the Dirac symbol $c\colon T^*M \to \mathrm{End}(S)$ is the complexification of a *real* Dirac symbol then the line bundle L_c is trivial. In this case the index formula reads quite simply

$$\mathrm{Index}(D) = \int_M \widehat{A}(M).$$

One of the interesting features of this formula is that in the real case, thanks to the fact that the reduction $\widetilde{F}$ is a covering space of the frame bundle F, there is a natural connection on $\widetilde{F}$ which gives rise to a natural affine connection on S, and ultimately a canonical operator (defined in terms of the Riemannian geometry of M) whose symbol is the Dirac symbol, namely

$$D = \sum \sigma(\omega_i)\nabla_{X_i},$$

where the sum is over a local frame $\{X_i\}$ and dual frame $\{\omega_i\}$. This operator has the following important property, known as the *Lichnerowicz formula*:

$$D^2 = \nabla^*\nabla + \frac{\kappa}{4},$$

where κ is the scalar curvature[3] function of M. Hence:

6.27. THEOREM. *Let M be a Riemannian manifold which admits a Spin structure. If the scalar curvature of M is everywhere positive then* $\int_M \widehat{A}(M) = 0$.

PROOF. If $\kappa > 0$ then by the Lichnerowicz formula the Dirac operator is bounded below, and is therefore invertible. Hence its index is zero. □

[3]Results of this type, known as *Bochner-Weitzenbock formulae*, can be proved for many natural geometric operators; the point about the Dirac operator associated to a Spin-structure is that the curvature term which appears in all such formulae is of a particularly simple sort.

CHAPTER 7

The Tangent Groupoid

In this chapter we will construct the homomorphism $\alpha_M \colon K(T^*M) \to \mathbb{Z}$ which maps the symbol class of an elliptic operator D to the index of D. The construction will be made using Connes' notion of *tangent groupoid*.

1. Smooth Groupoids

What is non-commutative geometry? Recall that a governing idea in all sorts of ordinary geometry is that features of geometric spaces are reflected within the algebras of their coordinate functions. For instance, if X is a compact Hausdorff space then one can recover X as the space of maximal ideals of the algebra $C(X)$ of continuous, complex-valued functions on X. Alain Connes' non-commutative geometry is concerned with aspects of the space–algebra correspondence which, on the algebra side, involve Hilbert space methods, particularly the spectral theory of operators on Hilbert space. Moreover it is a guiding principle of the theory that *non-commutative* algebras may often arise from geometric situations, and that one should as far as possible treat non-commutative and commutative algebras by *similar* geometric methods. One natural way to formalize this idea is by means of the theory of which we will develop in this chapter.

Let us begin with the following definition, which is short, but probably opaque to anyone who has not encountered it before.

7.1. DEFINITION. A *smooth groupoid* is a small category in which every morphism is invertible, and for which the set of all morphisms and the set of all objects are given the structure of smooth manifolds; the source and range maps are submersions; and the composition law and inclusion of identities are smooth maps.

In a more detail, a smooth groupoid consists of, to begin with, a manifold G, whose points constitute the morphisms (all of them, between any two objects) in some category; a smooth manifold B whose points are the objects in the category; and two maps $r, s \colon G \to B$ which associate to morphisms their range and source objects and which are required to be submersions[1]. It can be shown then that the set

$$G^2 = \{ (\gamma_1, \gamma_2) \in G \times G \,:\, s(\gamma_1) = r(\gamma_2) \}$$

of composable pairs of morphisms is a smooth submanifold of $G \times G$. With this in hand we require further that the composition operation $(\gamma_1, \gamma_2) \mapsto \gamma_1 \circ \gamma_2$ be a smooth map from G^2 to G. We also require that the map $e \colon B \to G$ which maps an object $x \in B$ to the identity morphism at x be smooth. Finally, we require that every morphism in the category be invertible; it may be shown that the map $\gamma \mapsto \gamma^{-1}$ from G to itself is automatically a diffeomorphism.

In noncommutative geometry it is customary to paint what might be called the *quotient space picture* of groupoid theory. In this view, one thinks of the morphisms in G as defining an equivalence relation on the object space B: two objects are equivalent if there is a morphism between them. Two objects might be equivalent for more than one reason, and the groupoid keeps track of this. It is customary in mathematics to form the quotient space from an equivalence relation, but even in rather simple examples the ordinary quotient space of general topology can be highly singular, and for example not at all a manifold. The groupoid serves as a smooth stand-in for the quotient space in these situations, and using it one can study the cohomology of the quotient space, and even its geometry. These ideas are developed extensively in Connes' book [].

A second view of groupoid theory, which is better suited to our present purposes, is what we shall call the *families picture*. We shall think of the groupoid first as the family of smooth manifolds

$$G_x = \{ \gamma \in G \,:\, s(\gamma) = x \}$$

parametrized by $x \in B$. If η is a morphism in G from x to y, then there is an associated diffeomorphism

$$R_\eta \colon G_y \to G_x$$

defined by $R_\eta(\gamma) = \gamma \circ \eta$. We shall therefore think of G as being a smooth family of smooth manifolds, provided with the collection of all the intertwining diffeomorphisms R_η. From this point of view, having been given a groupoid G it will be very natural to consider families of say differential operators D_x, one on each G_x, which are equivariant with respect to the R_η, in the obvious sense.

7.2. EXAMPLE. A Lie group G may be viewed as a smooth groupoid. The object set is a single-element set, and the set of morphisms from this

[1]Recall that a smooth map between manifolds is a submersion if in suitable local coordinates it has the form of a projection $(x_1, \dots, x_{p+q}) \mapsto (x_1, \dots, x_p)$.

single element to itself is G. In the families picture, we have one manifold — the underlying smooth manifold of G — and a family of self-maps of this manifold, given by the usual right-translation operators on a group. An equivariant operator in this example is a right-translation-invariant operator on the Lie group G. Thus if for example $G = \mathbb{R}^n$, then an equivariant differential operator is nothing but a constant coefficient operator on $\mathbb{R}^n$.

7.3. EXAMPLE. Let M be a smooth manifold. The *pair groupoid* of M has object space M, and morphism space $G = M \times M$. Its structure maps are as follows:

- Source map: $s(m_2, m_1) = m_1$.
- Range map $r(m_2, m_1) = m_2$.
- Composition: $(m_3, m_2) \circ (m_2, m_1) = (m_3, m_1)$.
- Inclusion of identities: $m \mapsto (m, m)$.

The spaces G_m all identify with M, and the translation operators $G_{m_2} \to G_{m_1}$ all become the identity map under these identifications. An equivariant family of operators in this example is nothing more than a single, but general, operator on the manifold M.

7.4. EXAMPLE. The previous two examples can be combined, after a fashion, as follows. Let A be a Lie group which acts (on the left) on a smooth manifold M. The *transformation groupoid* $A \ltimes M$ has object space M and the following morphism space:

$$\{ (m_2, a, m_1) \in M \times A \times M \, : \, m_2 = a m_1 \}.$$

Obviously the morphism space identifies with the product $A \times M$ by projection onto the last two factors, but the above description makes the structure maps more transparent:

- Source map: $s(m_2, a, m_1) = m_1$.
- Range map: $r(m_2, a, m_1) = m_2$.
- Composition: $(m_3, a_2, m_2) \circ (m_2, a_1, m_1) = (m_3, a_2 a_1, m_1)$.
- Inclusion of identities: $m \mapsto (m, e, m)$.

The inverse of (m_2, a, m_1) is (m_1, a^{-1}, m_2). However an equivariant family of operators is not, as one might guess, the same thing as an A-equivariant operator on M. Instead it is a family of operators D_m on A, parametrized by $m \in M$, for which the operator D_m is equivariant for the right translation action of the isotropy subgroup A_m on A.

2. Foliation Groupoids

3. The Tangent Groupoid

Let M be a smooth manifold. The tangent groupoid is a smooth groupoid whose object space is the product $M \times \mathbb{R}$. In the families picture the tangent groupoid of M consists of repeated copies of M, together with the tangent spaces $T_m M$. These will ultimately be joined together to form the fibers of a single smooth map $s\colon \mathbb{T}M \to M \times \mathbb{R}$. But let us begin by describing $\mathbb{T}M$ as a topological space.

7.5. DEFINITION. Let M be a smooth, open manifold. Denote by $\mathbb{T}M$ the set

$$\mathbb{T}M = TM \times \{0\} \cup M \times M \times \mathbb{R}^{\times}$$

(a disjoint union) equipped with the following topology:

(i) Any open subset of $M \times M \times \mathbb{R}^{\times}$ is deemed to be an open set in $\mathbb{T}M$.

(ii) Let X be a tangent vector on M, let $f\colon M \to \mathbb{C}$ be a smooth function and let $\varepsilon > 0$. The set $U_{f,\varepsilon} \subseteq \mathbb{T}M$ defined by

$$U_{f,\varepsilon} \cap TM \times \{0\} = \{ (Y, 0) \, : \, |X(f) - Y(f)| < \varepsilon \}$$

and

$$U_{f,\varepsilon} \cap M \times M \times \mathbb{R}^* = \left\{ (m_2, m_2, t) \, : \, \left| X(f) - \frac{f(m_2) - f(m_1)}{t} \right| < \varepsilon \right\}$$

is an open neighbourhood of X in $\mathbb{T}M$, and the set of finite intersections of such sets forms a neighborhood base at X.

7.6. REMARK. We are thinking here of a triple (m_2, m_1, t) as being an "approximate tangent vector" which is close to a real tangent vector X if the difference quotient $|f(m_2) - f(m_1)|/t$ is close to $X(f)$.

The topology on $\mathbb{T}M$ is easily seen to be Hausdorff. Moreover it is locally Euclidean:

7.7. LEMMA. *Let M be a smooth, open manifold. If U is an open subset of M then the set*

$$\mathbb{T}U = TU \times \{0\} \cup U \times U \times \mathbb{R}^{\times}$$

is an open subset of $\mathbb{T}M$. Moreover if $\phi\colon U \to \mathbb{R}^n$ is a diffeomorphism onto an open subset then the map

$$\Phi\colon \mathbb{T}U \to \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$$

defined by

$$\begin{cases} \Phi(X, m, 0) = (D\phi_m(X), \phi(m), 0) \\ \Phi(m_2, m_1, t) = (t^{-1}(\phi(m_2) - \phi(m_1)), \phi(m_1), t) \end{cases}$$

is a homeomorphism onto an open subset. □

7.8. REMARK. For clarity we are using the redundant notation (X, m), where $X \in T_m U$, to describe points of TU. We denote by $D\phi_m\colon T_m U \to \mathbb{R}^n$ the derivative of ϕ at $m \in U$.

