Quantization on Curves

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Abstract

Deformation quantization on varieties with singularities offers perspectives that are not found on manifolds. The Harrison component of Hochschild cohomology, vanishing on smooth manifolds, reflects information about singularities. The Harrison 2–cochains are symmetric and are interpreted in terms of abelian $*$–products. This paper begins a study of abelian quantization on plane curves over $\mathbb{C}$, being algebraic varieties of the form $\mathbb{C}^2/R$, where R is a polynomial in two variables; that is, abelian deformations of the coordinate algebra $\mathbb{C}[x,y]/(R)$. To understand the connection between the singularities of a variety and cohomology we determine the algebraic Hochschild (co-)homology and its Barr–Gerstenhaber–Schack decomposition. Homology is the same for all plane curves $\mathbb{C}[x,y]/R$, but the cohomology depends on the local algebra of the singularity of R at the origin.

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1 Introduction.

Deformation quantization is a term coined by Moshe Flato, who suggested that any nontrivial associative deformation of an algebra of functions should be interpreted as a kind of "quantization". Deformation quantization is [2] the study of associative $*$–products of the form $f*g = fg+\sum_{n>0}\hbar^n C_n(f,g)$, where $\hbar$ is a formal parameter. This concept has gained wide currency and has been intensively developed in recent years, but almost exclusively in the context of smooth Poisson manifolds [4, 14, 15]. In that case it is natural to consider deformations "in the direction of the Poisson bracket" (Drinfel'd); that is, taking $C_1(f,g) = \{f,g\}$, which is of course antisymmetric. But even if more general deformations were to be considered, independent of the symplectic structure, antisymmetry of C_1 entails no essential loss of generality for quantization on a smooth (finite dimensional) manifold. A famous result of Hochschild, Kostant and Rosenberg [13] implies that any $*$–product on a regular, commutative algebra is equivalent to one with antisymmetric C_1. For a related 'smooth' result, see [23].

It would seem, therefore, that the time has come to study deformation quantization on varieties with singularities. The cohomological implication of singularities should be interesting.

The Hochschild complex of any commutative algebra decomposes into smaller complexes; in the case of an algebra A generated by N generators, into N subcomplexes [1, 5, 11]. The topology of a smooth manifold is related to the restriction of the Hochschild complex to alternating maps $A^\wedge \to A$, dual to simplicial homology, and the only component with non-vanishing cohomology. But on varieties with singularities other components of the Hochschild complex come into play, which suggests the use of cohomological methods for the study of singularities.

Examples of quantization on singular varieties had been known in connection with geometric quantization (and $*$–quantization) on coadjoint orbits of Lie algebras, but the cohomological implications had not been recognized. (See [2, 6, 7].) The connection between singularities and cohomology was studied by Harrison [12], who was the first to describe the component of Hochschild cohomology that has become known, if not widely known, as Harrison cohomology. The 2–cochains of this complex are symmetric. On a commutative algebra every exact Hochschild 2–cochain is symmetric, so that triviality is not an issue if C_1 is antisymmetric. But it is an important consideration in the case of abelian $*$–products.

The BGS idempotents.

The p–chains of the Hochschild homology complex of a commutative algebra A are the p–tuples $a = \sum a_1 \otimes \cdots \otimes a_p \in A^{\otimes p}$, and the differential is defined by

$$da = a_1a_2 \otimes a_3 \otimes \cdots \otimes a_p - a_1 \otimes a_2a_3 \otimes a_4 \cdots \otimes a_p + \cdots + (-)^p a_1 \otimes \cdots a_{p-2} \otimes a_{p-1}a_p.$$

The p–cochains are maps $A^{\otimes p} \to A$, and the differential is

$$\delta C(a_1, \cdots, a_{p+1}) = a_1 C(a_2, \cdots, a_{p-1}) - C(da) - (-)^p C(a_1, \cdots, a_p) a_{p+1}.$$

After the pioneering work of Harrison [12] and Barr [1], the complete decomposition of the Hochschild cohomology of a commutative algebra was found by Gerstenhaber and Schack [11]. The Hochschild cochain complex splits into an infinite sum of direct summands. (If the algebra is generated by N generators then there are only N nonzero summands.) The decomposition is based on the action of S_n on n–cochains, and on the existence of n idempotents $e_n(k),\ k = 1, \cdots n$, in $\mathbb{C}S_n$, $\sum_k e_n(k) = 1$, with the property that $\delta \circ e_n(k) = e_{n+1}(k) \circ \delta$. Thus we have $\mathrm{Hoch}_n = \sum_{k=1}^n H_{n,k}$, $\mathrm{Hoch}^n = \sum_{k=1}^n H^{n,k}$ with $H_{n,1} = \mathrm{Harr}_n$ and $H^{n,1} = \mathrm{Harr}^n$.

A generating function was found by Garsia [8],

$$\sum_{k=1}^n x^k e_n(k) = \frac{1}{n!} \sum_{\sigma \in S_n} (x - d_\sigma)(x - d_\sigma + 1) \cdots (x - d_\sigma + n - 1)\mathrm{sgn}(\sigma)\sigma,$$

where d_σ is the number of descents, $\sigma(i) > \sigma(i+1)$, in $\sigma(1 \cdots n)$. [1] The simplest idempotents are

$$\begin{aligned} e_2(1)12 &= \frac{1}{2}(12 + 21), \\ e_3(1)123 &= \frac{1}{6}\big(2(123 - 321) + 132 - 231 + 213 - 312\big), \\ e_3(2)123 &= \frac{1}{2}(123 + 321) \\ e_n(n) &= \frac{1}{n!} \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma)\sigma. \end{aligned}$$

The Hochschild chains decompose in the same way, with $d \circ e_n(k) = e_{n-1}(k) \circ d$.

Summary.

Section 2 is concerned with abelian $*$–products on an arbitrary plane curve. The space of equivalence classes of first order abelian deformations of the algebra of polynomials on $\mathbb{C}[x,y]/(R)$ is isomorphic to the local algebra of the singularity of R at $x = y = 0$. The Harrison component $\mathrm{Harr}^3 = H^{3,1}$ of Hoch^3 vanishes, which implies that there are no obstructions to continuing a first order abelian $*$–product to higher orders. In this paper the strategy that leads to the calculation of Hochschild cohomology calls for a preparatory investigation of a homological complex that is not strictly Hochschild, but rather its restriction $A \to A_+$ to the non-unital subalgebra A_+ of positive degree; this has no effect on the cohomology.

In Section 3 the Hochschild homology is calculated for the case of a plane curve, with its BGS decomposition. In Section 4 the Hochschild cohomology is investigated; the result in Theorem 4.9. Section 5 contains a detailed calculation of the BGS decomposition for the singularity of $x^n = 0$ at $x = 0$.

The Appendix, by Maxim Kontsevich, explains in modern mathematical language a way to calculate Hochschild and Harrison cohomology groups for algebras of functions on singular planar curves etc. based on Koszul resolutions.

[1]Example: $\sigma(1234) = 3142$ has one descent, from 2 to 3.

