

Noncommutative Geometry for Pedestrians*

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Abstract

A short historical review is made of some recent literature in the field of noncommutative geometry, especially the efforts to add a gravitational field to noncommutative models of space-time and to use it as an ultraviolet regulator. An extensive bibliography has been added containing reference to recent review articles as well as to part of the original literature.

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1 Introduction

To control the divergences which from the very beginning had plagued quantum electrodynamics, Heisenberg already in the 1930's proposed to replace the space-time continuum by a lattice structure. A lattice however breaks Lorentz invariance and can hardly be considered as fundamental. It was Snyder [201, 202] who first had the idea of using a noncommutative structure at small length scales to introduce an effective cut-off in field theory similar to a lattice but at the same time maintaining Lorentz invariance. His suggestion came however just at the time when the renormalization program finally successfully became an effective if rather *ad hoc* prescription for predicting numbers from the theory of quantum electrodynamics and it was for the most part ignored. Some time later von Neumann introduced the term 'noncommutative geometry' to refer in general to a geometry in which an algebra of functions is replaced by a noncommutative algebra. As in the quantization of classical phase-space, coordinates are replaced by generators of the algebra [60]. Since these do not commute they cannot be simultaneously diagonalized and the space disappears. One can argue [148] that, just as Bohr cells replace classical-phase-space points, the appropriate intuitive notion to replace a 'point' is a Planck cell of dimension given by the Planck area. If a coherent description could be found for the structure of space-time which were pointless on small length scales, then the ultraviolet divergences of quantum field theory could be eliminated. In fact the elimination of these divergences is equivalent to coarse-graining the structure of space-time over small length scales; if an ultraviolet cut-off Λ is used then the theory does not see length scales smaller than Λ^{-1} . When a physicist calculates a Feynman diagram he is forced to place a cut-off Λ on the momentum variables in the integrands. This means that he renounces any interest in regions of space-time of volume less than Λ^{-4} . As Λ becomes larger and larger the forbidden region becomes smaller and smaller but it can never be made to vanish. There is a fundamental length scale, much larger than the Planck length, below which the notion of a point is of no practical importance. The simplest and most elegant, if certainly not the only, way of introducing such a scale in a Lorentz-invariant way is through the introduction of noncommuting space-time 'coordinates'.

As a simple illustration of how a 'space' can be 'discrete' in some sense and still covariant under the action of a continuous symmetry group one can consider the ordinary round 2-sphere, which has acting on it the rotational group SO_3 . As a simple example of a lattice structure one can consider two points on the sphere, for example the north and south poles. One immediately notices of course that by choosing the two points one has broken the rotational invariance. It can be restored at the expense of commutativity. The set of functions on the two points can be identified with the algebra of diagonal 2×2 matrices, each of the two entries on the diagonal corresponding to a possible value of a function at one of the two points. Now an action of a group on the lattice is equivalent to an action of the group on the matrices and there can

obviously be no non-trivial action of the group SO_3 on the algebra of diagonal 2×2 matrices. However if one extends the algebra to the noncommutative algebra of all 2×2 matrices one recovers the invariance. The two points, so to speak, have been smeared out over the surface of a sphere; they are replaced by two cells. An ‘observable’ is an hermitian 2×2 matrix and has therefore two real eigenvalues, which are its values on the two cells. Although what we have just done has nothing to do with Planck’s constant it is similar to the procedure of replacing a classical spin which can take two values by a quantum spin of total spin $1/2$. Only the latter is invariant under the rotation group. By replacing the spin $1/2$ by arbitrary spin s one can describe a ‘lattice structure’ of $n = 2s + 1$ points in an SO_3 -invariant manner. The algebra becomes then the algebra M_n of $n \times n$ complex matrices and there are n cells of area $2\pi\hbar$ with

$$n \simeq \frac{\text{Vol}(S^2)}{2\pi\hbar}.$$

In general, a static, closed surface in a fuzzy space-time as we define it can only have a finite number of modes and will be described by some finite-dimensional algebra [90, 92, 94, 95, 96]. Graded extensions of some of these algebras have also been constructed [97, 98]. Although we are interested in a matrix version of surfaces primarily as a model of an eventual noncommutative theory of gravity they have a certain interest in other, closely related, domain of physics. We have seen, for example, that without the differential calculus the fuzzy sphere is basically just an approximation to a classical spin r by a quantum spin r with \hbar in lieu of \hbar . It has been extended in various directions under various names and for various reasons [17, 58, 105, 22]. In order to explain the finite entropy of a black hole it has been conjectured, for example by ‘t Hooft [207], that the horizon has a structure of a fuzzy 2-sphere since the latter has a finite number of ‘points’ and yet has an SO_3 -invariant geometry. The horizon of a black hole might be a unique situation in which one can actually ‘see’ the cellular structure of space.