7.9. EXERCISE. Prove the lemma.

The maps Φ defined in the lemma determine an atlas of charts for the smooth manifold $\mathbb{T}M$:

7.10. LEMMA. *Let* $\Phi\colon \mathbb{T}U \to \mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}$ *and* $\Psi\colon \mathbb{T}V \to \mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}$ *be the maps associated to diffeomorphisms* $\phi\colon U \to \mathbb{R}^n$ *and* $\psi\colon V \to \mathbb{R}^n$, *as in the previous lemma. The composition* $\Psi\circ\Phi^{-1}$ *is defined on an open subset of* $\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}$, *and is a smooth map.*

PROOF. The inverse Φ^{-1} is given by the formula

$$\Phi^{-1}(v_2, v_1, t) = \begin{cases} (\phi^{-1}(tv_2+v_1), \phi^{-1}(v_1), t) & \text{if } t \neq 0 \\ (D\phi^{-1}_{v_1}(v_2), \phi^{-1}(v_1), 0) & \text{if } t = 0. \end{cases}$$

Using the notation $\theta = \psi\circ\phi^{-1}$, the composition $\Theta = \Psi\circ\Phi^{-1}$ is given by the formula

$$\Theta(w_2, w_1, t) = \begin{cases} (t^{-1}\big(\theta(tw_2+w_1) - \theta(w_1)\big), \theta(w_1), t) & \text{if } t \neq 0 \\ (D\theta_{w_1}(w_2), \theta(w_1), 0) & \text{if } t = 0. \end{cases}$$

By a version of the Taylor expansion, there is a smooth, matrix-valued function $\widetilde{\theta}(h, w)$ such that

$$\theta(h+w) = \theta(w) + \widetilde{\theta}(h,w)h \quad \text{and} \quad \widetilde{\theta}(0,w) = D\theta_w.$$

So we see that

$$\Theta(w_2, w_1, t) = \begin{cases} \big(\widetilde{\theta}(tw_2, w_1)w_2, \theta(w_1), t\big) & \text{if } t \neq 0 \\ \big(D\theta_{w_1}(w_2), \theta(w_1), 0\big) & \text{if } t = 0. \end{cases}$$

This is clearly a smooth function. □

We have therefore obtained a smooth manifold $\mathbb{T}M$. It is clear that the map

$$s\colon \mathbb{T}M \to M\times\mathbb{R}$$

defined by $s(X, m, 0) = (m, 0)$ and $s(m_2, m_1, t) = m_1$ is a submersion. The fibers of s are T_mM at $(m, 0)$ and M at (m, t), when $t \neq 0$. The remaining groupoid structure maps are as follows.

7.11. DEFINITION. Let M be a smooth, open manifold. The *tangent groupoid* of M is the groupoid with morphism set $\mathbb{T}M$, object set $M\times\mathbb{R}$, and the following structure maps (in which $t \neq 0$ in every formula):

- Source map: $s(X, m, 0) = (m, 0)$ and $s(m_2, m_1, t) = (m_1, t)$.
- Range map: $r(X, m, 0) = (m, 0)$ and $r(m_2, m_1, t) = (m_2, t)$.
- Composition: $(X, m, 0)\circ(Y, m, 0) = (X+Y, m, 0)$ and $(m_3, m_2, t)\circ(m_2, m_1, t) = (m_3, m_1, t)$.
- Inclusion of identities: $(m, 0) \mapsto (0, m, 0)$ and $(x, t) \mapsto (m, m, t)$.

Inverses are given by the formulas $(x,v)^{-1} = (x,-v)$ and $(x,y,t)^{-1} = (y,x,t)$.

It is clear that if we look at the subset $\mathbb{T}M$ consisting of those morphisms attached to a fixed t (including possibly $t=0$) then the above operations provide "slice" of $\mathbb{T}M$ with a groupoid structure on its own, whose object space is M. When $t=0$ we get the tangent bundle TM: the source and range maps are both the projection onto M, and the composition law is addition of tangent vectors. When $t \neq 0$ we get the pair groupoid of M. Thus, algebraically, $\mathbb{T}M$ is the union of multiple copies of the pair groupoid of M and one copy of the tangent bundle, viewed as a groupoid. Let us show that the overall package is a smooth groupoid, first in the special case where $M = \mathbb{R}^n$:

7.12. EXAMPLE. The map $\Phi\colon \mathbb{T}\mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ defined by

$$\Phi(v_2, v_1, 0) = (v_2, v_1, 0)$$
$$\Phi(v_2, v_1, t) = (t^{-1}(v_2 - v_1), v_1, t) \quad (t \neq 0)$$

is a diffeomorphism. Now consider the space

$$G = \{ (w_2, a, w_1) : w_1, w_2 \in \mathbb{R}^n \times \mathbb{R}, a \in \mathbb{R}^n, w_2 = a \triangle w_1 \},$$

where the operation $\triangle$ is defined by

$$a \triangle (v, t) = (v + ta, t).$$

Thus the $\triangle$ operation defines an action of the group $A = \mathbb{R}^n$ on $\mathbb{R}^n \times \mathbb{R}$, and our space G is the corresponding transformation groupoid. The space G identifies with $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ by dropping w_2 from (w_2, a, w_1). Using this, we can consider the diffeormorphism Φ to be a diffeomorphism $\Phi\colon \mathbb{T}M \to G$ by the formulas

$$\Phi(v_2, v_1, 0) = ((v_1, 0), v_2, (v_1, 0))$$
$$\Phi(v_2, v_1, t) = \big((v_2, t), t^{-1}(v_2 - v_1), (v_1, t)\big) \quad (t \neq 0).$$

Using these, it is evident that Φ is actually an isomorphism of groupoids, from which it follows that the groupoid structure on $\mathbb{T}\mathbb{R}^n$ is smooth.

To summarize:

7.13. PROPOSITION. *Denote by* $G = \mathbb{R}^n \ltimes \mathbb{R}^{n,1}$ *the transformation groupoid associated to the action of* $\mathbb{R}^n$ *on the space* $\mathbb{R}^{n,1} = \mathbb{R}^n \times \mathbb{R}$ *given by the formula*

$$a \triangle (v, t) = (v + ta, t) \qquad (a \in \mathbb{R}^n \text{ and } (v, t) \in \mathbb{R}^{n,1}).$$

The map $\Psi\colon \mathbb{TR}^n \to G$ *which is given by the formulas*

$$\Phi(v_2, v_1, 0) = ((v_1, 0), v_2, (v_1, 0))$$
$$\Phi(v_2, v_1, t) = \big((v_2, t), t^{-1}(v_2 - v_1), (v_1, t)\big) \quad (t \neq 0).$$

is an isomorphism of smooth groupoids. □

7.14. REMARK. The groupoid $\mathbb{TR}^n$ only depends on the smooth structure of $\mathbb{R}^n$, whereas, superficially at least, the groupoid $G = \mathbb{R}^n \ltimes \mathbb{R}^{n,1}$ depends very much on the vector space structure of $\mathbb{R}^n$. The proposition shows that this dependence is an illusion.

7.15. PROPOSITION. *The structure maps are all smooth, and the source and range maps are submersions. Thus* $\mathbb{T}M$ *is a smooth groupoid.*

PROOF. Since smoothness is a local property, we can check this in a coordinate neighbourhood U. Since the construction of $\mathbb{T}U$ is coordinate-independent we can assume that $U = \mathbb{R}^n$, and thereby reduce to the example just considered. □

4. Groupoid Algebras

We are going to associate to a smooth groupoid a convolution C^*-algebra, generalizing the reduced group C^*-algebra of a Lie group.

7.16. DEFINITION. A *right Haar system* on a smooth groupoid G is a system of smooth measures, one on each of the manifolds

$$G_x = \{\gamma \in G : s(\gamma) = x\},$$

with the properties that:

(i) If f is a smooth, compactly supported function on G then $\int_{G_x} f(\gamma)\, d\mu_x(\gamma)$ is a smooth function of x.

(ii) If η is a morphism from x to y then

$$\int_{G_x} f(\gamma)\, d\mu_x(\gamma) = \int_{G_y} f(\gamma \circ \eta)\, d\mu_y(\gamma).$$

7.17. PROPOSITION. *Every smooth groupoid admits a right Haar system.* □

The proposition can be proved by adapting the standard construction of Haar measures on Lie groups: pick a 1-density (basically a top degree differential form) on G_x at the point id_x and do so in a way which varies smoothly with x. Then right-translate the densities around G to define a 1-density at every point with the required properties.

7.18. DEFINITION. Let G be a smooth groupoid with right Haar system. Define a convolution multiplication and adjoint on the space $C_c^\infty(G)$ of smooth, compactly supported complex functions on the morphism space of G using the formulas

$$f_1 \star f_2(\gamma) = \int_{G_{s(\gamma)}} f_1(\gamma \circ \eta^{-1}) f_2(\eta)\, d\mu_{s(\gamma)}(\eta).$$

and

$$f^*(\gamma) = \overline{f(\gamma^{-1})}.$$

7.19. PROPOSITION. *Let* G *be a smooth groupoid with right Haar system. With the above operations,* $C_c^\infty(G)$ *is an associative* $*$*-algebra.* □

7.20. DEFINITION. Let G be a smooth groupoid with right Haar system. Define representations

$$\lambda_x \colon C_c^\infty(G) \to \mathcal{B}(L^2(G_x))$$

by the formulas

$$\lambda_x(f)h(\gamma) = f \star h(\gamma) = \int_{G_{s(\gamma)}} f(\gamma \circ \eta^{-1}) h(\eta)\, d\mu_{s(\gamma)}(\eta).$$

The *reduced groupoid* C^**-algebra* of G, denoted $C^*_\lambda(G)$, is the completion of $C_c^\infty(G)$ in the norm

$$\|f\| = \sup_x \|\lambda_x(f)\|_{B(L^2(G_x))}.$$

7.21. EXAMPLE. If $G = M \times M$ (the pair groupoid), then in any Haar system all the measures μ_m on $G_m = M \times \{m\} \cong M$ are equal to one another and conversely any smooth measure μ determines a Haar system. The convolution multiplication and adjoint are

$$f_1 \star f_2(m_2, m_1) = \int_M f_1(m_2, m) f_2(m, m_1)\, d\mu(m).$$

and

$$f^*(m_2, m_1) = \overline{f(m_1, m_2)}.$$

The groupoid C^*-algebra $C^*_\lambda(G)$ is the C^*-algebra of compact operators on $L^2(M)$.

7.22. EXAMPLE. If $G = TM$ (the tangent bundle) then a Haar system is a smoothly varying system of translation-invariant measures on the vector spaces T_mM. Since a translation-invariant measure on T_mM is the same thing as a point in $\bigwedge^n T^*_mM$, we see that a smooth Haar system on TM is determined by a smooth measure on M. The convolution multiplication and adjoint in the groupoid algebra are

$$f_1 \star f_2(X, m) = \int_{T_m M} f_1(X - Y, m) f_2(Y, m)\, d\mu(Y).$$

and

$$f^*(X, m) = \overline{f(-X, m)}.$$

The groupoid C^*-algebra is therefore, so to speak, a bundle of C^*-algebras over M, whose fiber at $m \in M$ is the group C^*-algebra of $T_m M$. Consider the tangent bundle TM. But to get a clearer picture of it, let us invoke some Fourier theory, as follows. The Fourier transform

$$\widehat{h}(\eta) = \int_{T_m M} e^{-i\eta(X)} h(X)\, dX \qquad (\eta \in T^*_m M)$$

determines an isometric isomorphism $L^2(T_m M) \cong L^2(T^*_m M)$ (for a suitable Haar measure on $T^*_m M$). If $f \in C_c(TM)$ then let

$$\widehat{f}(\eta, m) = \int_{T_m M} e^{-i\eta(Y)} f(Y, m)\, dy.$$

The function $\widehat{f}$ is continuous and vanishes at infinity on the cotangent bundle T^*M. Since $\widehat{f \star h} = \widehat{f} \cdot \widehat{h}$ (pointwise multiplication) we obtain, with a little more work, a Fourier isomorphism

$$C^*_\lambda(TM) \cong C_0(T^*M).$$

5. The C*-Algebra of the Tangent Groupoid

Let M be a smooth manifold without boundary. To define a smooth Haar system on the tangent groupoid $\mathbb{T}M$, first fix a smooth measure μ on M. As we noted above, μ determines a family of translation invariant measures μ_m on the vector spaces $T_m M$. We define smooth measures on the fibers $\mathbb{T}M_{(m,t)}$ of the source map by the formulas

$$\mu_{m,0} = \mu_m \quad \text{on } \mathbb{T}M_{(m,0)} \cong T_x M$$

and

$$\mu_(m, t) = t^{-n}\mu \quad \text{on } \mathbb{T}M_{(m,t)} \cong M.$$

7.23. LEMMA. *The above formulas define a smooth right Haar system on* $\mathbb{T}M$.

PROOF. The measures certainly constitute a translation-invariant system (compare Examples 7.21 and 7.22 above). To prove they are smooth we shall make use of the diffeomorphisms Φ introduced in the previous section, or rather their inverses $\Theta = \Phi^{-1}\colon \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{T}U$. Let us choose coordinates on $U \subseteq M$ so that the diffeomorphism $\phi\colon U \to \mathbb{R}^n$ from which Φ is defined is the identity in these local coordinates. Then

$$\Theta(X, m, 0) = (X, m, 0) \text{ and } \Theta(m_2, m_1, t) = (m_1 + tm_2, m_1, t), \text{ if } t \neq 0.$$