2 Associative $*$–products and cohomology.

2.1 Formal $*$–products. A formal, abelian $*$–product on a commutative algebra A is a commutative, associative product on the space of formal power series in a formal parameter $\hbar$ with coefficients in A, given by a formal series

$$f*g = fg + \sum_{n>0} \hbar^n C_n(f,g). \tag{2.1}$$

Associativity is the condition that $f*(g*h) = (f*g)*h$, or

$$\sum_{m,n=0}^{k} \hbar^{m+n}\Big(C_m(f,C_n(g,h)) - C_m(C_n(f,g),h)\Big) = 0, \tag{2.2}$$

where $C_0(f,g) = fg$. This must be interpreted as an identity in $\hbar$; thus

$$\sum_{m,n=0}^{k} \delta_{m+n,k}\Big(C_m(f,C_n(g,h)) - C_m(C_n(f,g),h)\Big) = 0, \;\; k = 1,2,\cdots . \tag{2.3}$$

The formal $*$–product (2.1) is associative to order p if Eq(2.3) holds for $k = 1,\cdots p$.

A first order abelian $*$–product is a product

$$f*g = fg + \hbar C_1(f,g), \quad C_1(f,g) = C_1(g,f), \tag{2.4}$$

associative to first order in $\hbar$, which is the requirement that C_1 be closed,

$$\delta C_1(f,g,h) := fC_1(g,h) - C_1(fg,h) + C_1(f,gh) - C_1(f,g)h = 0.$$

Suppose that a formal $*$–product is associative to order $p \geq 1$; this statement involves $C_1,\cdots,C_p$ only, and we suppose these cochains fixed. Then the condition that must be satisfied by C_{p+1}, in order that the $*$–product be associative to order $p+1$, is

$$\sum_{\substack{m,n=1\\ m+n=p+1}}^{p} \Big(C_m(f,C_n(g,h)) - C_m(C_n(f,g),h)\Big) = -\delta C_{p+1}(f,g,h). \tag{2.5}$$

The left hand side is closed, and thus it is seen that the obstructions to promote associativity from order p to order $p+1$ are in Hoch^3.

There is an important difference between the two cases of symmetric and antisymmetric C_1. If $C_1,\cdots,C_p$ are symmetric, then the left hand side of (2.5) has the symmetry of the idempotent $e_3(1)$ (a Harrison cochain) and it is the symmetric part of C_{p+1} that is relevant, while the antisymmetric part of C_{p+1} must simply be closed. Symmetry of the $*$–product can therefore be maintained to all orders. If C_1 is antisymmetric, and $p = 1$, then the left hand side has the symmetry of $e_3(1) + e_3(3)$. The first part must be balanced on the right hand side by means of the symmetric part of C_2; the second part must vanish, and this condition is the Jacobi identity for C_1.

The obstructions against continuing a formal, first order, abelian $*$–product to higher orders are in Hoch^3; more precisely, they are in $H^{3,1} = \mathrm{Harr}^3(A,A)$.

A formal $*$–product is trivial if there is an invertible map $E : A \to A$, in the form of a formal series $E(f) = f + \sum_{n>0} \hbar^n E_n(f)$ such that $E(f * g) = E(f)E(g)$. A first order, abelian $*$–product is trivial if there is a 1–cochain E_1 such that

$$C_1(f,g) = \delta E_1(f,g) = fE_1(g) - E_1(fg) + E_1(f)g.$$

2.2. Deformations on a curve. In view of the theorem of Hochschild, Kostant and Rosenberg [13] cited earlier, there can be no nontrivial, abelian $*$–products on a smooth manifold. It is natural to turn to varieties with singularities, and especially algebraic varieties. It is the aim of this paper to explore the phenomena, with elementary methods of calculation, in the case of plane curves over $\mathbb{C}$, $M = \mathbb{C}^2/R$, where R is a $\mathbb{C}$–polynomial. The algebras of interest are the coordinate algebra

$$A = \mathbb{C}[x,y]/(R), \tag{2.6}$$

with generators x, y and a single polynomial relation R. The polynomial R can be transformed, by a linear change of variables, to either of the forms $R = x^m - P(x,y)$ or $R = y^n - Q(x,y)$, where the polynomial P is of order less than m in x and the polynomial Q is of order less than n in y. Either form gives rise to a Poincaré–Witt basis for A, for example, $x^i y^j$, $i = 0,1,\cdots\infty$, $j = 0,1,\cdots,n-1$.

The deformed algebra has a Poincaré–Witt basis of the same form. Let W be the map that takes a $*$–monomial of this basis to the same ordinary monomial of the original basis. Let $R_\hbar := W(R^*)$ and let $M_\hbar := \mathbb{C}^2/R_\hbar$. Then, morally, the $*$–product is trivial if there is a bijection $E : M_\hbar \to M$ such that $R_\hbar \mapsto R$. However, since $\hbar$ is a formal parameter, the following definition is preferred.

2.3. Definition. A $*$–product, as defined in this section, is trivial if there is a mapping by a formal power series in $\hbar$, $E = \mathrm{Id} + \sum_{n>1} \hbar^n E_n$, such that $R_\hbar \mapsto R$.

2.4. First order $*$–product on a curve. Consider a first order, associative and abelian $*$–product on the algebra (2.6), with the polynomial R in the form $R = y^n - Q(x,y)$. A change of variables ensures that $(x^i y^j) * (x^k y^l) = x^{i+k} y^{j+l}$ for $j + l < n$ and

$$y^i * y^{n-i} = Q(x,y) + \hbar Q_1(x,y), \quad 1 \le i \le n-1, \tag{2.7}$$

The first order deformation (2.7) is trivial if there is a derivation E such that $Q_1 = E(R)$. See Subsection 4.6.

2.5. Example. Let $A=\mathbb{R}[x,y]/(R),\ R=y^2-x^2-r^2,\ \ r^2\in\mathbb{C}$, decompose $f\in A$ as $f=f_++yf_-,\ f_\pm\in\mathbb{R}[x]$, and define a $*$–product on A by setting $f*g=fg+\hbar f_-g_-$. Then $Q_1=1$ and we seek E such that $E(x^2+r^2-y^2)=1$. The general solution to this equation is $2E=\frac{-1}{r^2}(x\partial_x+y\partial_y)+\alpha(y\partial_x+x\partial_y)$, with $\alpha\in A$.

Of course, this breaks down if $r^2=0$, and the simple reason why there is no solution in this case is that there is no differential operator E such that the polynomial $E(x^2-y^2)$ contains a constant term.

2.6. Proposition. *Let X be the space of polynomials in x and y, of degree less than n in y, and let DR be the gradient ideal of R. As vector spaces, X coincides with A and DR consists of all differentials of R. The space of equivalence classes of essential, first order $*$–products on A is the space X/DR,* $\mathrm{Harr}^2(A,A)=X/DR$.

2.7. Example. Let $M=\mathbb{C}^2,\ \ R=y^2-x^3$. A full set of representatives of X/DR is $a+bx,\ a,b\in\mathbb{C}$. The deformed algebras are $A_\hbar=\mathbb{C}[x,y]/R_\hbar$ with $R_\hbar=y^2-x^3-\hbar(ax+b)$. Expand $f(x,y)=f_+(x)+yf_-(x)$. Then $f*g=fg+\hbar C_1(f,g)$, where $C_1(f,g)=(ax+b)f_-g_-$.