It is to be stressed that we shall here modify the structure of Minkowski space-time but maintain covariance under the action of the Poincaré group. A fuzzy space-time looks then like a solid which has a homogeneous distribution of dislocations but no disclinations. We can pursue this solid-state analogy and think of the ordinary Minkowski coordinates as macroscopic order parameters obtained by coarse-graining over scales less than the fundamental scale. They break down and must be replaced by elements of some noncommutative algebra when one considers phenomena on these scales. It might be argued that since we have made space-time ‘noncommutative’ we ought to do the same with the Poincaré group. This logic leads naturally to the notion of a q -deformed Poincaré (or Lorentz) group which act on a very particular noncommutative version of Minkowski space called q -Minkowski space [141, 142, 28, 10, 30]. The idea of a q -deformation goes back to Sylvester [200]. It was taken up later by Weyl [212] and Schwinger [197] to produce a finite version of quantum mechanics.

It has also been argued, for conceptual as well as practical, numerical reasons, that a lattice version of space-time or of space is quite satisfactory if one uses a random lattice structure or graph. The most widely used and successful modification of space-time is in fact what is called the lattice approximation. From this point of view the Lorentz group is a classical invariance group and is not valid at the microscopic level. Historically the first attempt to make a finite approximation to a curved manifold was due to Regge and this developed into what is now known as the Regge calculus. The idea is based on the fact that the Euler number of a surface can be expressed as an integral of the gaussian curvature. If one applies this to a flat cone with a smooth vertex then one finds a relation between the defect angle and the mean curvature of the vertex. The latter is encoded in the former. In recent years there has been a burst of activity in this direction, inspired by numerical and theoretical calculations of critical exponents of phase transitions on random surfaces. One chooses a random triangulation of a surface with triangles of constant fixed length, the lattice parameter. If a given point is the vertex of exactly six triangles then the curvature at the point is flat; if there are less than six the curvature is positive; if there are more than six the curvature is negative. Non-integer values of curvature appear through statistical fluctuation. Attempts have been made to generalize this idea to three dimensions using tetrahedra instead of triangles and indeed also to four dimensions, with euclidean signature. The main problem, apart from considerations of the physical relevance of a theory of euclidean gravity, is that of a proper identification of the curvature invariants as a combination of defect angles. On the other hand some authors have investigated random lattices from the point of view of noncommutative geometry. For an introduction to the lattice theory of gravity from these two different points of view we refer to the books by Ambjørn & Jonsson [5] and by Landi [136]. Compare also the loop-space approach to quantum gravity [11, 82, 7].

One typically replaces the four Minkowski coordinates x^μ by four generators q^μ of a noncommutative algebra which satisfy commutation relations of the form

$$[q^\mu, q^\nu] = i\tilde{\kappa}q^{\mu\nu}. \quad (1.1)$$

The parameter $\tilde{\kappa}$ is a fundamental area scale which we shall suppose to be of the order of the Planck area:

$$\tilde{\kappa} \simeq \mu_P^{-2} = G\hbar.$$

There is however no need for this assumption; the experimental bounds would be much larger. Equation (1.1) contains little information about the algebra. If the right-hand side does not vanish it states that at least some of the q^μ do not commute. It states also that it is possible to identify the original coordinates with the generators q^μ in the limit $\tilde{\kappa} \rightarrow 0$:

$$\lim_{\tilde{\kappa} \rightarrow 0} q^\mu = x^\mu. \quad (1.2)$$

For mathematical simplicity we shall suppose this to be the case although one could include a singular ‘renormalization constant’ Z and replace (1.2) by an equation of the form

$$\lim_{\hbar \rightarrow 0} q^\mu = Z x^\mu. \quad (1.3)$$

If, as we shall argue, gravity acts as a universal regulator for ultraviolet divergences then one could reasonably expect the limit $\hbar \rightarrow 0$ to be a singular limit.