If we restrict to one of the fibers of the source map then we obtain the maps

$$\Theta_{m_1,0}(X) = X \in \mathbb{T}_{v_1} U$$

and

$$\Theta_{m_1,t}(v_2) = v_1 + tv_2 \in U.$$

The derivatives of these maps (expressed as matrices, using our chosen coordinates) are I in the first case and tI in the second. Now, to transfer the measures from the fibers of $\mathbb{T}U$ to the fibers (under projection onto the last two factors) of $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$, we must multiply by the determinant of these derivative matrices. That is, if $\Theta\colon A \to B$ is a diffeomorphism between open sets in $\mathbb{R}^k$, and if $\mu(b) = m(b)db$ is a smooth measure on B, then

$$\int_B f(b)\, d\mu(b) = \int_B f(b) m(b)\, db = \int_A f(\Theta(a)) m(\Theta(a)) \det(D\Theta_a)\, da.$$

In our case we see that the factor t^{-n} in the definition of $\mu_{m,t}$ cancels with $\det(D\Theta) = t^n$, and we obtain smoothly varying measures, as required. □

The C^*-algebra of the tangent groupoid comes equipped with a family of restriction $*$-homomorphisms

$$\varepsilon_0\colon C^*_\lambda(\mathbb{T}M) \to C_0(T^*M)$$

and, for $t \neq 0$,

$$\varepsilon_t\colon C^*_\lambda(\mathbb{T}M) \to \mathcal{K}(L^2(M)).$$

On the subalgebra $C_c^\infty(\mathbb{T}M)$ these are defined by restricting functions on $\mathbb{T}m$ to the "slice" of $\mathbb{T}M$ over t, which is either the tangent bundle TM (when $t = 0$) or the pair groupoid $M \times M$ (when $t \neq 0$). Strictly speaking, when $t \neq 0$ the restriction $*$-homomorphism lands in the compact operators on the Hilbert space $L^2(M, S)$ associated to the measure μ scaled by t^{-n}. But this Hilbert space is obviously unitarily equivalent to the Hilbert space associated to μ itself: just multiply by $t^{\frac{n}{2}}$. The restriction homomorphisms will be put to good use in the next section to finally construct the index homomorphism in K-theory.

6. The Index Homomorphism

The $*$-homomorphism $\varepsilon_0\colon C^*_\lambda(\mathbb{T}M) \to C_0(T^*M)$ is surjective. Let us fix a set-theoretic section $\sigma\colon C_0(T^*M) \to C^*_\lambda(T^*M)$. The map σ need have no other property than that $\varepsilon_0 \circ \sigma = \text{id}$, although it is for example possible to choose σ to be linear and continuous.

7.24. LEMMA. *The family of maps* $\alpha_t\colon C_0(T^*M) \to \mathcal{K}(L^2(M))$ *defined by the formula*

$$\alpha_t(h) = \varepsilon_{t^{-1}}(\sigma(h)) \qquad (t \in [1,\infty), h \in C_0(T^*M)),$$

is an asymptotic morphism.

PROOF. Exercise. □

7.25. DEFINITION. The index homomorphism $\alpha\colon K(T^*M) \to K(\mathrm{pt})$ is the K-theory map induced from the above asymptotic morphism $\alpha\colon C_0(T^*M) \to \mathcal{K}(L^2(M))$.

7. Elliptic Operators and the Tangent Groupoid

Let M be a smooth manifold and let D be a partial differential operator on M. To keep within the framework developed in Chapter 2, let us assume right away that D has order 1, although this assumption will only be essential later in the section.

Recall that we had associated to D a family of "model operators" D_m, which are translation-invariant partial differential operators on the tangent spaces T_mM. They were obtained by freezing the coefficients of D at m. Thus if $D = \sum a_j X^j + b$, where the a_j and b are smooth functions on M and the X^j are vector fields, then $D_m = \sum a_j(m) X^j_m$, where we view the tangent vector X^j_m as a directional derivative on T_mM.

Let us now associate to D a family of partial differential operators on the fibers of the source map for the tangent groupoid. To define the family, we shall identify the fibers associated to $t = 0$ with the tangent spaces T_mM and the fibers associated to $t \neq 0$ with M itself, in the obvious way. Having done so, we define

$$\begin{cases} D_{m,0} = D_m & \text{on } T_mM \\ D_{m,t} = tD & \text{on } M. \end{cases}$$

7.26. PROPOSITION. *The operators* $D_{m,t}$*, for* $m \in M$ *and* $t \in \mathbb{R}$*, constitute a smoothly varying family of operators on the fibers of the source map* $s\colon \mathbb{T}M \to M \times \mathbb{R}$.

7.27. REMARK. By "smoothly varying" we mean that if the family is applied fiber-wise to a smooth function on $\mathbb{T}M$ then the result is another smooth function on $\mathbb{T}M$.

PROOF. Let us verify this locally by transferring the problem to $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ using our standard diffeomorphism $\Phi\colon \mathbb{T}U \to \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$. As in the last section, let us choose coordinates on U so that Φ has the simple form

$$\Phi(X, m, 0) = (X, m, 0) \quad \text{and} \quad \Phi(m_2, m_1, t) = (t^{-1}(m_2 - m_1), m_1, t).$$

Write $D = \sum a_j \partial_j + b$. Then under Φ the family of operators $D_{m,t}$ corresponds to the family $E_{m,t} \colon C^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$ given by the formulas

$$\begin{cases} E_{m,0} = \sum a_j(m)\partial_j \\ E_{m,t} = \sum a_j(tv + m)\partial_j + tb_j(tv + m) \end{cases}$$

(these act on functions of the variable $v \in \mathbb{R}^n$). It is now clear that we have a smooth family. □

It is clear that the operators $D_{m,t}$, for all t and m, constitute an equivariant family: compare Examples 7.2 and 7.3.

We are now come to an important general fact about smooth, equivariant families of *first order, elliptic* operators on the fibers of a groupoid. Let G be a smooth groupoid. Just as we endowed G with a smooth Haar system by right-translating a smooth density on M (thought of as the space of identity morphisms) over all of the G_x, it is possible to equip the fibers G_x with a smoothly varying family of Riemannian metrics which is right-translation invariant. If we now assume that the object space of G is a *compact* manifold then the Riemannian metrics we construct on the G_x by this process are all *complete*.

If $D = \{D_x\}$ is an equivariant family of first order, elliptic operators on the fibers of G, then by invariance, and by the compactness of the object space of G, the principal symbols of the D_x have the property that, when evaluated on cotangent vectors of length one, they return values whose norm is uniformly bounded (over all the G_x). Because of this we can appeal to a theorem of Chernoff to deduce that D_x is essentially self-adjoint, and moreover is equipped with a sharp version of the elliptic package:

7.28. THEOREM (Chernoff). *Let D be a symmetric, first order, elliptic operator on a complete Riemmanian manifold W, and assume that the symbol of D is uniformly bounded by a constant $C > 0$ on cotangent vectors of length one. Then D is essentially self-adjoint. Moreover, if $f \colon \mathbb{R} \to \mathbb{C}$ has smooth, compactly supported Fourier transform, then the operator $f(D)$ is represented by a smooth kernel $k_D(w_2, w_1)$, that is,*

$$f(D)h)(w_2) = \int_W k_d(w_2, w_1)h(w_1)\, dw_1,$$

and

$$\text{Support}(k_D) \subseteq \{\, (w_2, w_1) \,:\, d(w_2, w_1) \leq C \,\}.$$

□

7.29. REMARK. For brevity, we shall say that the operator D has *finite propagation* C.

7.30. EXAMPLE. We shall not be able to prove this here, however it is perhaps helpful to look at the case where $W = \mathbb{R}$ and $D = -i d/dx$. Here, by Fourier theory, $f(D)$ is represented by the kernel $k_D(y, x) = \check{f}(y - x)$, where $\check{f}$ is the inverse Fourier transform (which is equal to the Fourier transforms, up to signs).

A second theorem in partial differential equations guarantees that the function k_D varies smoothly with the coefficients of D:

7.31. THEOREM. *Let* $s \colon W \to X$ *be a submersion of smooth manifolds and assume that the fibers of* s *have been equipped with a smoothly varying family of complete Riemannian metrics (on the other hand* X *may for example be a manifold with boudary). Let* $\{D_x\}$ *be a family of symmetric, first-order, elliptic operators on the fibers of* s*, and assume that the family has uniformly bounded finite propagation speed. Then the kernel functions* k_{D_x} *vary smoothly with* x*.* □

7.32. REMARK. “Smooth” means the k_{D_x} constitute a smooth function on the manifold $\{ (w_2, w_1) : s(w_2) = s(w_1) \}$.

Now let us return to our groupoid. If f has compactly supported Fourier transform then we can form the kernel functions $k_{D_x}(\gamma_2, \gamma_1)$, defined on $G_x \times G_x$. From the equivariance of the family $\{D_x\}$ it follows that

$$k_{D_x}(\gamma_2, \gamma_1) = k_{D_y}(\gamma_2 \circ \eta, \gamma_1 \circ \eta),$$

for every morphism $\eta \colon y \to x$. So if we define a function $h \colon G \to \mathbb{C}$ by the formula $h(\gamma) = k_{D_x}(\gamma, id_x)$, where $x = s(\gamma)$, then

$$k_{D_x}(\gamma_2, \gamma_1) = h(\gamma_2 \circ \gamma_1^{-1}) \quad \forall \gamma_1, \gamma_2 \in G_x.$$

The function h is compactly supported, by the finite propagation speed argument. Checking the definitions, we arrive at the following theorem:

7.33. THEOREM. *Let* $D = \{D_x\}$ *be a smooth, right-translation invariant family of elliptic operators on the leaves* G_x *of a smooth groupoid* G *with compact object space. There is a* $*$*-homomorphism*

$$\phi_D \colon C_0(\mathbb{R}) \to C^*_\lambda(G)$$

with the property that if x *is any object, and* $\lambda_x \colon C^*_\lambda(G) \to \mathcal{B}(L^2(G_x))$ *is the regular representation, then*

$$\lambda_x(\phi_D(f)) = f(D_x) \colon L^2(G_x) \to L^2(G_x)$$

for every $f \in C_0(\mathbb{R})$*.* □

Now let us turn to the index homomorphism which we defined in the last section. For the purposes of the index homomorphism we can replace the tangent groupoid, as we have defined it, with the closed subset which lies over $[0,1] \subseteq \mathbb{R}$. This is a smooth groupoid in its own right (the object space has a boundary, but the G_x are manifolds without boundary, so the analysis above applies.

7.34. PROPOSITION. *Let* D *be a first order, elliptic operator on a closed manifold* M. *Let* $f \in C_0(T^*M)$. *The index asymptotic morphism* $\alpha_t \colon C_0(T^*M) \to \mathcal{K}(L^2(M))$ *maps* $f(\sigma_D) \in C_0(T^*M)$ *to the family*

$$\alpha_t(f(\sigma_D)) \sim f(tD) \in \mathcal{K}(L^2(M)).$$

7.35. REMARK. In the context of asymptotic morphisms $\alpha_t \colon A \to B$, the notation $\alpha_t(a) \sim b_t$ means $\lim_{t\to\infty} \|\alpha_t(a) - b_t\| = 0$.