3 Homology.

This section deals with the homology of a modified Hochschild complex. The strategy that is used in this paper, to calculate the Hochschild cohomology of A, begins by a determination of the homology of the algebra A_+, the subalgebra with positive degree of A. The n–chains of this homology of A_+ are n–tuples $a=a_1\otimes a_2\otimes\cdots a_n,\ \ a_i\in A_+,\ \ i=1,\cdots N$.

3.1. 2–chains. Every 'Hochschild' 2–chain is homologous to a 2–chain of the form $x\otimes a+y\otimes b$. It will be convenient to re-label the generators, $x,y\mapsto x_1,x_2$, then $a\approx\sum x_i\otimes a_i,\ a_i\in A_+,\ \ i=1,2$. It is closed if $\sum x_ia_i=0$. We shall suppose that R has no constant term and no linear terms, then a has the representation

$$a\approx\sum x_i\otimes x_j\varepsilon^{ij}b+\sum_{i=1}^{2}x_i\otimes R_ic,$$

where $\varepsilon^{ij}=-\varepsilon^{ji},\varepsilon^{12}=1,\ \sum x_iR_i=R$ and where b,c are in the unital augmentation A of A_+. The first term is exact if $b\in A_+$, the second term is exact if $c\in A_+$ and (a section of) $H_2=Z_2/B_2$ is spanned (over $\mathbb{C}$) by the chains $x_1\wedge x_2$ and $\sum x_i\otimes R_i$. The second one is homologous to a symmetric chain that is a basis for $\mathrm{Harr}_2=H_{2,1}$.

3.2. Example. If $R=y^2-x^n$, then Harr_2 has dimension 1 and every symmetric, closed 2–chain is homologous to a $\mathbb{C}$–multiple of $x\otimes x^{n-1}+x^{n-1}\otimes x-2y\otimes y$.

3.3. 3–chains. Every 3–chain is homologous to one of the form $a = \sum x_i \otimes b_j \otimes c^{ij}$. If a is closed it takes the form $a \approx \sum x_i \otimes x_j \varepsilon^{ij} b \otimes b' + x_i \otimes R_i c \otimes c',\ \ b,c \in A$ which is homologous to $a \approx \sum x_i \otimes x_j \varepsilon^{ij} \otimes bb' + x_i \otimes R_i \otimes cc'$, with $x_2 bb' + R_1 cc' = 0$ and $-x_1 bb' + R_2 cc' = 0$. A simple case-by-case study shows that we then have:

$$bb' = \alpha R_1 + \beta R_2, \quad cc' = -\alpha x_2 + \beta x_1,$$

with α, β in A. Thus any closed 3–chain is homologous to one of the form

$$\Big((x_1 \wedge x_2) \otimes R_1 c_1 - \sum x_i \otimes R_i \otimes x_2 c_1\Big) - \Big((x_1 \wedge x_2) \otimes R_2 c_2 + \sum x_i \otimes R_i \otimes x_1 c_2\Big). \tag{3.1}$$

The first (second) term is exact unless $c_1 (c_2)$ is in $\mathbb{C}$. Adding an exact, alternating 3–cycle we get an alternative section of Z_3/B_3 with a basis that consists of the two chains (the GS idempotents were defined in the introduction)

$$\begin{aligned} \alpha_1 &= e_3(2)\big(x_1 \otimes x_2 \otimes R_1 - x_2 \otimes R_1 \otimes x_1 - x_2 \otimes x_1 \otimes R_1 - x_2 \otimes R_2 \otimes x_2\big), \\ \alpha_2 &= e_3(2)\big(x_2 \otimes x_1 \otimes R_2 - x_1 \otimes R_2 \otimes x_2 - x_1 \otimes x_2 \otimes R_2 - x_1 \otimes R_1 \otimes x_1\big). \end{aligned} \tag{3.2}$$

Thus $\mathrm{Hoch}_3 = H_{3,2}$ has dimension 2 and $\mathrm{Harr}_3 = 0$.

Another way to reach this conclusion is to differentiate (3.1). The result is $(c_1 x_2 + c_2 x_1) \wedge R$, which is in $Z_{2,2}$ and which implies that (3.1) $\in Z_{3,2}$.

3.4. Example. If $R = y^2 - x^2$, set $u = x + y,\ v = x - y$. The basis (3.2) is then $\{u \otimes v\} \otimes u,\ v \otimes \{u \otimes v\}$ and the dimension of Hoch_3 is 2. More precisely, dim $H_{3,k}$ is $0,2,0$ for $k = 1,2,3$.

3.5. Example. If $R = y^2 - x^3$, then the chains (3.2) become

$$y \otimes x \otimes y - x \otimes y \otimes y - y \otimes y \otimes x + x \otimes x^2 \otimes x$$

and

$$e_3(2)\big(x \otimes y \otimes x^2 - y \otimes x^2 \otimes x - x^2 \otimes x \otimes y + y \otimes y \otimes y\big).$$

It is straightforward to prove the following.

3.6. Proposition. *Let $P^1 = \{x_1, x_2\}$, $P^{n+1} = P^n \otimes M_n$, and*

$$M_{2k+1} = \begin{pmatrix} R_1 & -x_2 \\ R_2 & x_1 \end{pmatrix}, \quad M_{2k} = \begin{pmatrix} x_1 & x_2 \\ -R_2 & R_1 \end{pmatrix}.$$

Then for $n > 1$ every closed n–chain is homologous to an n–chain in the linear span of the two linearly independent polynomials in P^n.

3.7. Example. If $R = y^2 - x^2$, set $u = x + y,\ v = x - y$. The dimension of Hoch_n is 2; the basis is $\{u \otimes v \otimes u \cdots,\ \ v \otimes u \otimes v \otimes u \cdots\}$.

3.8. Theorem. *$Hoch_{2k} = H_{2k,k} + H_{2k,k+1}$, each component one-dimensional over $\mathbb{C}$,* and *$Hoch_{2k-1} = H_{2k-1,k}$, two-dimensional over $\mathbb{C}$, $k = 1, 2, \ldots$.*

Proof. For $k = 1, ..., p-1$, $P^{p+1} = P^k \otimes M_k \otimes M_{k+1} \otimes \ldots \otimes M_p$ and thus

$$dP^{p+1} = P^1 M_1 \otimes M_2 \otimes \ldots \otimes M_p + \sum_{k=1}^{p-1} (-)^k P^k \otimes M_k M_{k+1} \otimes \ldots \otimes M_p.$$

We have $M_k M_{k+1} = R$ times the unit matrix and $P^1 M_1 \otimes M_2 = R \otimes P^1$; consequently $dP^1 = 0,\ dP^2 = \{R, 0\}$ and $dP^{p+1} = R \textcircled{sh} P^{p-1},\ \ p \geq 2$. If $a \in C_{p,k}$, then $da \in C_{p-1,k}$, and $R \textcircled{sh} a$ is homologous to some $b \in C_{p+1,k+1}$. Hence if $P^{p-1} \in C_{p-1,k}$, then P^{p+1} is homologous to a $C_{p+1,k+1}$ chain.The action of these maps between spaces with cohomology is shown in the diagram.

$$\begin{array}{ccccccc} & C_{2,1} & & & & C_{2k,k} & \\ \swarrow & & \searrow & & \swarrow & & \searrow \\ C_{1,1} & & C_{3,2} & \cdots & C_{2k-1,k} & & C_{2k+1,k+1} \\ \searrow & & \swarrow & & \searrow & & \swarrow \\ & C_{2,2} & & & & C_{2k,k+1} & \end{array} \tag{3.4}$$

A southeast arrow represents the map $a \mapsto R \textcircled{sh} a$; a southwest arrow is the action of the differential. The projections of $\{P_i^{2k+1}\}_{i=1,2}$ form a basis for $H_{2k+1,k+1}$ and the projections of P_1^{2k} (resp. P_2^{2k}) are bases for $H_{2k,k}$ (resp. $H_{2k,k+1}$).