Let \mathcal{A}_\hbar be the algebra generated in some sense by the elements q^μ . We shall be here working on a formal level so that one can think of \mathcal{A}_\hbar as an algebra of polynomials in the q^μ although we shall implicitly suppose that there are enough elements to generate smooth functions on space-time in the commutative limit. Since we have identified the generators as hermitian operators on some Hilbert space we can identify \mathcal{A}_\hbar as a subalgebra of the algebra of all operators on the Hilbert space. We have added the subscript \hbar to underline the dependence on this parameter but of course the commutation relations (1.1) do not determine the structure of \mathcal{A}_\hbar . We in fact conjecture that every possible gravitational field can be considered as the commutative limit of a noncommutative equivalent and that the latter is strongly restricted if not determined by the structure of the algebra \mathcal{A}_\hbar . We must have then a large number of algebras \mathcal{A}_\hbar for each value of \hbar .

Interest in Snyder’s idea was revived much later when mathematicians, notably Connes [42] and Woronowicz [214, 215], succeeded in generalizing the notion of differential structure to noncommutative geometry. Just as it is possible to give many differential structures to a given topological space it is possible to define many differential calculi over a given algebra. We shall use the term ‘noncommutative geometry’ to mean ‘noncommutative differential geometry’ in the sense of Connes. Along with the introduction of a generalized integral [50] this permits one in principle to define the action of a Yang-Mills field on a large class of noncommutative geometries.

One of the more obvious applications was to the study of a modified form of Kaluza-Klein theory in which the hidden dimensions were replaced by noncommutative structures [145, 146, 67]. In simple models gravity could also be defined [146, 147] although it was not until much later [171, 69, 117] that the technical problems involved in the definition of this field were to be to a certain extent overcome. Soon even a formulation of the standard model of the electroweak forces could be given [48]. A simultaneous development was a revival [161, 52, 145] of the idea of Snyder that geometry at the Planck scale would not necessarily be described by a differential manifold.

One of the advantages of noncommutative geometry is that smooth, finite examples [148] can be constructed which are invariant under the action of a continuous symmetry group. Such models necessarily have a minimal length associated to them and quantum field theory on them is necessarily finite [90, 92, 94, 24]. In general this minimal length is usually considered to be in some

way or another associated with the gravitational field. The possibility which we shall consider here is that the mechanism by which this works is through the introduction of noncommuting ‘coordinates’. This idea has been developed by several authors [103, 148, 62, 124, 73, 123, 31] from several points of view since the original work of Snyder. It is the left-hand arrow of the diagram

$$\begin{array}{ccc}
 \mathcal{A}_{\bar{k}} & \longleftarrow & \Omega^*(\mathcal{A}_{\bar{k}}) \\
 \downarrow & & \uparrow \\
 \text{Cut-off} & & \text{Gravity}
 \end{array} \tag{1.4}$$

The $\mathcal{A}_{\bar{k}}$ is a noncommutative algebra and the index \bar{k} indicates the area scale below which the noncommutativity is relevant; this would normally be taken to be the Planck area.

The top arrow is a mathematical triviality; the $\Omega^*(\mathcal{A}_{\bar{k}})$ is a second algebra which contains $\mathcal{A}_{\bar{k}}$ and is what gives a differential structure to it just as the algebra of de Rham differential forms gives a differential structure to a smooth manifold. There is an associated differential d , which satisfies the relation $d^2 = 0$. The couple $(\Omega^*(\mathcal{A}), d)$ is known as a differential calculus over the algebra \mathcal{A} . The algebra \mathcal{A} is what in ordinary geometry would determine the set of points one is considering, with possibly an additional topological or measure theoretic structure. The differential calculus is what gives an additional differential structure or a notion of smoothness. On a commutative algebra of functions on a lattice, for example, it would determine the number of nearest neighbours and therefore the dimension. The idea of extending the notion of a differential to noncommutative algebras is due to Connes [42, 45, 48, 49] who proposed a definition based on a formal analogy with an identity in ordinary geometry involving the Dirac operator \mathcal{D} . Let ψ be a Dirac spinor and f a smooth function. Then one can write

$$i\gamma^\alpha e_\alpha f \psi = \mathcal{D}(f\psi) - f\mathcal{D}\psi.$$

Here e_α is the Pfaffian derivative with respect to an orthonormal moving frame θ^α . This equation can be written