PROOF. We can define the section $\sigma \colon C_0(T^*M) \to C^*_\lambda(\mathbb{T}M)$ used in the definition of α by the partial formula $\sigma(f(\sigma_D)) = \phi_D(f)$. (This defines σ on the elements $f(\sigma_D) \in C_0(T^*M)$; we don't care how it is defined on the rest of $C_0(T^*M)$.) Having made this particular choice, we get, by the preceding theorem, the *exact* relation $\alpha_t(f(\sigma_D)) = f(tD)$. □

Hence:

7.36. THEOREM. *Let* D *be a first order, elliptic operator on a closed manifold* M. *The index map* $\alpha \colon K(T^*M) \to K(\mathrm{pt})$ *maps the symbol class of* D *to the index of* D. □

8. Groupoid Algebras with Coefficients in a Bundle

In the previous section we argued as if the operator D acted on scalar functions, whereas in all interesting examples D acts not on functions but on sections of some Hermitian vector bundle S over M. In this section we shall indicate the changes needed to treat this case properly.

Let G be a smooth groupoid and let S be a smooth Hermitian vector bundle over the manifold of objects. Form the vector bundle $\mathrm{End}(S)$ over G whose fiber over a morphism $\gamma \colon x \to y$ is the vector space $\mathrm{Hom}(S_x, S_y)$. This is isomorphic to the pullback of S^* along the source map s, tensored with the pullback of S along the range map. The groupoid algebra $C_c^\infty(G, \mathrm{End}(S))$ is the algebra of smooth, compactly supported sections of $\mathrm{End}(S)$, equipped with the convolution multiplication

$$f_1 \star f_2(\gamma) = \int_{G_{s(\gamma)}} f_1(\gamma \circ \eta^{-1}) f_2(\eta) \, d\mu_{s(\gamma)}(\eta).$$

In the formula, the product $f_1(\gamma \circ \eta^{-1}) f_2(\eta)$ is a composition of operators

$$S_{s(\eta)} \to S_{r(\eta)} \to S_{r(\gamma)}.$$

The adjoint operation on $C_c^\infty(G, \mathrm{End}(S))$ is of course $f^*(\gamma) = f(\gamma^{-1})^*$. This algebra has natural regular representations on $L^2(G_x, S)$, for each object x, and using these we define the C^*-algebra completion, just as we did for $C^*_\lambda(G)$.

Repeating the arguments we gave in the previous section, we arrive at the following result:

7.37. THEOREM. *Let G be a smooth groupoid with compact object space. Let S be a Hermitian vector bundle over the object space of G and let $D = \{D_x\}$ be a smooth, right-translation invariant family of elliptic operators on the leaves G_x, acting on sections of the pullback along the range map r of S. There is a $*$-homomorphism*

$$\phi_D \colon C_0(\mathbb{R}) \to C^*_\lambda(G, \mathrm{End}(S))$$

with the property that if x is any object, and $\lambda_x \colon C^*_\lambda(G, \mathrm{End}(S)) \to \mathcal{B}(L^2(G_x, S))$ is the regular representation, then

$$\lambda_x(\phi_D(f)) = f(D_x) \colon L^2(G_x, S) \to L^2(G_x, S)$$

for every $f \in C_0(\mathbb{R})$. □

7.38. EXERCISE. In the case of the tangent groupoid, if S is pulled back from a bundle on M, show that the K-theory map

$$K(C_0(T^*M, \mathrm{End}(S))) \to K(\mathcal{K}(L^2(M, S)))$$

obtained from the asymptotic morphism associated to $C^*_\lambda(G, \mathrm{End}(S))$ identifies with the analytic index map upon composing with the isomorphism

$$K(C_0(T^*M, \mathrm{End}(S))) \cong K(T^*M)$$

obtained from Morita invariance of K-theory.

CHAPTER 8

Bott Periodicity and the Thom Isomorphism

In this lecture we shall evaluate the index map $\alpha\colon K(T^*\mathbb{R}^n)$ on the Bott element $b \in K(T^*\mathbb{R}^n)$, and use this computation to prove the Bott periodicity theorem.

1. The Bott Element

This section is a quick review of some ideas encountered in Lecture 5.

Let V be a finite-dimensional euclidean vector space. The tangent bundle TV may be identified with $V \times V$ (to be precise, the first factor of V will represents points of the manifold V and the second represents tangent vectors). Using the inner product we obtain an isomorphism

$$T^*V \cong V \times V^* \cong V \times V.$$

In this way we shall identify T^*V with the complex vector space $W = V + iV$. The inner product on the real vector space V provides the complex vector space W with a hermitian inner product. Denote by $\bigwedge^* W$ the complex exterior algebra of W, equipped with its inherited hermitian inner product and define a real-linear map

$$b\colon W \to \mathrm{End}(\textstyle\bigwedge^* W)$$

by the formula $b(w)(z) = w \wedge z + w \vee z$, where the operator $w\vee$ is adjoint to the exterior product operator $w\wedge$ on $\bigwedge^* W$. This is an elliptic endomorphism, in the sense of the term used in Lecture 3, and so defines a K-theory class $b \in K(W)$ by the procedure given there. Namely, b is the K-theory class of the $*$-homomorphism

$$\beta\colon \mathcal{S} \to C_0(W, \mathrm{End}(\textstyle\bigwedge^* W))$$

given by the formula $\beta(f) = f(b)$.

8.1. DEFINITION. Let V be a finite-dimensional Euclidean vector space. The *Bott element* for V is the class $b \in K(T^*V)$ defined above.

8.2. EXAMPLE. Let W be $T^*\mathbb{R}$, viewed as a one-dimenensional complex vector space as described above. If $\bigwedge^* W$ is given the basis consisting of the zero-form $1 \in \wedge^0 W$ and the 1-form $1 \in \mathbb{R} \subset \wedge^1 W$, then the operator b has the form

$$b(x,\xi) = \begin{pmatrix} 0 & x - i\xi \\ x + i\xi & 0 \end{pmatrix}.$$

More generally, if V is a real inner product space, if $W = V + iV$ and if we identify $\wedge^* W$ with the complexification of $\wedge^* V$, then the operator b has the form

$$b(v_1, v_2)\omega = v_1 \wedge \omega + v_1 \vee \omega + iv_2 \wedge \omega - iv_2 \vee \omega.$$

Having made our terminology precise, we can state again our goal:

8.3. THEOREM. *Let V be a finite-dimensional euclidean vector space, let $b \in K(T^*V)$ be the Bott element, and let $\alpha\colon K(T^*\mathbb{R}^n) \to K(\mathrm{pt})$ be the index homomorphism. Then $\alpha(b) = 1$.*

2. Index Computation

In this section we shall prove Theorem 8.3. As we shall see, this boils down to a Fredholm index computation. We shall concentrate on the case where V is one-dimensional, and discuss the modifications needed to handle the general case at the end.

The proof hinges on the following technical lemma. For $t \neq 0$, let

$$D_t = \begin{pmatrix} 0 & x - td \\ x + td & 0 \end{pmatrix},$$

where $d = \frac{d}{dx}$. This is an elliptic differential operator on a noncompact manifold — namely the real line.

8.4. LEMMA. *Let $b\colon \mathbb{R} \to M_2(\mathbb{C})$ be the elliptic element which defines the Bott element. If $\alpha_t\colon C_0(T^*\mathbb{R}) \to \mathcal{K}(L^2(\mathbb{R}))$ is the index asymptotic morphism then for every $f \in \mathcal{S}$,*

$$\alpha_t(f(b)) \sim f(D_t).$$

8.5. REMARK. In the lemma we have extended α_t to an asymptotic morphism

$$\alpha_t\colon M_2(C_0(T^*\mathbb{R})) \to M_2(\mathcal{K}(L^2(\mathbb{R})))$$

in the obvious way, in order to apply α_t to $f(b)$ (compare Example 8.2).

We shall also need a simple, basically algebraic, lemma.

8.6. LEMMA. *If $f \in \mathcal{S}$ then $f(D_t) \in \mathcal{K}(L^2(\mathbb{R}) \oplus L^2(\mathbb{R}))$. The kernel of D_t is one-dimensional, and is spanned by $\left[\begin{smallmatrix} v \\ 0 \end{smallmatrix}\right]$, where $v(x) = e^{-x^2/2}$.*

8.7. REMARK. It follows from Chernoff's theorem in the previous lecture that D_t is essentially self-adjoint, although the arguments given below could easily be adapted to prove this.

PROOF. Let us write $L = x + td$ and $R = x - td$, so that

$$D = \begin{pmatrix} 0 & R \\ L & 0 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} RL & 0 \\ 0 & LR \end{pmatrix}.$$

It is easy to compute that $RL = LR - 2t$, and that $Lv = 0$. From this we get that

$$RL \cdot Rv = R \cdot LRv = R \cdot (RL + 2t)v = R \cdot 2tv = 2tRv,$$

and more generally,

$$RL \cdot R^n v = 2ntR^n v.$$

In other words the functions $R^n v$ are eigenfunctions for RL with eigenvalue $2nt$. Now it is easy to check, by induction, that $R^n v$ is a polynomial of degree n, times v, from which it follows that the functions $R^n v$ span $L^2(\mathbb{R})$. Thus there is an orthonormal basis for $L^2(\mathbb{R})$ consisting of eigenfunctions of RL, with corresponding eigenvalue sequence $\{0, 2t, 4t, \ldots\}$. Since $LR = RL + 2t$, exactly the same thing is true for it, except that the eigenvalue sequence starts at $2t$, not 0. It follows that $f(D^2)$ is compact, for every $f \in \mathcal{S}$, from which it follows that $f(D)$ is compact, for every f, too. □

PROOF OF THEOREM 8.3, ASSUMING THE TECHNICAL LEMMA. For brevity, let us write $\mathcal{K}$ for $\mathcal{K}(L^2(\mathbb{R}))$. The asymptotic morphism α gives rise to a $*$-homomorphism

$$\tilde{\alpha}\colon M_2(C_0(T^*\mathbb{R})) \to \mathcal{M}_2(Q(\mathcal{K})),$$

where $\mathcal{Q}(\mathcal{K})$ is the "asymptotic algebra" of bounded functions from $[1, \infty)$ into $\mathcal{K}$, modulo functions which vanish at infinity. According to the recipe given in Lecture 3, to compute $\alpha(b)$ we must first compose the graded $*$-homomorphism $\beta\colon \mathcal{S} \to M_2(C_0(T^*\mathbb{R}))$ with $\tilde{\alpha}$, to obtain a graded $*$-homomorphism

$$\tilde{\alpha} \circ \beta\colon \mathcal{S} \to M_2(\mathcal{Q}(\mathcal{K})),$$

which represents an element of $K(\mathcal{Q}(\mathcal{K}))$. We must then lift this K-theory element to an element in the K-theory of $\mathcal{A}(\mathcal{K})$, the bounded continuous functions on $[1, \infty)$. The homotopy property of K-theory guarantees that we can find such a lift. Then we use the map

$$K(\mathcal{A}(\mathcal{K})) \to K(\mathcal{K})$$

induced from evaluation of functions at $1 \in [1, \infty)$, to obtain $\alpha(b)$. In our case, the technical lemma asserts that $\tilde{\alpha} \circ \beta$ lifts to the graded $*$-homomorphism

$$\phi\colon \mathcal{S} \to M_2(\mathcal{A}(\mathcal{K}))$$

given by the formula $\phi(f)(t) = f(D_t)$. So we have in this situation an explicit lifting. Evaluating at $t = 1$ we see that $\alpha(b)$ is represented by the graded $*$-homomorphism $f \mapsto f(D_1)$. Since D_1 has index one, this represents the element $1 \in K(\mathrm{pt})$ (see the proof of Proposition 3.20). $\square$

PROOF OF THE TECHNICAL LEMMA (SKETCH). Define a family of operators on the fibers of the source map $s\colon \mathbb{TR} \to \mathbb{R} \times \mathbb{R}$ by the formulas

$$D_{m,0} = \begin{pmatrix} 0 & m+d \\ m-d & 0 \end{pmatrix} \quad \text{and} \quad D_{m,t} = \begin{pmatrix} 0 & x+td \\ x-td & 0 \end{pmatrix} \quad \text{if } t \neq 0.$$