4 Cohomology.

4.1. The reduction process. The chains considered in this section are restricted to positive degree. The cochains are valued in A. A p–cochain is closed if

$$\delta C(a_1, \cdots, a_{p+1}) = a_1 C(a_2, \cdots, a_{p+1}) - C(da) - (-)^p C(a_1 \cdots, a_p) a_{p+1} = 0. \tag{4.1}$$

One may attempt to interpret this relation as fixing the value $C(da)$, recursively in the degree of the argument. The obstruction to this is $da = 0$, but if a is exact then (4.1) is satisfied automatically by virtue of its being true for arguments of lower degree. (One can show that, in this context, if a is exact then there is b of the same degree such that $a = db$.) It is enough, therefore, to verify closure for a basis of representatives of Hoch_{p+1}.

A closed p–cochain C is a coboundary if there is a $(p-1)$–cochain E such that

$$C(a) = a_1E(a_2,\cdots,a_p) - E(da) + (-)^pE(a_1,\cdots a_{p-1})a_p. \tag{4.2}$$

This relation can be solved for $E(da)$, recursively by increasing degree, except for the obstruction presented by $da = 0$. But if $a = db$ then $C(a)$ is determined by $\delta C(b) = 0$. So it is enough to examine (4.2) for a complete set of representatives of Hoch_p.

The most useful interpretation is this. Given any closed p–cochain a "gauge transformation" is the addition of an exact p–cochain, $C \to C + \Delta C$, with

$$\Delta C(a_1...a_p) = a_1E(a_-) + (-)^pa_pE(a_+) - E(da). \tag{4.3}$$

The space Hoch^p is the spac of closed, gauge-invariant p–cochains.

If any BGS component $H_{p,k}$ of $Hoch_p$ vanishes then the corresponding component $H^{p,k}$ of $Hoch^p$ is zero. There are no obstructions to continuing a first order, abelian $*$–product to higher orders.

4.2. Closure for $p = 1$. The 2–homology is spanned by $x_1 \wedge x_2$ and $x_i \otimes R_i$. We shall replace the latter by $\hat{R} = \sum A_{ij}x_1^i \otimes x_2^j,\ \ R = \sum A_{ij}x_1^ix_2^j$. The relation $\delta C(x_1 \wedge x_2) = 0$ is trivial. The formula $\delta C(x_1^i \otimes x_2^j) = x_1^iC(x_2^j) + x_2^jC(x_1^i) - C(x_1^ix_2^j)$ tells us that, if C is closed, then for any polynomial f, $C(f) = C(x_i)\partial_i f$. Hence (this is the result 2.6)

$$\delta C(P_1^2) = C(x_i)\partial_iR,\ \ \delta C(P_2^2) = 0. \tag{4.4}$$

For the algebra $\mathbb{C}[x,y]$, Z^1 is the space of vector fields with coefficients in the unital augmentation of the same algebra, but for $A = \mathbb{C}[x,y]/R$, Z^1 is the algebra of vector fields that annihilate R (the algebra of vector fields tangential to the curve).

4.3. Closure for $p = 2$. For homology we use the basis (3.3); it is enough to examine one of the two,

$$P_1^3 = \hat{R} \otimes x_1 + x_1 \wedge x_2 \otimes R_2,$$
$$\delta C(P_1^3) = x_1C(R_1 \wedge x_1) + x_2C(R_2 \wedge x_1) - R_2C(x_1 \wedge x_2).$$

The first two arguments are exact; a certain amount of calculation is needed to verify that these terms are of the same form as the third one. We need the following simple formula, satisfied by closed 2–cochains: $C(x_2 \wedge f) = C(x_2 \wedge x_1)\partial_1 f,\ f \in A$. Now it follows easily that $\delta C(P_1^3) = -C(x_1 \wedge x_2)\partial_2R$, $\delta C(P_2^3) = C(x_1 \wedge x_2)\partial_1R$. Therefore, we can interpret the condition $\delta C(a) = 0$ as fixing the value $C(da)$, provided only that $C(P_2^2)\partial_iR = 0$, $i = 1,2$. (That is satisfied if $R = x^2y^3$, $C(x \wedge y) = xy$.)

4.4. Theorem. *Closure of a p–cochain C implies that its values for exact arguments are given recursively in the polynomial degree as in (4.1). Conversely, (4.1) can be solved recursively for all $C(da)$, if and only if the following conditions hold*

$$C \in Z^{2k,k+1} \quad : \qquad C(P_2^{2k})\partial_iR = 0,\ i = 1,2;$$

$$\begin{array}{rcc} C \in Z^{2k+1,k+1} & : & \sum C(P_i^{2k+1})\partial_i R = 0; \\ C \in Z^{2k,k} & : & always. \end{array}$$

4.5. Gauge invariance for $p = 1$. Trivial, all 1–cochains are gauge invariant, $H^1 = Z^1$.

4.6. Gauge invariance for $p = 2$. We must examine evaluations on the homology basis. To begin with, $\Delta C(x_1 \wedge x_2) = 0$, so that the evaluation $C(x_1 \wedge x_2)$ is gauge invariant. To examine the supplementary homology space, set $R = \sum A_{ij} x_1^i x_2^j$, $\hat{R} = \sum A_{ij} x_1^i \otimes x_2^j$. Then we have

$$\sum \frac{1}{2} A_{ij} \big(\Delta C(x_1^i \otimes x_2^j) + x_1^i \sum_{k=2}^{j-2} x_2^k \Delta C(x_2 \otimes x_2^{j-1-k}) + x_2^j \sum_{k=0}^{i-2} \Delta C(x_1 \otimes x_1^{i-k-1}) \big)$$
$$= E(x_i)\partial_i R.$$

Hence, in a gauge where C vanishes on arguments of lower degrees, $\Delta C(\hat{R}) \in DR$ and we have recovered Proposition 2.6.