$$\gamma^\alpha e_\alpha f = -i[\mathcal{D}, f]$$

and it is clear that if one makes the replacement

$$\gamma^\alpha \mapsto \theta^\alpha$$

then on the right-hand side one has the de Rham differential. Inspired by this fact, one defines a differential in the noncommutative case by the formula

$$df = i[F, f]$$

where now f belongs to a noncommutative algebra \mathcal{A} with a representation on a Hilbert space \mathcal{H} and F is an operator on \mathcal{H} with spectral properties which

make it look like a Dirac operator. The triple $(\mathcal{A}, F, \mathcal{H})$ is called a spectral triple. It is inspired by the K -cycle introduced by Atiyah [9] to define a dual to K -theory [8]. The simplest example is obtained by choosing $\mathcal{A} = \mathbb{C} \oplus \mathbb{C}$ acting on \mathbb{C}^2 by left multiplication and

$$F = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The 1-forms are then off-diagonal 2×2 complex matrices. The differential is extended to them using the same formula as above but with a bracket which is an anticommutator instead of a commutator. Since $F^2 = 1$ it is immediate that $d^2 = 0$. The algebra \mathcal{A} of this example can be considered as the algebra of functions on 2 points and the differential can be identified with the finite-difference operator.

One can argue [59, 156, 152], not completely successfully, that each gravitational field is the unique ‘shadow’ in the limit $\hbar \rightarrow 0$ of some differential structure over some noncommutative algebra. This would define the right-hand arrow of the diagram. A hand-waving argument can be given [21, 154] which allows one to think of the noncommutative structure of space-time as being due to quantum fluctuations of the light-cone in ordinary 4-dimensional space-time. This relies on the existence of quantum gravitational fluctuations. A purely classical argument based on the formation of black-holes has been also given [62]. In both cases the classical gravitational field is to be considered as regularizing the ultraviolet divergences through the introduction of the noncommutative structure of space-time. This can be strengthened as the conjecture that the classical gravitational field and the noncommutative nature of space-time are two aspects of the same thing. If the gravitational field is quantized then presumably the light-cone will fluctuate and any two points with a space-like separation would have a time-like separation on a time scale of the order of the Planck time, in which case the corresponding operators would no longer commute. So even in flat space-time quantum fluctuations of the gravitational field could be expected to introduce a non-locality in the theory. This is one possible source of noncommutative geometry on the order of the Planck scale. The composition of the three arrows in (1.4) is an expression of an old idea, due to Pauli, that perturbative ultraviolet divergences will somehow be regularized by the gravitational field [57, 107]. We refer to Garay [84] for a recent review.

One example from which one can seek inspiration in looking for examples of noncommutative geometries is quantized phase space, which had been already studied from a noncommutative point of view by Dirac [60]. The minimal length in this case is given by the Heisenberg uncertainty relations or by modifications thereof [124]. In fact in order to explain the supposed Zitterbewegung of the electron Schrödinger [193] had proposed to mix position space with momentum space in order to obtain a set of center-of-mass coordinates which did not commute. This idea has inspired many of the recent attempts to introduce minimal

lengths. We refer to [73, 123] for examples which are in one way or another connected to noncommutative geometry. Another concept from quantum mechanics which is useful in concrete applications is that of a coherent state. This was first used in a finite noncommutative geometry by Grosse & Prešnajder [91] and later applied [123, 33, 39] to the calculation of propagators on infinite noncommutative geometries, which now become regular 2-point functions and yield finite vacuum fluctuations. Although efforts have been made in this direction [39] these fluctuations have not been satisfactorily included as a source of the gravitational field, even in some ‘quasi-commutative’ approximation. If this were done then the missing arrow in (1.4) could be drawn. The difficulty is partly due to the lack of tractable noncommutative versions of curved spaces.

The fundamental open problem of the noncommutative theory of gravity concerns of course the relation it might have to a future quantum theory of gravity either directly or via the theory of ‘strings’ and ‘membranes’. But there are more immediate technical problems which have not received a satisfactory answer. We shall mention the problem of the definition of the curvature. It is not certain that the ordinary definition of curvature taken directly from differential geometry is the quantity which is most useful in the noncommutative theory. Cyclic homology groups have been proposed by Connes as the appropriate generalization to noncommutative geometry of topological invariants; the definition of other, non-topological, invariants is not clear. It is not in fact even obvious that one should attempt to define curvature invariants.