This is an equivariant family of operators, and it is easy to check that it is smooth (we will see this in a moment anyway). We would like to appeal to the general arguments we developed in the last lecture, but unfortunately the object space of $\mathbb{T}$R is noncompact, not only by virtue of the noncompactness of the t-parameter space, which is easily overcome, but by virtue of the noncompactness of the manifold $\mathbb{R}$. So we resort to some *ad hoc* computations. Under the usual diffeomorphism from $\mathbb{TR}$ to $\mathbb{R} \times \mathbb{R} \times \mathbb{R}$, the family $\{D_{m,t}\}$ is transformed to the family

$$E_{m,t} = \begin{pmatrix} 0 & m + t(x-m) + d \\ m + t(x-m) - d & 0 \end{pmatrix}$$

(which incidentally proves smoothness). If f has smooth and compactly supported Fourier transform then the general finite propagation and other arguments proved last time show that $f(E_{m,t})$ is represented by a kernel $k_{m,t}(y,x)$ which is smooth in all its arguments, and which vanishes if x and y are further than some distance C apart. We should like to say that the corresponding function on the groupoid,

$$h(y,x,t) = k_{x,t}(y,x),$$

is compactly supported, at least if we restrict to say $t \in [0,1]$. Unfortunately this is not so, but a little analysis does show that h is at least of rapid decay in x and y. An approximation argument now shows that all such h lie in the groupoid algebra. $\square$

In higher dimensions we can proceed in much the same way. If V is a finite-dimensional, euclidean vector space, then we can consider the operators

$$D_t = d_t + d_t^*,$$

acting on the trivial bundle over V with fiber $\wedge^*_{\mathbb{C}} V$, where $d_t\omega = \eta \wedge \omega + t d\omega$ and η is the differential of the function $\frac{1}{2}\|v\|^2$. It is again true that $f(D)$ is compact, when $f \in \mathcal{S}$, and the kernel of D_t is spanned the 0-form $\exp(-\|v\|^2/2)$. (In fact the analysis of this operator reduces very quickly to the one-dimensional case because after squaring D_t one can separate variables). Thus:

8.8. LEMMA. *Let* $b\colon V \to \wedge^*_{\mathbb{C}} V$ *be the elliptic element which defines the Bott element. If* $\alpha_t\colon C_0(T^*V) \to \mathcal{K}(L^2(V))$ *is the index asymptotic morphism then for every* $f \in \mathcal{S}$,

$$\alpha_t(f(b)) \sim f(D_t).$$

□

8.9. LEMMA. *If* $f \in \mathcal{S}$ *then* $f(D_t) \in \mathcal{K}(L^2(V, \wedge^*_{\mathbb{C}} V))$. *The kernel of* D_t *is one-dimensional, and is spanned by the* 0*-form* $e(v) = \exp -\|v\|^2/2$. □

With these, the rest of the proof is the same.

3. Bott Periodicity

In this section we shall prove the famous Bott periodicity theorem:

8.10. THEOREM (Bott Periodicity). *The Bott element* $b \in K(T^*\mathbb{R}^n)$ *freely generates* $K(T^*\mathbb{R}^n)$ *as an abelian group.*

In order to prove the theorem we shall use the following small generalization of the index map, which we present as an exercise.

8.11. EXERCISE. Let A be any C^*-algebra. There is an asymptotic morphism

$$\alpha^A_t\colon C_0(T^*\mathbb{R}^n) \otimes A \to \mathcal{K}(L^2(\mathbb{R}^n)) \otimes A$$

such that $\alpha^A_t(f \otimes a) \sim \alpha_t(f) \otimes a$, for all $f \in C_0(T^*\mathbb{R}^n)$ and $a \in A$. It has the property that

$$\alpha^A(x \otimes z) = \alpha(x) \otimes z$$

for all $x \in K(C_0(T^*\mathbb{R}^n))$ and all $z \in K(A)$.

¿From now on we shall drop the superscript A. We shall also specialize to the case where $A = C_0(Z)$, and write the induced K-theory map just as

$$\alpha\colon K(T^*\mathbb{R}^n \times Z) \to K(Z),$$

for which we have the relation $\alpha(x \otimes z) = \alpha(x) \otimes z$ for all $x \in K(T^*\mathbb{R}^n)$ and all $z \in K(Z)$.

We shall also need to manufacture from the Bott element maps

$$\beta\colon K(\text{pt} \times Z) \to K(T^*\mathbb{R}^n \times Z).$$

This we do by the formula $\beta(z) = b \otimes z$, using the product in K-theory. We get that $\beta(y \otimes z) = \beta(y) \otimes z$, for all $y \in K(\text{pt})$, and $z \in K(Z)$.

PROOF OF THE BOTT PERIODICITY THEOREM. With the generalized versions of α and β in hand, let us recast the problem of proving that $\beta(\alpha(x)) = x \in K(T^*\mathbb{R}^n)$ as the problem of proving the equivalent identity

$$\beta(\alpha(x \otimes 1)) = x \otimes 1 \in K(T^*\mathbb{R}^n \times \text{pt}),$$

where 1 denotes the generator of $K(\text{pt})$. Using the multiplicative properties of α and β we get

$$\beta(\alpha(x \otimes 1)) = \beta(\alpha(x) \otimes 1) = \beta(1 \otimes \alpha(x)) = \beta(1) \otimes \alpha(x),$$

where the middle equality holds because $u \otimes v = v \otimes u$ in $K(\text{pt} \times \text{pt})$. We therefore need to show that

$$\beta(1) \otimes \alpha(x) = x \otimes 1 \in K(T^*\mathbb{R}^n \times \text{pt}),$$

or equivalently

$$\alpha(x) \otimes \beta(1) = 1 \otimes x \in K(\text{pt} \times T^*\mathbb{R}^n).$$

Now the crucial observation is that

$$\alpha(x) \otimes \beta(1) = \alpha(x \otimes \beta(1)) = \alpha(\beta(1) \otimes x).$$

The last equality is a special case of the general identity

$$u \otimes v = v \otimes u \quad \text{in} \quad K(T^*\mathbb{R}^n \times T^*\mathbb{R}^n),$$

which holds because the map from $T^*\mathbb{R}^n \times T^*\mathbb{R}^n$ to itself which exchanges the two copies of $T^*\mathbb{R}^n$ in the product is homotopic to the identity through proper maps (to see this, identify $T^*\mathbb{R}^n$ with $\mathbb{R}^{2n}$ and note that the exchange matrix $\left(\begin{smallmatrix} 0 & I_{2n} \\ I_{2n} & 0 \end{smallmatrix}\right)$ is homotopic to the identity through orthogonal matrices). The argument finishes with an appeal to Theorem 8.3:

$$\alpha(\beta(1) \otimes x) = \alpha(\beta(1)) \otimes x = 1 \otimes x.$$

□

This remarkable argument is due to Atiyah.

4. A Remark on Categories

This section is optional.

The proof in the preceding section is best viewed in the context of a suitable category which includes "generalized" morphisms between C^*-algebras. Let us assume that we have a category with the following features:

(a) The objects are C^*-algebras.[1] Every $*$-homomorphism $\phi \colon A \to B$ determines in a functorial way a morphism from A to B, which depends only on the homotopy class of ϕ. (Thus there is a functor from the homotopy category of C^*-algebras into our category, which is the identity on objects.)

[1] It is customary to work with *separable* C^*-algebras, but this detail need not concern us here.

(b) The category has a natural product operation, so that morphisms $\sigma_1 \colon A_1 \to B_1$ and $\sigma_2 \colon A_2 \to B_2$ may be multiplied in a functorial way to produce a morphism

$$\sigma_1 \otimes \sigma_2 \colon A_1 \otimes A_2 \to B_1 \otimes B_2.$$

The product should be compatible with tensor product of $*$-homomorphisms, and should have the property that $\sigma \otimes 1 \colon A \otimes \mathbb{C} \to B \otimes \mathbb{C}$ identifies with $\sigma \colon A \to B$, once $A \otimes \mathbb{C}$ is identified with A and $B \otimes \mathbb{C}$ is identified with B. It should also be compatible with the flip isomorphisms $A \otimes B \to B \otimes A$ in the natural way.

(c) A morphism $\sigma \colon A \to B$ induces in a functorial way a homomorphism from $K(A)$ to $K(B)$, which is the standard induced homomorphism when σ is determined by a $*$-homomorphism. (Thus there K-theory functor should factor through our category.)

With this category in hand, the rotation argument may be expressed as follows (we shall write X in place of $C_0(X)$, and $\times$ in place of $\otimes$ in this commutative context). In view of the diagram

$$\begin{array}{ccccc} \mathbb{R}^{2n} \times \mathrm{pt} & \xrightarrow{\alpha\times 1} & \mathrm{pt} \times \mathrm{pt} & \xrightarrow{\beta\times 1} & \mathbb{R}^{2n} \times \mathrm{pt} \\ & & {\scriptstyle \mathrm{flip}}\downarrow{\scriptstyle =} & & {\scriptstyle \cong}\downarrow{\scriptstyle \mathrm{flip}} \\ & & \mathrm{pt} \times \mathrm{pt} & \xrightarrow[1\times\beta]{} & \mathrm{pt} \times \mathbb{R}^{2n} \end{array}$$

to prove $\beta \circ \alpha$ is an isomorphism it suffices to show $(1 \times \beta) \circ (\alpha \times 1)$ is an isomorphism. But consider now the commuting diagram

$$\begin{array}{ccc} \mathbb{R}^{2n} \times \mathrm{pt} & \xrightarrow{\alpha\times 1} & \mathrm{pt} \times \mathrm{pt} \\ {\scriptstyle 1\times\beta}\downarrow & \searrow^{\alpha\times\beta} & \downarrow{\scriptstyle 1\times\beta} \\ \mathbb{R}^{2n} \times \mathbb{R}^{2n} & \xrightarrow[\alpha\times 1]{} & \mathrm{pt} \times \mathbb{R}^{2n} \end{array}$$

It shows that it suffices to show $(\alpha \times 1) \circ (1 \times \beta)$ is an isomorphism. Now we can use the diagram

$$\begin{array}{ccccc} \mathbb{R}^{2n} \times \mathrm{pt} & \xrightarrow{1\times\beta} & \mathbb{R}^{2n} \times \mathbb{R}^{2n} & \xrightarrow{\alpha\times 1} & \mathrm{pt} \times \mathbb{R}^{2n} \\ & & {\scriptstyle \mathrm{flip}}\downarrow{\scriptstyle =\mathrm{id}} & & {\scriptstyle \cong}\downarrow{\scriptstyle \mathrm{flip}} \\ & & \mathbb{R}^{2n} \times \mathbb{R}^{2n} & \xrightarrow[1\times\alpha]{} & \mathbb{R}^{2n} \times \mathrm{pt} \end{array}$$

to complete the argument, bearing in mind that the left-hand flip induces the identity map in K-theory.

The morphisms constructed in Lecture 3 provide a suitable category: one sets

$$\mathrm{Hom}(A,B) = \left\{ \begin{array}{l} \text{Homotopy classes of graded asymp-} \\ \text{totic morphisms from } \mathcal{S}\otimes A \text{ to } B\otimes\mathcal{K} \end{array} \right\}.$$

We didn't show it, but these *are* the morphism sets in a category with a suitable product. This is the E-theory category of operator K-theory.

5. Compatibility with the Thom Homomorphism

In this section we shall prove the following theorem, after which the proof of the Atiyah-Singer index theorem will be complete.