4.7. Gauge invariance for $p = 3$. We have

$$\begin{aligned} \delta C(P_1^3) &= \Delta C(\hat{R} \otimes x_1 + x_1 \wedge x_2 \otimes R_2) \\ &= x_1 E(R_1 \wedge x_1) + x_2 E(R_2 \wedge x_1)) - R_2 E(x_1 \wedge x_2) \\ &= \sum \frac{1}{2} A_{ij} \big\{ x_1 E(x_1^{i-1} x_2^j \wedge x_1) + x_2 E(x_1^i x_2^{j-1} \wedge x_1) \big\} - R_2 E(x_1 \wedge x_2). \end{aligned} \tag{4.5}$$

With the help of the identity

$$\sum_{k=1}^{i-1} x_1^k \Delta C(x_1 \otimes x_1^{i-k-1} x_2^j \otimes x_1) = x_1^i E(x_2^j \wedge x_1) - x_1 E(x_1^{i-1} x_2^j \wedge x_1), \; j \geq 1,$$

and another one, similar, we can reduce (4.5) to

$$\Delta C(P_1^3) + \sum_{k=1}^{i-1} A_{ij} x_1^k \Delta C(x_1 \otimes x_1^{i-k-1} x_2^j \otimes x_1) + \sum_{k=1}^{i} A_{ij} x_2 x_1^{k-1} \Delta C(x_1 \otimes x_1^{i-k} x_2^{j-1} \otimes x_1)$$
$$= \sum A_{ij} \big\{ x_1^i E(x_2^j \wedge x_1) + x_1^i E(x_2^{j-1} \wedge x_1) \big\} - R_2 E(x_1 \wedge x_2).$$

A similar, further reduction leads to the result that, if δC vanishes on arguments of lower orders, $\Delta C(P_1^3) + ... = -(\partial_2 R) E(x_1 \wedge x_2)$, $\Delta C(P_2^3) + ... = (\partial_1 R) E(x_1 \wedge x_2)$. We recall that $\Delta C(a) = \delta E(a_1)$ and remember from Subsection 4.3 that $\delta E = 0$ implies that $\partial_i R E(x_1 \wedge x_2) = 0$. The above result is thus natural; the calculation is needed only to fix the numerical coefficients.

4.8. Proposition. If the gauge is fixed by the condition that $C(a) = 0$ for arguments a of lower degree, then the remaining gauge transformations take the following form,

$$
\begin{aligned}
\Delta C(P^1) &= 0, \quad \Delta C(P_1^{2k}) = \sum E_i \partial_i R, \\
\Delta C(P_2^{2k}) &= 0, \quad \Delta C(P^{2k+1}) = E dR^*, \;\; k > 0.
\end{aligned}
$$

Proof (outline). (a) The statement reflects the structure of (3.4). The dimension of $H^{p,k}$, over the local algebra, more or less, coincides with the dimension of $H_{p,k}$. 'More or less' comes from the existence of homologies of lower orders, as the complete calculation in Subsection 4.7 shows.

(b) We have

$$
\begin{aligned}
\Delta C(P_1^{2k}) &= \sum R_i E(P_i^{2k-1}) + \sum x_i E(Q_i) + \dots\, , \\
\Delta C(P_i^{2k+1}) &= \sum \varepsilon_{ij} R_j E(P_2^{2k-1}) + \sum x_j E(S_{ij}), \\
\Delta C(P_2^{2k}) &= \sum x_i E(T_i).
\end{aligned}
$$

The chains Q_i, S_{ij}, T_i are closed and, unless R_1 or R_2 is linear, exact. The reduction exemplified in (4.4) and in (4.5) is then available. The result is

$$
\Delta C(P_1^{2k}) + \dots = E(P_i^{2k-1})\partial_i R, \quad \Delta C(P_i^{2k+1}) + \dots = E(P_2^{2k-1})\varepsilon_{ij}\partial_j R, \quad \Delta C(P_2^{2k}) = 0.
$$

(c) The last case ($P_2^{2k} \in C_{2k,k+1}$) is simpler than the others and we give the details in that case only. Let $\tau \in S_p$ be the reversing permutation. Garsia's formula tells us that the chains $C_{p,k}$ correspond to the character $\tau \mapsto (-)^k$, so the projection $e_{2k}(k+1)P_2^{2k}$ has $\tau \mapsto (-)^{k+1}$. Now $\Delta C(P_2^{2k}) = \sum_{i=1}^2 x_i E(a_i)$, with $a_i \in C_{2k-1}$ closed and with the same symmetry: $\tau \mapsto (-)^{k+1}$. The symmetry of $C_{2k-1,k}$ (where the homology is) is $(-)^k$; therefore a_1 and a_2 are exact. The reduction process encounters no homology and leads to zero.

Putting it all together we get the following result (for notations, see Propositions 2.6 and 4.8).

4.9. Theorem. *Let V_R be the space of vector fields, with values in A, that annihilate R. Then as vector spaces,*

$$
\begin{aligned}
H^1 &= V_R, \\
H^{2k,k} &= A/DR, \\
H^{2k,k+1} &= \{a \in A_+, a\partial_1 R = a\partial_2 R = 0\}, \\
H^{2k+1,k+1} &= V_R/\{AdR^*\}, \;\; k > 0.
\end{aligned}
$$

5 Deformation of $x^n = 0$.

Here we complete the calculation of Hochschild cohomology of the algebra $A = \mathbb{C}[x]/x^n,\ n \geq 2$. This purely algebraic problem, though not associated with a curve, is nevertheless very similar to that posed by curves. In the context of singularity theory it is one of the standard forms. The chains are restricted to positive degree. This subalgebra of A is denoted A_+.

5.1. Homology. For convenience, $xxx^2...$ shall stand for either $x \otimes x \otimes x^2...$ or $x, x, x^2, ...$. The spaces H_p are one-dimensional for $p \geq 1$ and representative elements of Z_p are x, xx^{n-1}, $xx^{n-1}x$,..., or $(xx^{n-1})^k$ for $p = 2k$ and $(xx^{n-1})^k x$ for $p = 2k+1$.

5.2. Closed cochains. A p–cochain C is closed if

$$\delta C(a_1...a_{p+1}) := a_1 C(a_-) + (-)^{p+1} a_{p+1} C(a_+) - C(da) = 0, \tag{5.1}$$

with $a_- = a_2 \ldots a_{p+1},\ a_+ = a_1 \ldots a_p$. We interpret this relation, in the first place, as a recursion relation that determines the cochain C on exact arguments, in terms of its values on arguments of lower degree. For example, if the 1–cochain C is closed, then $C(x^k) = kx^{k-1}C(x),\ k = 2, ..., n$. Hence $C(x^k)$ is determined for $k = 2, ..., n-1$ by $C(x)$, and $C(x) \in A_+$ (thus restricted to positive degree).