There is an interesting theory of gravity, due to Sakharov and popularized by Wheeler, called induced gravity, in which the gravitational field is a phenomenological coarse-graining of more fundamental fields. Flat Minkowski space-time is to be considered as a sort of perfect crystal and curvature as a manifestation of elastic tension, or possibly of defects, in this structure. A deformation in the crystal produces a variation in the vacuum energy which we perceive as gravitational energy. ‘Gravitation is to particle physics as elasticity is to chemical physics: merely a statistical measure of residual energies.’ The description of the gravitational field which we are attempting to formulate using noncommutative geometry is not far from this. We have noticed that the use of noncommuting coordinates is a convenient way of making a discrete structure like a lattice invariant under the action of a continuous group. In this sense what we would like to propose is a Lorentz-invariant version of Sakharov’s crystal. Each coordinate can be separately measured and found to have a distribution of eigenvalues similar to the distribution of atoms in a crystal. The gravitational field is to be considered as a measure of the variation of this distribution just as elastic energy is a measure of the variation in the density of atoms in a crystal.

We shall here accept a noncommutative structure of space-time as a mathematical possibility. One can however attempt to associate the structure with other phenomena. A first step in this direction was undoubtedly taken by Bohr & Rosenfeld [21] when they deduced an intrinsic uncertainty in the position

of an event in space-time from the quantum-mechanical measurement process. This idea has been since pursued by other authors [4] and even related to the formation of black holes [62, 137] and to the influence of quantum fluctuations in the gravitational field [154, 7]. An uncertainty relation in the measurements of an event is one of the most essential aspects of a noncommutative structure. The possible influence of quantum-mechanical fluctuations on differential forms was realized some time ago by Segal [198]. A related idea is what one might refer to as ‘spontaneous lattization’. A quantum operator is a very singular object in general and the correct definition of the space-time coordinates, considered as quantum operators, could give rise to a preferred set of events in space-time which has some of the aspects of a ‘lattice’ in the sense that each operator, has a discrete spectrum [196, 124, 73, 123, 122]. The work of Yukawa [218] and Takano [205] could be considered as somewhat similar to this, except that the fuzzy nature of space-time is emphasized and related to the presence of particles. Finkelstein [75] has attempted a very philosophical derivation of the structure of space-time from the notion of ‘simplicity’ (in the group-theoretic sense of the word) which has led him to the possibility of the ‘superposition of points’, something very similar to noncommutativity. We shall mention below the attempts to derive a noncommutative structure of space-time from string theory.

When referring to the version of space-time which we describe here we use the adjective ‘fuzzy’ to underline the fact that points are ill-defined. Since the algebraic structure is described by commutation relations the qualifier ‘quantum’ has also been used [201, 62, 156]. This latter expression is unfortunate since the structure has no immediate relation to quantum mechanics and also it leads to confusion with ‘spaces’ on which ‘quantum groups’ act. To add to the confusion the word ‘quantum’ has also been used [87] to designate equivalence classes of ordinary differential geometries which yield isomorphic string theories and the word ‘lattice’ has been used [201, 73, 207] to designate what we here qualify as ‘fuzzy’.

2 A simple example

The algebra $\mathcal{P}(u, v)$ of polynomials in $u = e^{ix}$, $v = e^{iy}$ is dense in any algebra of functions on the torus, defined by the relations $0 \leq x \leq 2\pi$, $0 \leq y \leq 2\pi$, where x and y are the ordinary cartesian coordinates of \mathbb{R}^2 . If one considers a square lattice of n^2 points then $u^n = 1$ and $v^n = 1$ and the algebra is reduced to a subalgebra \mathcal{P}_n of dimension n^2 . Introduce a basis $|j\rangle_1$, $0 \leq j \leq n-1$, of \mathbb{C}^n with $|n\rangle_1 \equiv |0\rangle_1$ and replace u and v by the operators

$$u|j\rangle_1 = q^j|j\rangle_1, \quad v|j\rangle_1 = |j+1\rangle_1, \quad q^n = 1.$$

Then the new elements u and v satisfy the relations

$$uv = qvu, \quad u^n = 1, \quad v^n = 1$$

and the algebra they generate is the matrix algebra M_n instead of the commutative algebra \mathcal{P}_n . There is also a basis $|j\rangle_2$ in which v is diagonal and a ‘Fourier’ transformation between the two [197].