8.12. THEOREM. *Let V be a Euclidean vector bundle over a smooth manifold M. Choose a splitting of the tangent bundle of V into horizontal and vertical subbundles, and using this splitting, view T^*V as a complex vector bundle over T^*M. If $\phi\colon K(T^*M) \to K(T^*V)$ denotes the Thom homomorphism then the following diagram commutes:*

$$\begin{array}{ccc} K(T^*M) & \xrightarrow{\alpha} & K(pt) \\ \phi\big\downarrow & & \big\downarrow = \\ K(T^*V) & \xrightarrow{\alpha} & K(pt). \end{array}$$

Here is how we shall prove the theorem. Let $S = \wedge^*_{\mathbb{C}} V$. This is a Hermitian vector bundle over M, and also, by pullback, a Hermitian vector bundle over V. We shall realize the Thom homomorphism using the $*$-homomorphism

$$\phi\colon \mathcal{S}\otimes C_0(T^*M) \to C_0(T^*V, \mathrm{End}(S))$$

which was discussed in Lecture 5. We shall realize the index map $\alpha\colon K(T^*V) \to K(pt)$ using the asympotic morphism

$$\alpha^V_t\colon C_0(T^*V, \mathrm{End}(S)) \to \mathcal{K}(L^2(V,S))$$

which is associated to the C^*-algebra of the groupoid $\mathbb{T}V$ with coefficients in S (this small elaboration of the groupoid C^*-algebra $C^*_\lambda(\mathbb{T}V)$ was discussed in Lecture 7). We have introduced the superscript "V" in an attempt to avoid confusion with the index map for M. For the latter we shall use the notation the notation

$$\alpha^M_t\colon \mathcal{S}\otimes C_0(T^*M) \to \mathcal{K}(L^2(M))$$

for the underlying asymptotic morphism. These various C^*-algebra homomorphisms and asymptotic morphisms fit into a diagram

$$
\begin{array}{ccccc}
\mathcal{S} \otimes C_0(T^*M) & \xrightarrow{\varepsilon} & C_0(T^*M) & \xrightarrow{\alpha_t^M} & \mathcal{K}(L^2(M)) \\
\phi \downarrow & & & & \downarrow \mathrm{Ad}_T \\
C_0(T^*V, \mathrm{End}(S)) & & \xrightarrow[\alpha_t^V]{} & & \mathcal{K}(L^2(V, S))
\end{array}
\tag{4}
$$

in which $\varepsilon\colon \mathcal{S} \to \mathbb{C}$ is evaluation at 0 and the left vertical map is induced from a certain isometry $T\colon L^2(M) \to L^2(V, S)$ that we will define in a moment. We shall show that the diagram commutes up to homotopy of asymptotic morphisms. Since ε and Ad_T both induce the identity map in K-theory, this will prove the theorem.

The isometry $T\colon L^2(M) \to L^2(V, S)$ is defined as follows. Just as we did in the last lecture, let us denote by $e\colon V \to \mathbb{C}$ the function

$$e(v) = (2\pi)^{\frac{n}{2}} \exp(-\tfrac{\|v\|^2}{2}),$$

which we shall view as a zero-form on V. The significance of e is that in each fiber of V it spans the kernel of the operator B discussed in the previous lecture. The significance of the constant is that in each fiber of V the L^2-norm of e is 1. We define T by the formula

$$(Th)(v) = h(\pi(v))e(v) \qquad (h \in L^2(M), \quad v \in V).$$

Thus T pulls back functions $h \in L^2(M)$ to V, then multiplies them pointwise with e. It is easy to check that T is an isometry.

Before going on, it is instructive to consider the case in which V is a *trivial* bundle $V = \mathbb{R}^k \times M$. In this case the diagram (4) is quite easy to analyze in view of the following result.

8.13. DEFINITION. Two asymptotic morphisms $\phi_t, \phi'_t\colon A \to B$ are *asymptotically equivalent* if $\lim_{t\to\infty} \|\phi_t(a) - \phi'_t(a)\| = 0$, for every $a \in A$.

8.14. LEMMA. *In the case of a product of smooth manifolds,* $M_1 \times M_2$*, the index asymptotic morphism*

$$\alpha_t^{M_1 \times M_2}\colon C_0(T^*M_1 \times T^*M_2) \to \mathcal{K}(L^2(M_1 \times M_2))$$

decomposes as a tensor product,

$$\alpha_t^{M_1 \times M_2} \sim \alpha_t^{M_1} \otimes \alpha_t^{M_2}$$

up to asymptotic equivalence, where $\alpha_t^{M_1}$ *and* $\alpha_t^{M_2}$ *are the index asymptotic morphisms for* M_1 *and* M_2. □

8.15. EXERCISE. The reader should check that if $\phi_{1,t}\colon A_1 \to B_1$ and $\phi_{2,t}\colon A_2 \to B_2$ are asymptotic morphisms, then there is an asymptotic morphism, $\phi_t\colon A_1 \otimes A_2 \to B_1 \otimes B_2$, unique up to asymptotic equivalence, such that

$$\phi_t(a_1 \otimes a_2) \sim \phi_{1,t}(a_1) \otimes \phi_{2,t}(a_2).$$

Here we need to assume that there is a *unique* C^*-algebra tensor product of A_1 and A_2, of we need to use the maximal tensor product. This is not a problem in our present, commutative, situation.

Because of the lemma, diagram (4) assumes the following form in the case of a trivial bundle:

$$\begin{array}{ccccc}
\mathcal{S} \otimes C_0(T^*M) & \xrightarrow{\varepsilon\otimes\mathrm{id}} & \mathbb{C} \otimes C_0(T^*M) & \xrightarrow{\alpha_t^M} & \mathbb{C} \otimes \mathcal{K}(L^2(M)) \\
\downarrow{\scriptstyle \phi\otimes\mathrm{id}} & & & & \downarrow{\scriptstyle \mathrm{Ad}_T} \\
C_0(T^*\mathbb{R}^k, \mathrm{End}(S)) \otimes C_0(T^*M) & & \xrightarrow[\alpha_t^{\mathbb{R}^k}\otimes\alpha_t^M]{} & & \mathcal{K}(L^2(\mathbb{R}^k, S)) \otimes \mathcal{K}(L^2(M))
\end{array}$$

But in the previous lecture we computed the composition

$$\mathcal{S} \xrightarrow{\phi} C_0(\mathbb{R}^k, \mathrm{End}(S)) \xrightarrow{\alpha_t^{\mathbb{R}^k}} \mathcal{K}(L^2(\mathbb{R}^k, S)),$$

which we found to be (asymptotic to) the family of $*$-homomorphisms $f \mapsto f(D_t)$. Inserting this fact into the diagram above, and altering the latter family by a homotopy to obtain first the family $f \mapsto f(D_1)$, and then the family $f \mapsto f(tD_1)$, we obtain, in the end, a commutative diagram, as required.

To deal with general case we should like to use the fact that the index asymptotic morphisms are canonical — that they don't depend on any choices of coordinates — to reduce to the trivial case just considered, taking advantage of the fact that every bundle is at least locally trivial. This strategy works well, except for the fact that the Thom isomorphism is not completely canonical. Its dependence on a choice of metric on the fibers of the bundle V is not an issue since every bundle is locally isomorphic to $\mathbb{R}^k \times M$ as a Euclidean bundle. But there is also a dependence on the way that T^*V is realized as a complex vector bundle over T^*M.

In order to analyze the situation we need to describe a bit more carefully how T^*V may be made into a bundle over T^*M. If $U \subseteq M$ is an open set over which there is a trivialization of V (respected the Euclidean structure on V), say

$$\theta\colon V|_U \xrightarrow{\cong} \mathbb{R}^k \times U,$$

then using θ we obtain isomorphisms

$$\theta_1\colon T^*(V|_U) \xrightarrow{\cong} T^*\mathbb{R}^k \times T^*U \quad \text{and} \quad \theta_2\colon \pi^*(V|_U \oplus V|_U) \xrightarrow{\cong} T^*\mathbb{R}^k \times T^*U,$$

where $\pi\colon T^*M \to M$ is the projection, and in the second map we identify the second copy of V with vertical tangent vectors along V (after applying the derivative of θ we get a tangent vector along $\mathbb{R}^k$, which we identify with a cotangent vector using the standard metric on $\mathbb{R}^k$. Unfortunately the local isomorphism

$$\theta_2^{-1}\theta_1\colon T^*(V|_U) \xrightarrow{\cong} \pi^*(V|_U \times V|_U)$$

we obtain in this way is not independent of θ and so does not immediately globalize to all of T^*V. However different choices of θ give rise to local isomorphisms which differ only by "transition functions" on $\pi^*(V|_U \times V|_U)$ of the form

$$(v_1, v_2, \xi, u) \mapsto (v_1, v_2, s(u)v_2 + \xi, u),$$

where s is a (local) vector bundle map from V to T^*M. What this means is that by using a partition of unity and a family of local isomorphisms we can assemble a global *diffeomorphism*

$$\Theta\colon T^*V \xrightarrow{\cong} \pi^*(V \oplus V)$$

which is not however an isomorphism of vector bundles over T^*M (this is unsurprising since T^*V is not naturally a vector bundle over T^*M). Let us use Θ to identify T^*V with $\pi^*(V \oplus V)$. Then in a local trivialiation θ of V, which gives a (canonical) local identification $T^*(V|_U) \cong T^*\mathbb{R}^k \times T^*U$, the Thom $*$-homomorphism

$$\phi\colon \mathcal{S} \otimes C_0(T^*U) \to C_0(T^*(V|_U), \mathrm{End}(S)) \cong C_0(T^*\mathbb{R}^k \times T^*U, \mathrm{End}(S))$$

has the form

$$\phi_s(f \otimes h)(v) = f(b)h(\tilde{\pi}(v)) \qquad (v \in T^*M).$$

where $b\colon T^*\mathbb{R}^k \to \mathrm{End}(S)$ is the standard Bott element, but $\tilde{\pi}\colon T^*\mathbb{R}^k \times T^*U \to T^*U$ is *not* the projection onto the second factor. it is just some smooth map which is the identity on $\{0\} \times T^*U$.

To circumvent this difficulty we consider the composition

(5)
$$\begin{array}{ccc} \mathcal{S} \otimes C_0(T^*M) & & \\ \downarrow{\scriptstyle \phi_s} & & \\ C_0(T^*V, \mathrm{End}(S)) & \xrightarrow[\alpha_t^V]{} & \mathcal{K}(L^2(V, S)) \end{array}$$

which is part of our diagram (4), except that we have replaced the Thom $*$-homomorphism ϕ by the one-parameter family of $*$-homomorphisms

$$\phi_s\colon \mathcal{S} \otimes C_0(T^*M) \to C_0(T^*V, \mathrm{End}(S)),$$

indexed by $s \in (0,1]$ and defined by the formula

$$\phi_s(f \otimes h)(v) = f(s^{-1}b)h(\tilde{\pi}(v)) \qquad (v \in T^*V).$$

Here b is the Thom element and $\tilde{\pi}\colon T^*V \to T^*M$ is the projection defined by identification $T^*V \cong \pi^*(V \oplus V)$ we have specified. The advantage of including the parameter s is that as $s \to 0$ the $*$-homomorphism ϕ_s becomes independent of the choice of identifications involved in the definition of $\tilde{\pi}$: different choices give asymptotic families of $*$-homomorphisms. Using this fact, and working in locally over coordinate patches U in M , where $V \cong \mathbb{R}^d \times U$, we can compute the composition. In a product space $\mathbb{R}^k \times U$ the index asymptotic morphism has the form

$$\alpha_t^{\mathbb{R}^k} \otimes \alpha_t^U\colon C_0(T^*\mathbb{R}^k, \mathrm{End}(S)) \otimes C_0(T^*U) \to \mathcal{K}(L^2(\mathbb{R}^k, S)) \otimes \mathcal{K}(L^2(U)).$$

Using Lemma 8.8 we conclude that as $s \to 0$, the composition $\alpha_t^V \circ \phi_s$ in diagram (5) becomes asymptotic (uniformly in t) to the map given locally by

$$\mathcal{S} \otimes C_0(T^*U) \ni f \otimes h \mapsto f(s^{-1}D_t) \otimes \alpha_t(h) \in \mathcal{K}(L^2(\mathbb{R}^k, S)) \otimes \mathcal{K}(L^2(U)).$$

In fact the asymptotic morphism $\alpha_t^V \circ \phi$ is homotopic to the asymptotic morphism

$$\mathcal{S} \otimes C_0(T^*U) \ni f \otimes h \mapsto f(t^{-1}D_t) \otimes \alpha_t(h) \in \mathcal{K}(L^2(\mathbb{R}^k, S)) \otimes \mathcal{K}(L^2(U))$$

(the local formula gives a globally well-defined asymptotic morphism). This in turn is homotopic to the asymptotic morphism

$$\mathcal{S} \otimes C_0(T^*U) \ni f \otimes h \mapsto f(t^{-1}D) \otimes \alpha_t(h) \in \mathcal{K}(L^2(\mathbb{R}^k, S)) \otimes \mathcal{K}(L^2(U)),$$

which is nothing but the local form of the composition

$$\begin{array}{ccccc} \mathcal{S} \otimes C_0(T^*M) & \xrightarrow{\varepsilon} & C_0(T^*M) & \xrightarrow{\alpha_t^M} & \mathcal{K}(L^2(M)) \\ & & & & \downarrow{\scriptstyle \mathrm{Ad}_T} \\ & & & & \mathcal{K}(L^2(V,S)) \end{array}$$

from diagram (4). This is because as $t \to \infty$, the operator $f(t^{-1}D_1)$ converges to $f(0)P_{\mathrm{Kernel}(D_1)}$. Commutativity of the diagram (4) up to homotopy of asymptotic morphisms is proved.