The obstruction to this interpretation of (5.1) is $da = 0$; in this case closure requires the relation

$$\delta C(a) = a_1 C(a_-) + (-)^{p+1} a_{p+1} C(a_+) = 0. \tag{5.2}$$

But if $a = db$, then this last relation is automatic, since

$$\begin{aligned}
\delta C(db) &= b_1 b_2 C(b_3...) - b_1 C(db_-) + (-)^{p+1} b_{p+2} C(db_+) - b_{p+1} b_{p+2} C(b_1...b_p) \\
&= b_1 b_2 C(b_3...) - b_1 \Big(b_2 C(b_3...) + (-)^{p+1} b_{p+2} C(b_2...b_{p+1}) \Big) \\
&\quad - b_{p+1} b_{p+2} C(b_1...b_p) + b_{p+2} \Big((-)^{p+1} b_1 C(b_2...b_{p+1}) + b_{p+1} C(b_1...b_p) \Big) \\
&= 0.
\end{aligned}$$

The real obstruction is thus the presence of homology. When $a = xx^{n-1}x...$, then (5.2) reduces to

$$p = 2k: \qquad xC(x^{n-1}...x - x...x^{n-1}) = 0, \tag{5.3}$$

$$p = 2k-1: \ xC(x^{n-1}...x^{n-1}) + x^{n-1}C(x...x) = 0. \tag{5.4}$$

5.3. Proposition. The obstructions to interpreting the closure condition (5.1) as recursively fixing the value of $C(da)$ in terms of values of C on arguments of lower degrees are: [2]

$$p = 2k: \text{none}, \quad p = 2k-1: \ \ x^{n-1}C(x...,x). \tag{5.5}$$

[2] From now on dots indicate a sequence in which x and x^{n-1} alternate.

Homology selects the argument here also. The truth of the Proposition is obvious except for the possibility of accidental cancellations. Here, nevertheless, is a direct proof.

Proof of Proposition 5.3, case $p=2k$. For $p=2k$ and $m=1,2,...,\alpha,\quad \alpha=k(n-2)+1$, let

$$\phi^m := \sum_{\substack{1\leq p_1,\dots,p_k\leq m\\ p_1+\dots+p_k=k+m-1}} xx^{p_1}xx^{p_2}\dots xx^{p_k}x. \tag{5.6}$$

It may be shown by induction that

$$d\phi^{\alpha-1}=x^{n-1}\dots x-x\dots x^{n-1},\;\; d\phi^m=\phi_-^{m+1}-\phi_+^{m+1},\; m<\alpha-1.$$

Posing $\delta C(\phi^m)=0$ for $m<\alpha$, we find that the left hand side of (5.3)vanishes identically:

$$xC(x^{n-1}\dots x-x\dots x^{n-1})=xC(\phi_-^\alpha-\phi_+^\alpha)=xC(d\phi^{\alpha-1})=x^2C(\phi_-^{\alpha-1}-\phi_+^{\alpha-1})=\dots\,.$$

Iteration ends with $x^nC(a_-^{\alpha+1-n}-a_+^{\alpha+1-n})=0$.

Proof of Proposition 5.3, case $p=2k-1$. For $m=1,2,\dots\alpha=k(n-2)+1$, set

$$\psi^m := \sum_{\substack{1\leq p_1,\dots,p_k\leq m\\ p_1+\dots+p_k=k+m-1}} xx^{p_1}xx^{p_2}\dots xx^{p_k}. \tag{5.7}$$

Then $d\psi^{\alpha-1}=x^{n-1}\dots x^{n-1}=\psi_-^\alpha$ and for $m<\alpha-1,\;\; d\psi^m=\psi_-^{m+1}-\phi^{m+1}$, and

$$(x^l\psi_{p+1}^{\alpha-l})\otimes\psi_+^{\alpha-l}=x^{n-1}\otimes\phi^{\alpha+2-n},\;\; l=0,1,\dots n-2. \tag{5.8}$$

If $\delta C(\psi^m)=0,\;\; m<\alpha$, then the left hand side of (5.4) is

$$\begin{aligned}
xC(x^{n-1}\dots x^{n-1}) \;&+\; x^{n-1}C(x\dots x)\\
&=\; xC(d\psi^{\alpha-1}+x^{n-1}C(\phi^{\alpha+2-n})\\
&=\; x^2C(\psi_-^{\alpha-1})+2x^{n-1}C(\phi^{\alpha+2-n})=\dots\\
&=\; x^{n-1}C(\psi_-^{\alpha+2-n})+(n-1)x^{n-1}C(\phi^{\alpha+2-n})\\
&=\; x^{n-1}C(d\psi^{\alpha+1-n}+\phi^{\alpha+2n})+(n-1)x^{n-1}C(\phi^{\alpha+2-n})\\
&=\; nx^{n-1}C(x\dots x).
\end{aligned}$$

The proof of Proposition 5.3 is complete. The implication is that, if a $(2k-1)$–cochain C is closed, then $C(x\dots x)\in A_+$.

5.4. Exact cochains. Exact p–cochains have the form

$$C(a_1\dots a_p)=a_1E(a_-)+(-)^pa_pE(a_+)-E(da). \tag{5.9}$$

The obstruction to interpreting this relation as a recursion relation to determine the $E(da)$ is $da = 0$. Here too, the real obstruction, when C is closed, is the existence of homology. The most useful interpretation is this. Given any closed p–cochain a “gauge transformation” is the addition of an exact p–cochain, $C \to C + \Delta C$, with

$$\Delta C(a_1...a_p) = a_1 E(a_-) + (-)^p a_p E(a_+) - E(da). \tag{5.10}$$

The space H^p is the space of gauge invariant evaluations of closed p–cochains.

To illustrate, here is the situation for 2–cochains, when $n = 3$. Closure,

$$\delta C(xxx) = C(xx^2) - C(x^2x) = 0, \quad \delta C(xx^2x) = xC(x^2x - xx^2) = 0.$$

Gauge transformation

$$\Delta C(xx) = 2xE(x) - E(x^2), \quad \Delta C(xx^2) = xE(x^2) + x^2E(x) = \Delta C(x^2x),$$

By means of gauge transformations we can, for example, reduce $C(xx)$ to zero. Cohomology is the existence of the gauge invariant object $C(xx^2) + xC(xx)$ Mod x^2.

5.5. Theorem. The space of the gauge-equivalent evaluations, and the associated cohomology spaces on $Z_p(A,A)$ are as follows

$$\begin{array}{rcl}
p = 0 & : & A \\
 & & H^0(A,A) = \text{span}\{1, x, ..., x^{n-1}\}, \text{ dim.} = 1; \\
p = 1 & : & C(x) \\
 & & H^1(A,A) = \text{span}\{x, ..., x^{n-1}\}, \text{ dim.} = n-1; \\
p = 2k-1 & : & \sum_{l=0}^{n-2} x^l C(\phi^{\alpha - l}) \quad (k > 1) \\
 & & H^{2k-1}(A,A) = \text{span}\{x, ..., x^{n-1}\}, \text{ dim.} = n-1; \\
p = 2k & : & \sum_{l=0}^{n-2} x^l C(\psi^{\alpha - l}) \ \text{Mod}\ \mathbb{C}x^{n-1} \\
 & & H^{2k}(A,A) = \text{span}\{1, x, ..., x^{n-2}\}, \text{ dim.} = n-1.
\end{array}$$

Proof. By a direct and straightforward calculation we obtain, for $p = 2k$, $\sum_{l=0}^{n-2} x^l \Delta C(\psi^{\alpha - l}) = nx^{n-1}E(x...x)$, and for $p = 2k-1$, $\sum_{l=0}^{2} x^l \Delta C(\phi^{\alpha - 1}) = 0$.