CHAPTER 9

K-Homology and Other Index Theorems

As we have tried to indicate above, the proof of the index theorem depends on a contemplation of certain 'wrong way maps' in K-theory: the Thom homomorphism (induced by the inclusion of the zero-section in a complex vector bundle), and the map induced by the inclusion of an open subset $U \subseteq X$. Indeed, the analytic index map $\alpha\colon K(TM) \to \mathbb{Z}$ itself may be thought of as a 'wrong way map', induced by the collapse $TM \to$ pt.

If one wishes to study more elaborate versions of the Index Theorem (as one does in non-commutative geometry), it is helpful to have some systematic theory into which these various 'wrong way' maps can be fitted and within which they can be computed. In this chapter we shall sketch the beginning of such a theory.

1. K-Homology

Let D be an elliptic operator on a compact manifold M (first-order, symmetric, and so on, according to our usual conventions). Recall that we may define the index of D as the K-theory class defined by the graded $*$-homomorphism

$$f \mapsto f(D)$$

from $\mathcal{S}$ to the compacts.

9.1. LEMMA. *Let* $\phi \in C(M)$ *be a continuous function. Regard* ϕ *as an operator on* L^2 *via pointwise multiplication. Then for all* $f \in \mathcal{S}$ *the commutator* $[f(tD), \phi] := f(tD)\phi - \phi f(tD)$ *tends to zero as* $t \to 0$.

PROOF. Remember the commutator identities

$$[A + B, C] = [A, C] + [B, C], \quad [AB, C] = A[B, C] + [A, C]B.$$

Using these and the spectral theorem, it is easy to see that the collection $\mathfrak{A}$ of functions $f \in \mathcal{S}$ which satisfy the conclusion of the lemma is a C^*-subalgebra of $\mathcal{S}$. According to the Stone-Weierstrass theorem, then, it suffices to show that $\mathfrak{A}$ separates points on $\mathbb{R}$.

Consider then the case $f(\lambda) = 1/(\lambda + a)$, where $a \in \mathbb{C} \setminus \mathbb{R}$. Then $f(tD) = (tD + a)^{-1}$ and

$$[f(tD), \phi] = -t(tD + a)^{-1}[D, \phi](tD + a)^{-1}.$$

Here $(tD + a)^{-1}$ has norm bounded by $1/|\Im a|$, by the spectral theorem, and the term $[D, \phi]$ is also a *bounded* operator (namely $\sigma_D(d\phi)$). Thus $\|[f(tD), \phi]\|$ is of order t and $f \in \mathfrak{A}$. For varying a, these f separate points on $\mathbb{R}$, so the proof is complete. □

9.2. REMARK. One can show that if D is a self-adjoint first order operator on a *non*-compact manifold M then the lemma still holds true, in the sense that for every $\phi \in C_0(M)$, the operator $[f(tD), \phi]$ is compact and tends to zero in norm as $t \to \infty$.

A more sophisticated way to state the result of Lemma 9.1 is

9.3. COROLLARY. *With hypotheses as above, the family of maps*

$$f \otimes \phi \mapsto f(tD)\phi$$

defines an asymptotic morphism *(Definition 3.29) from* $\mathcal{S} \otimes C(M)$ *to the compact operators.* □

According to Proposition 3.33, this asymptotic morphism gives rise to a homomorphism

$$K(M) \to K(\mathrm{pt}) = \mathbb{Z}.$$

In other words, an elliptic operator gives rise to a *functional* on K-theory.

9.4. REMARK. In fact this homomorphism can be described in terms of the objects that we have introduced already. Namely, since $K(T^*M)$ is a module over $K(M)$, we get a homomorphism $K(M) \to K(T^*M)$ generated by the symbol of D. The diagram

$$\begin{array}{ccc} K(M) & \longrightarrow & K(TM) \\ & \searrow & \downarrow \alpha \\ & & \mathbb{Z} \end{array}$$

in which the right-hand vertical map is the analytic index map, is commutative. (The proof, which involves an asymptotically commuting diagram of asymptotic morphisms, is left as an instructive exercise for the reader.)

Notice that if our elliptic operator is the Dirac operator associated to a Spin^c structure, then the map $K(M) \to K(TM)$ above is an isomorphism. In this situation, then, the map $K(M) \to \mathbb{Z}$ associated to the Dirac operator encodes exactly the same information as the analytic index map α.

Our discussion above leads to two conclusions.

(a) The collection of all elliptic operators on M is in some sense 'dual' to $K(M)$.

(b) The key functional-analytic properties which make this duality possible are encoded in the notion of asymptotic morphism.

It is now a short step to the following definition.

9.5. DEFINITION. Let X be a locally compact space. The K-*homology* of X, written $K_0(X)$, is the group $[\![\mathcal{S} \otimes C_0(X), \widehat{\mathcal{K}}]\!]$ of asymptotic homotopy classes of asymptotic morphisms from $\mathcal{S} \otimes C_0(X)$ to the compacts.

Now that we have K-homology available to us, we shall distinguish ordinary K-theory by writing it with a superscript, $K^0(X) = K(X)$. Proposition 3.33 gives us a pairing

$$K_0(X) \otimes K^0(X) \to \mathbb{Z}$$

between K-homology and K-theory. One can prove that K-homology is indeed a *generalized homology theory* — it satisfies the homotopy invariance and excision properties that are summarized in the Eilenberg-Steenrod axioms.

9.6. REMARK. The idea that one can develop a homology theory dual to K-theory by abstracting the functional-analytic properties of elliptic operators is due to Atiyah (around 1970). The idea was implemented by Brown-Douglas-Fillmore and (independently) Kasparov, in a technically different way to that described above. The 'asymptotic morphism' definition of K-homology is due to Connes and Higson.

9.7. REMARK. If X is a non-compact space, then there is a sometimes confusing nuance in the statement that $K_0(X)$ is a homology theory. Namely, one can see that

$$K_0(X) = \text{Kernel}(K_0(X^+) \to K_0(\text{pt})),$$

where X^+ denotes the one-point compactification of X. This property is certainly not enjoyed by homology theory as it is usually defined; but it *is* enjoyed by the variant of ordinary homology called 'closed' or 'locally finite' homology (see Bott and Tu). Thus we should describe $K_0(X)$ in algebraic-topological terms as the *locally finite* K-homology of X. If we want to recover the usual (compactly supported) K-homology we can do this as $\lim_{\to} K_0(L)$, where we take the direct limit over compact subsets L of X. Of course these issues can be ignored provided we work with compact spaces only.

Since we have definitions both of K-homology and of K-theory in terms of classes of asymptotic morphisms, it is natural to combine them.

9.8. DEFINITION. The group $E(A, B)$ is defined to be $[\![\mathcal{S} \otimes A, B \otimes \widehat{\mathcal{K}}]\!]$.

Thus $K_0(X) = E(C_0(X), \mathbb{C})$ and $K^0(X) = E(\mathbb{C}, C_0(X))$. Moreover, an element of $E(A, B)$ gives rise to a homomorphism $K(A) \to K(B)$.

2. Wrong Way Functoriality

Let M and N be manifolds.

9.9. DEFINITION. A smooth map $f\colon M \to N$ is K-*oriented* if there is given a Spin^c structure on the real vector bundle $TM \oplus f^*(TN)$. (Each equivalence class of such Spin^c structures is called a K-orientation for the map.)

9.10. EXAMPLE. If M is a Spin^c manifold the the collapse map $M \to \mathrm{pt}$ is K-oriented. In particular, the collapse $TM \to \mathrm{pt}$ is K-oriented for *any* manifold M.

9.11. EXAMPLE. If U is an open subset of a manifold M then the inclusion $U \to M$ is canonically K-oriented.

9.12. EXAMPLE. If V is a Spin^c vector bundle over M then both the inclusion $M \to V$ by the zero-section and the projection $V \to M$ are K-oriented.

The examples above should look familiar.

9.13. THEOREM (Connes and Skandalis). *To each K-oriented map $f\colon M \to N$ of manifolds one can associate a functorial 'wrong way map' $f_!\colon K^0(M) \to K^0(N)$. This map has the following special cases:*

(i) *If $f\colon TM \to \mathrm{pt}$ is the collapse map then $f_!\colon K^0(TM) \to \mathrm{pt}$ is the analytic index map.*
(ii) *If $\iota\colon M \to V$ is the inclusion of the zero-section in a* Spin^c *vector bundle V, then $\iota_!\colon K^0(M) \to K^0(V)$ is the Thom isomorphism.*
(iii) *If $j\colon U \to M$ is the inclusion of an open subset then $j_!\colon K^0(U) \to K^0(M)$ is the 'extension by zero' map that we have previously discussed.*

□

Using the information given in the theorem one can calculate the wrong-way homomorphisms associated to some other K-oriented maps of spaces.

9.14. EXAMPLE. Let $\pi\colon V \to M$ be the projection of a Spin^c vector bundle. Since $\pi \circ \iota$ is the identity on M, $\pi_!$ is left inverse to $\iota_!$. But $\iota_!$ is the Thom isomorphism, and therefore $\pi_!$ is the inverse to the Thom isomorphism.

9.15. EXERCISE. Show that if M is a Spin^c manifold, then the map $f_!\colon K^0(M) \to \mathbb{Z}$ induced by collapsing M to a point is the homomorphism associated to the K-homology class of the Dirac operator.

More to the point, the existence of functorial 'wrong way' maps of the sort described above gives a proof of the index theorem. For recall that in Theorem 5.20 we showed that the Index Theorem will follow from the existence of an analytical index map which is compatible in an appropriate way with Thom homomorphisms and with open inclusions[1]. But the desired compatibility is certainly true if the analytical index map, open inclusion map, and Thom homomorphism are all particular instances of one general functorial construction.

One should not think of this as giving a new proof of the index theorem — the construction of the 'wrong way maps' involves all the techniques that we have discussed thus far. But it has the potential to suggest new forms of the index theorem.

3. The Index Theorem for Families

Imagine that we have a manifold M, and over M we have not one but a whole collection of elliptic operators $\{D_b\}$, parameterized by points of some other space B.

9.16. EXERCISE. Show that as b varies continuously in B, the index $\operatorname{Index}(D_b)$ remains constant.

This fact might lead us to believe that the index theory of a family of elliptic operators contains no more information than the index theory of a single operator. However, this is far from the case. Consider the analogous situation of a vector bundle. A vector bundle over a space B is a family of vector spaces over B. All the individual fibers are isomorphic as vector spaces, but the whole bundle need not be a product.

9.17. REMARK. The previous discussion could lead us to try to define the index of a family of elliptic operators as a $\mathbb{Z}/2$-graded 'kernel bundle'. With some effort, this can be made to work, as is done in Atiyah and Singer's paper on the index for families. However, it will be easier for us to use the $\mathcal{S}$ technology that we have already developed.

Now to the details. Let B be a manifold.

9.18. DEFINITION. By a *family of manifolds* over B we mean a locally trivial fiber bundle $\pi\colon E \to B$, with fiber a smooth manifold M and with structural group $\operatorname{Diff}(M)$, the diffeomorphisms of M.

Usually we shall assume that both B and the fiber M are compact.

[1]Together with the calculation of the index of a single 'Bott operator' on Euclidean space.

9.19. DEFINITION. Let $\pi\colon E \to B$ be a family of manifolds over B. A *family of elliptic operators* on this family of manifolds consists of

(i) A $\mathbb{Z}/2$-graded hermitian vector bundle S over E;
(ii) For each $x \in B$, an elliptic differential operator[2] D_x on the manifold $E_x = \pi^{-1}(x)$, acting on sections of the restriction of the bundle S to E_x, such that:
(iii) The operators D_x vary smoothly with x.

The condition of smooth variation may be expressed, for instance, by requiring that the operators D_x are the restrictions to the M_x of a single differential operator (*not* elliptic) on the manifold E.

In the language of groupoids, a family of elliptic operators over B is the same thing as a leafwise elliptic operator on the following groupoid.

9.20. DEFINITION. Let $\pi\colon E \to B$ be a family of manifolds over B. Then the *groupoid of the family* G_π is the smooth groupoid defined as follows:

- The object space is E;
- The morphism space is $\{(x,y) \in E \times E : \pi(x) = \pi(y) \in B\}$;
- The source and range maps are $s(x,y) = x$, $r(x,y) = y$;
- The composition law is $(x,y)\cdot(y,z) = (x,z)$;
- The inverse $(x,y)^{-1} = (y,x)$;
- The inclusion of identities is $x \mapsto (x,x)$.

In other words, G_π is a family of pair groupoids, parameterized by B. A Haar system for this groupoid is a smoothly varying family of Lebesgue measures on the fibers of π.

9.21. LEMMA. *The C^*-algebra of the groupoid G_π above is Morita equivalent to $C(B)$.*

PROOF. The Morita equivalence bimodule is $\{L^2(\pi^{-1}(b))\}$, considered as a continuous family of Hilbert spaces over B. □

According to the results of Chapter 7, the index of an elliptic family gives rise to a $*$-homomorphism $\mathcal{S} \to C^*_r(G_\pi)$ and thus to an element of $K(C^*_r(G_\pi)) = K^0(B)$. The *index problem for families* is to compute this element.

9.22. PROPOSITION. *In the above situation the map*

$$\mathcal{S} \otimes C(E) \to C^*(G_\pi))$$

defined by $f \otimes \phi \mapsto f(tD)\phi$ *is an asymptotic morphism.*

[2]Odd, symmetric and first-order, in accordance with our standing conventions.

PROOF. A simple generalization of the proof of Proposition 9.1. □

In the index theory of families, the rôle of the tangent bundle is played by the 'vertical tangent bundle' or 'tangent bundle to the fibers'

$$T_\pi E = \mathrm{Kernel}(\pi_*\colon TE \to TB).$$

If this bundle is provided with a Spin^c structure then there is a natural family of Dirac operators on $E \to B$. Applying the previous proposition to this operator we get an E-theory element in $E(C(E), C(B))$ and thus a map $K^0(E) \to K^0(B)$.

9.23. PROPOSITION. *The map* $K^0(E) \to K^0(B)$ *just defined is the wrong way map* $\pi_!$ *associated to the* K*-oriented map* $\pi\colon E \to B$.

9.24. REMARK. In general (that is, in the absence of a Spin^c structure) we can use a fiberwise version of the tangent groupoid construction to define an analytical index map $K^0(T_\pi E) \to K^0(B)$, and this will again be a wrong way map, this time associated to the projection $T_\pi E \to TB$, which is always K-oriented.

How shall we compute this analytic index? The key idea in the proof of the ordinary index theorem was to factor the collapse map $M \to \mathrm{pt}$ (or $TM \to \mathrm{pt}$) into a composite of maps each of which was of one of the 'easy' forms for which we have an explicit understanding of the induced wrong way homomorphism on K-theory, namely

(a) The inclusion of the zero-section in a Spin^c vector bundle;
(b) The inclusion of an open set;
(c) The projection of a Spin^c vector bundle (over a point!).

Then we made use of the functoriality of wrong way maps.

We can do the same sort of thing in the families case. Namely, we can find an embedding of the family $E \to B$ fiberwise into a vector bundle $Z \to B$ (we can in fact take Z to be trivial). Using the tubular neighborhood theorem we can factor the embedding $E \to Z$ into the composite of the inclusion of the zero-section of the normal bundle, followed by the inclusion of an open set into Z. We therefore have the same computational techniques available to us as in the previous case.

The reader may expect that we are now going to use this factorization to obtain a cohomological formula for the index, and indeed this is possible (the next exercise states the formula). However there is an important nuance here. The cohomological formula of necessity computes the Chern character, $\mathrm{ch}(\mathrm{Index}\, D) \in H^*(B)$. When B is a point, as in the case of the ordinary index theorem, $\mathrm{ch}\colon K(\mathrm{pt}) \to H^*(\mathrm{pt})$ is injective so that the Chern character captures all the information about the index. However, in general the Chern character is not injective (it loses all torsion information). For this reason, it is better to regard the K-theoretic statement, that the index can be computed by the factorization process described above, as constituting the 'correct' form of the index theorem for families; the cohomological statement is just a homomorphic image of this.

9.25. EXERCISE. Derive the cohomological form of the index theorem for families

$$\operatorname{ch}(\text{Index}\, D) = (-1)^n \oint \operatorname{ch}(\sigma_D)\, \operatorname{Todd}(T_\pi E \otimes \mathbb{C})$$

from the discussion above. Here $\oint$ denotes 'integration along the fiber', an operation that passes from $H^*(E)$ to $H^*(B)$.

4. The Longitudinal Index Theorem for Foliations

CHAPTER 10

Higher Index Theory

1. The Higher Index

We finish by taking a look at another kind of index theory which, superficially at least, seems to generalize in a quite different direction.

10.1. DEFINITION. Let M be a compact manifold with fundamental group $\Gamma = \pi_1 M$. Construct a groupoid G as follows:

- Space of objects is M;
- Space of morphisms is the space of orbits of the diagonal action of Γ on $\tilde{M} \times \tilde{M}$, where $\tilde{M}$ denotes the universal cover. In other words, a morphism is an equivalence class of pairs $(x, y) \in \tilde{M} \times \tilde{M}$, two such pairs (x, y) and (x', y') being considered equivalent if there is $\gamma \in \Gamma$ such that $\gamma x = x'$ and $\gamma y = y'$.
- Source and range maps $s(x, y) = \pi(x)$, $r(x, y) = \pi(y)$;
- Inclusion of identities $p \mapsto (x, x)$ for any $x \in \pi^{-1}(p)$ (well-defined).
- Composition and inverses as in the pair groupoid (well-defined).

10.2. EXERCISE. Check that this is a smooth groupoid.

10.3. LEMMA. *The* C^**-algebra of the above groupoid* G *is Morita equivalent to* $C^*_r(\Gamma)$.

SKETCH PROOF. This time, the equivalence bimodule can be described as follows. Consider the vector space U of compactly supported continuous functions on $\tilde{M}$. We can equip this with a $C^*_r(\Gamma)$-valued inner product by defining

$$\langle f, g \rangle = \sum_{\gamma \in \Gamma} \left(\int f(x) g(\gamma^{-1} x) dx \right) \cdot [\gamma] \in C^*_r(\Gamma).$$

Completing this U, we obtain a $C^*_r(\Gamma)$-Hilbert module whose algebra of compact operators can be seen to be exactly $C^*_r(G)$. □

Suppose that D is an elliptic operator on the compact manifold M. We can lift it (using local charts) to an elliptic operator $\tilde{D}$ on $\tilde{M}$, which is equivariant with respect to Γ. Such an operator gives an (equivariant) elliptic family on the groupoid G. Notice that, in particular, $\tilde{D}$ is essentially self-adjoint. We have as a special case of our general results for groupoids:

10.4. PROPOSITION. *The assignment* $f \mapsto f(D)$ *gives a* $*$*-homomorphism* $\mathcal{S} \to C^*_r(G)$.

The associated element of $K(C^*_r(G)) = K(C^*_r(\Gamma))$ is called the *higher index* of D.

10.5. EXERCISE. Using Lichnerowicz' formula from Chapter 6, show that if M is a spin manifold carrying a metric of positive scalar curvature, the higher index of the Dirac operator (and not just the ordinary index) vanishes.

Suppose that M is Spin^c. Then, again using the argument of 9.1, we will get a map

$$K(M) \to K(C^*_r(\Gamma))$$

which is called the *assembly map*. (In general we can get an assembly map $K(TM) \to K(C^*_r(\Gamma))$ using an appropriate variation of the tangent groupoid construction.)

10.6. CONJECTURE (Baum and Connes). *If* M *is compact and* aspherical *(that is,* $\tilde{M}$ *is contractible), then the assembly map is an isomorphism.*

The Baum-Connes conjecture in its various forms has been a central theme in the development of non-commutative geometry and topology. To gain perspective, note that the homotopy type of an aspherical space is completely determined by its fundamental group. Thus, both the left and the right hand sides of the conjecture depend on Γ only.

2. Higher Index Theory for the Torus

Let us investigate higher index theory for $M = \mathbb{T}^n$. In this case $\Gamma = \pi_1(M) = \mathbb{Z}^n$ and $C^*_r(\Gamma) = C(P)$, where P is *another* n-torus. The assembly map is then a map from the K-theory of the first torus to the K-theory of the second one.

10.7. LEMMA. *The torus* P *parameterizes the flat connections on a trivial line bundle over* M.

PROOF. A point of P is a $*$-homomorphism $C^*_r(\Gamma) \to \mathbb{C}$, that is, a one-dimensional unitary representation of the fundamental group Γ. But in general, homomorphisms $\pi_1(M) \to U(n)$ correspond to flat connections on a trivial n-dimensional bundle over a manifold M (one sends the connection to the corresponding holonomy representation of the fundamental group). □

This means that from the canonical Dirac operator D on M, we can manufacture a *family* of elliptic operators, parameterized by P, by twisting D with the various line bundles with connection. Although all the individual operators in this family have the same index, the family itself is nevertheless non-trivial.

10.8. THEOREM (Lusztig). *Consider the composite*

$$K(M) \longrightarrow K(M \times P) \longrightarrow K(P)$$

where the first map is induced from the projection $M \times P \to M$ *and the second is the analytic index map for families of Proposition 9.23. This composition is the Baum-Connes assembly map.*

This allows us to use the index theorem for families to compute the effect of the assembly map, at least on the cohomology level. Remember that the cohomology of an n-torus is an exterior algebra on n 1-dimensional generators.

10.9. LEMMA. *Let* $\mathbb{L}$ *be the universal line bundle over* $M \times P$ *— it is a line bundle with connection, whose restriction to the copy of* M *lying over* $p \in P$ *just is the flat line bundle described by* p*. Then*

$$c_1(\mathbb{L}) = x_1 y_1 + \cdots + x_n y_n$$

in the cohomology of $M \times P$*.*

PROOF. □

10.10. PROPOSITION. *The analytic assembly map for* $M = \mathbb{T}^n$ *is rationally an isomorphism.*

PROOF. use the index theorem for families □

10.11. COROLLARY (Gromov-Lawson). *There is no metric of positive scalar curvature on the torus.*