5.6. Proposition. The BGS ‘decomposition’ for $k \geq 1$ is

$$H_{2k} = H_{2k,k}, \quad H_{2k+1} = H_{2k+1,k+1}.$$

Proof. The element $x...x^{n-1} \in Z_{2k}$ lifted to $Z_{2k}(\mathbb{C}[x], \mathbb{C}[x])$, is

$$d(xx^{n-1})^k = x^n \circledS\!h (xx^{n-1})^{k-1}.$$

If $(xx^{n-1})^{k-1}$ is of type $H_{2k-2,k-1}$, then the right hand side is of type $H_{2k-1,k}$, and $(xx^{n-1})^k$ is of type $H_{2k,k}$. Since xx^{n-1} is indeed of type $H_{2,1}$ the result follows by induction. Similarly, $d(xx^{n-1})^k x = x^n \circledS\!h (xx^{n-1})^{k-1}x$, and the same argument applies *mutatis mutandi*.

Appendix

Hochschild and Harrison cohomology of complete intersections

I will explain here a way to calculate Hochschild and Harrison cohomology groups for algebras of functions on singular planar curves etc. based on Koszul resolutions. This calculation is standard and definitely known to specialists.

A1. Reminder on complete intersections and Koszul resolution

Results of this section can be found e.g. in the classical textbook [18].

Suppose that we are given a system of polynomial equations (say, over the field of complex numbers $\mathbf{C}$, one can replace it by an arbitrary field):

$$f_1(z_1, \ldots, z_n) = 0, \ldots, f_m(z_1, \ldots, z_n) = 0$$

Denote by A the quotient algebra $P/(f_1, \ldots, f_m)$ where P denotes the ring of polynomials $\mathbf{C}[z_1, \ldots, z_n]$.

We say that we have a complete intersection if the dimension of the set of solutions of the system above is $n - m$. A sufficient condition for this is that $f_1, \ldots, f_m$ form a regular sequence in P, i.e. for any $k \leq n$ element f_k is not a divisor of zero in the quotient of P by the ideal generated by $f_1, \ldots, f_{k-1}$.

Theorem 1 *Assume (in the previous notations) the condition of the complete intersection. Let us consider* $\mathbf{Z}_{\leq 0}$*–graded supercommutative superalgebra*

$$\tilde{A} := P \otimes \wedge(\{\alpha_j\}_{j=1,\ldots,m})$$

where subalgebra P is in degree 0 and generators α_j are in degree -1, endowed with differential

$$d_{\tilde{A}} := \sum_j f_j \frac{\partial}{\partial \alpha_j}.$$

Then cohomology of this differential is zero in negative degrees and isomorphic to $P/(f_1, \ldots, f_m)$ in degree 0.

In the above theorem one can replace $P = \mathbf{C}[z_1, \ldots, z_n]$ by the algebra of functions on arbitrary smooth n–dimensonal affine algebraic variety. Complex $(\tilde{A}, d_{\tilde{A}})$ is called the Koszul resolution of A. Slightly abusing notations we will write $\tilde{A} = \mathbf{C}[z_1, \ldots, z_n; \alpha_1, \ldots, \alpha_m]$ meaning that (α_i) are fermionic (odd) variables. Here and later variables denoted by Latin (resp. Greek) letters are even (resp. odd).

A2. Generalities on Hochschild and Harrison cohomological complexes for differential graded algebras

In what follows all complexes will be $\mathbf{Z}$–graded with the differential of degree $+1$. A morphism of complexes is called a quasi-isomorphism iff it induces an isomorphism of cohomology groups. A vector space can be considered as a complex concentrated in degree 0 and endowed with zero differential.

Definitions of homological and cohomological Hochschild complexes extend immediately to the case of differential graded algebras (dga in short), the same for Harrison (co-)homological complexes in the graded commutative case. The underlying $\mathbf{Z}$–graded space for the cohomological Hochschild complex for a dga F with coeffcients in a dg bimodule M is defined as the infinite product (in the category of $\mathbf{Z}$–graded spaces)

$$C^\bullet(F,M) := \prod_{n \geq 0} \underline{Hom}(F[1]^{\otimes n}, M)$$

where $\underline{Hom}$ is inner Hom–space in tensor category of $\mathbf{Z}$–graded spaces,

$$\left(\underline{Hom}(U,V)\right)^k := \prod_{n \in \mathbf{Z}} Hom(U^n, V^{n+k})$$

and $F[1]$ denotes the complex obtained from F by the shift of the grading, $F[1]^k := F^{k+1}$. The formula for the differential in $C^\bullet(F,M)$ is the sum of a super-version of the formula for the differential in the an ordinary algebra (in degree 0), and a term arising from the differential in F itself (see e.g. section 5.3 from [17] for a similar case of the homological Hochschild complex).

Lemma 1 *If $\phi : \tilde{F} \to F$ is a quasi-isomorphism between two dga's, then the corresponding cohomological Hochschild complexes $C^\bullet(F,F)$ and $C^\bullet(\tilde{F},\tilde{F})$ are quasi-isomorphic.*

Proof: An algebra F can be considered as a differential graded bimodule over $\tilde{F}$ via the homomorphism $\phi : \tilde{F} \to F$. Let us consider three complexes and natural homomorphisms between them:

$$C^\bullet(\tilde{F},\tilde{F}) \to C^\bullet(\tilde{F},F) \leftarrow C^\bullet(F,F)$$

All three complexes carry complete decreasing filtrations with the associated quotients (and maps between them)

$$\underline{Hom}(\tilde{F}[1]^{\otimes k},\tilde{F}) \to \underline{Hom}(\tilde{F}[1]^{\otimes k},F) \leftarrow \underline{Hom}(F[1]^{\otimes k},F)$$

We see that associated quotients are quasi-isomorphic, and applying spectral sequences we conclude that $C^\bullet(\tilde{F},\tilde{F})$ and $C^\bullet(F,F)$ are quasi-isomorphic. *Q.E.D.*

For a graded supercommutative F one can define the Hodge decomposition for Hochschild cochains, and Harrison cohomology in the same way as in the usual non-graded case. In the above lemma the quasi-isomorphism between Hochschild cohomology of the resolution and of algebra itself is manifestly compatible with the Hodge decomposition.

A3. Calculation of Hochschild and Harrison cohomology for complete intersections

The cohomological Hochschild–Kostant–Rosenberg theorem says that the Hochschild cohomology of the algebra $\mathcal{O}_X$ of functions on an algebraic affine manifold X is the algebra T_X^{poly} of polyvector fields on X. Moreover, there is a canonical quasi-isomorphism $T_X^{poly} \to C^\bullet(\mathcal{O}_X,\mathcal{O}_X)$ mapping polyvector field $f v_0 \wedge \ldots \wedge v_n$ where $f \in \mathcal{O}_X$, $(v_i)_{i=1,n}$ are derivations of $\mathcal{O}_X$, to the polylinear operator

$$a_1 \otimes \ldots \otimes a_n \mapsto f \sum_{\sigma \in \Sigma_n} sign(\sigma) \prod_i v_{\sigma(i)}(a_i)$$

The super-version of this theorem is also true, e.g. for supermanifold $Y = \mathbf{C}^{n|m}$, we have $\mathcal{O}_Y = \mathbf{C}[z_1,\ldots,z_n;\alpha_1,\ldots,\alpha_m]$ and its Hochschild cohomology is the algebra T_Y^{poly}:

$$T_Y^{poly} = \mathbf{C}[z_1,\ldots,z_n;\eta_1,\ldots,\eta_n;\alpha_1,\ldots,\alpha_m;b_1,\ldots,b_m],$$

$$deg(z_i) = 0 \;\; deg(\eta_i) = +1, \;\; deg(\alpha_j) = -1, \;\; deg(b_j) = +2$$

Here the new variables η_i, b_j have the meaning of derivations $\partial/\partial z_i$, $\partial/\partial\alpha_j$. Strictly speaking, here we should consider not polynomials but formal power series with respect to variables η_i, b_j, but it gives the same result in the category of $\mathbf{Z}$-graded spaces because there are only finitely many monomials in η_i, b_j in any given degree.

The dga $\tilde{A}$ is obtained from $\mathcal{O}_Y$ by "switching on" the differential $d_{\tilde{A}}$. Here we describe the corresponding HKR description of the Hochschild cohomology of $\tilde{A}$, and therefore of $H^\bullet(A,A)$ by lemma 1.

Proposition 1 *Complex $C^\bullet(\tilde{A},\tilde{A})$ is quasi-isomorphic to $T := T_Y^{poly}$ endowed with the differential*

$$d_T := \sum_{i,j} \frac{\partial f_j}{\partial z_i} b_j \frac{\partial}{\partial \eta_i} + \sum_j f_j \frac{\partial}{\partial \alpha_j}$$

The Hodge grading is given by counting variables η_i, b_j.

Proof: The formula for d_T is just the formula for the Lie derivative of a polyvector field on $Y = \mathbf{C}^{n|m}$ with respect to the odd vector field $d_{\tilde{A}} = \sum_j f_j \frac{\partial}{\partial \alpha_j}$. It is easy to see that the formulas from above give a homomorphism of complexes

$$\chi : (T, d_T) \to C^{\bullet}(\tilde{A}, \tilde{A})$$

We have to prove that it is a quasi-isomorphism. Let us introduce $\mathbf{Z}_{\geq 0}$–grading deg_α on $\mathcal{O}_Y$ by the total number of variables α_j (incidentally, it coincides with minus the standard $\mathbf{Z}$–grading on $\mathcal{O}_Y$). A Hochschild cochain $\mathcal{O}_Y^{\otimes n} \to \mathcal{O}_Y$ is called homogeneous of $\deg_\alpha$ degree $N \in \mathbf{Z}$ if it is homogenous with respect to grading deg_α of degree N. The whole Hochschild complex $C^{\bullet}(\mathcal{O}_Y, \mathcal{O}_Y)$ is the product over all $N \in \mathbf{Z}$ of subcomplexes consisting of $\deg_\alpha$ degree N cochains. The Hochschild differential of algebra $\mathcal{O}_Y$ preserves the $\deg_\alpha$ grading. The correction to the differential coming from $d_{\tilde{A}}$ decreases this grading by 1. Finally, it is obvious that for a non-zero cochain its $\deg_\alpha$ is bounded from below (by $-m$). Therefore we have a convergent spectral sequence proving that χ is a quasi-isomorphism. The statement about the Hodge grading is obvious. *Q.E.D.*

Now we introduce a smaller complex

$$\tilde{T} := A[\eta_1, \ldots, \eta_n; b_1, \ldots, b_m], \; d_{\tilde{T}} := \sum_{i,j} \frac{\partial f_j}{\partial x_i} b_j \frac{\partial}{\partial \eta_i}$$

where the variables have the same grading as before, $deg(\eta_i) = +1$, $deg(b_j) = +2$.

Theorem 2 *Under the previous assumptions the Hochschild cohomology of A is isomorphic to the cohomology of complex $(\tilde{T}, d_{\tilde{T}})$. The Hodge grading is given by counting variables η_i, b_j.*

Proof: The obvious map $(T_F, d_T) \to (\tilde{T}, d_{\tilde{T}})$ induces a quasi-isomorphism on graded quotients for the filtration by the total number of variables η_i. *Q.E.D.*

The conclusion for the only non-trivial Harrison cohomology are in degrees 1 and 2 and are given by kernel and cokernel of the map

$$A^n \stackrel{(\partial f_j / \partial_{z_i})}{\longrightarrow} A^m$$

In particular, there is no obstruction for commutative deformations as $Harr^3(A) = 0$. It is easy to see that a miniversal commutative deformation of A is given by any deformation $\tilde{f}_1(z,t), \ldots, \tilde{f}_m(z,t)$ of polynomials $f_1(z), \ldots, f_m(z)$ depending on formal parameters $t_1, \ldots, t_N$ where $N = rk\, Harr^2(A)$, such that vectors

$$v_k := \left(\frac{\partial \tilde{f}_1}{\partial t_k}_{|t=0}, \ldots, \frac{\partial \tilde{f}_m}{\partial t_k}_{|t=0} \right), \; k = 1, \ldots, N$$

form a basis in $Harr^2(A) = A^m / \left(\frac{\partial f_j}{\partial z_i}\right) A^n$. The deformed algebra is

$$\mathbf{C}[[t_1,\dots,t_N]][z_1,\dots,z_n]/I$$

where I is the completion with respect to the topology on $\mathbf{C}[[t_1,\dots,t_N]]$ associated with the maximal ideal, of the ideal generated by $\tilde{f}_1(z,t),\dots,\tilde{f}_m(z,t)$.

In particular, if we have only one equation $f(z) = f_1(z) = 0$ then $Harr^2(A)$ is the quotient $\mathbf{C}[z_1,\dots,z_n]/(f,\partial f/\partial z_1,\dots,\partial f/\partial z_n)$.

In the case $n = 2$ and $m = 1$, Hochschild cohomology groups consists of an unstable part in lower degrees and 2–periodically repeated block

$$A \overset{(\partial_{z_1} f_1, \partial_{z_2} f_1)}{\longrightarrow} A \oplus A \overset{(\partial_{z_2} f_1, -\partial_{z_1} f_1)}{\longrightarrow} A$$

Finally, for $n = m = 1$, $A = \mathbf{C}[z]/(z^k)$ we have

$$H^O(A,A) = A \simeq \mathbf{C}^k,\ H^l(A,A) \simeq \mathbf{C}^{k-1} \text{ for } l = 1,2,\dots$$

A4. Calculation of Hochschild a homology with coefficients with the diagonal bimodule, for complete intersections

Similarly, one can calculate Hochschild homology $H_*(A,A)$ for complete intersections. Here is the final result:

Theorem 3 *In previous notations and under the assumption of complete intersection the Hochschild homology $H_*(A,A)$ of A is isomorphic to the cohomology of complex $\tilde{\Omega} := A[\xi_1,\dots,\xi_n;a_1,\dots,a_m]$ where degrees of variables are $deg(\xi_i) = -1,\ deg(a_j) = -2$ endowed with the differential $d_{\tilde{\Omega}} := \sum_{i,j} \frac{\partial f_j}{\partial z_i} \xi_i \frac{\partial}{\partial a_j}$. The Hodge grading is given by counting variables ξ_i, a_j.*

The proof is parallel to one for the cohomological complex. An example of this calculation for the case of truncated polynomial ring can be found in [17], exercise E.4.1.8 and Proposition 5.4.15.

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