Introduce the forms [157]

$$\begin{aligned}\theta^1 &= -i\left(1 - \frac{n}{n-1}|0\rangle_2\langle 0|\right)u^{-1}du, \\ \theta^2 &= -i\left(1 - \frac{n}{n-1}|n-1\rangle_1\langle n-1|\right)v^{-1}dv.\end{aligned}$$

In this simple example the differential calculus can be defined by the relations

$$\theta^a f = f\theta^a, \quad \theta^a\theta^b = -\theta^b\theta^a$$

of ordinary differential geometry. It follows that

$$\Omega^1(M_n) \simeq \bigoplus_1^2 M_n, \quad d\theta^a = 0.$$

The differential calculus has the form one might expect of a noncommutative version of the torus. Notice that the differentials du and dv do not commute with the elements of the algebra.

One can choose for q the value

$$q = e^{2\pi il/n}$$

for some integer l relatively prime with respect to n . The limit of the sequence of algebras as $l/n \rightarrow \alpha$ irrational is known as the rotation algebra or the noncommutative torus [184]. This algebra has a very rich representation theory and it has played an important role as an example in the development of noncommutative geometry [50].

3 Noncommutative electromagnetic theory

The group of unitary elements of the algebra of functions on a manifold is the local gauge group of electromagnetism and the covariant derivative associated to the electromagnetic potential can be expressed as a map

$$\mathcal{H} \xrightarrow{D} \Omega^1(V) \otimes_{\mathcal{A}} \mathcal{H} \tag{3.1}$$

from a $\mathcal{C}(V)$ -module \mathcal{H} to the tensor product $\Omega^1(V) \otimes_{\mathcal{C}(V)} \mathcal{H}$, which satisfies a Leibniz rule

$$D(f\psi) = df \otimes \psi + fD\psi, \quad f \in \mathcal{C}(V), \quad \psi \in \mathcal{H}.$$

We shall often omit the tensor-product symbol in the following. As far as the electromagnetic potential is concerned we can identify \mathcal{H} with $\mathcal{C}(V)$ itself;

electromagnetism couples equally, for example, to all four components of a Dirac spinor. The covariant derivative is defined therefore by the Leibniz rule and the definition

$$D1 = A \otimes 1 = A.$$

That is, one can rewrite (3.1) as

$$D\psi = (\partial_\mu + A_\mu)dx^\mu \psi.$$

One can study electromagnetism on a large class of noncommutative geometries [146, 67, 48, 53] and there exist many recent reviews [209, 152, 116]. Because of the noncommutativity however the result often looks more like non-abelian Yang-Mills theory.

4 Metrics

We shall define a metric as a bilinear map

$$\Omega^1(\mathcal{A}) \otimes_{\mathcal{A}} \Omega^1(\mathcal{A}) \xrightarrow{g} \mathcal{A}. \quad (4.1)$$

This is a ‘conservative’ definition, a straightforward generalization of one of the possible definitions of a metric in ordinary differential geometry:

$$g(dx^\mu \otimes dx^\nu) = g^{\mu\nu}.$$

The usual definition of a metric in the commutative case is a bilinear map

$$\mathcal{X} \otimes_{\mathcal{C}(V)} \mathcal{X} \xrightarrow{g} \mathcal{C}(V)$$

where \mathcal{X} is the $\mathcal{C}(V)$ -bimodule of vector fields on V :

$$g(\partial_\mu \otimes \partial_\nu) = g_{\mu\nu}.$$

This definition is not suitable in the noncommutative case since the set of derivations of the algebra, which is the generalization of \mathcal{X} , has no natural structure as an \mathcal{A} -module. The linearity condition is equivalent to a locality condition for the metric; the length of a vector at a given point depends only on the value of the metric and the vector field at that point. In the noncommutative case bilinearity is the natural (and only possible) expression of locality. It would exclude, for example, a metric in ordinary geometry defined by a map of the form

$$g(\alpha, \beta)(x) = \int_V g_x(\alpha_x, \beta_y) G(x, y) dy.$$

Here $\alpha, \beta \in \Omega^1(V)$ and g_x is a metric on the tangent space at the point $x \in V$. The function $G(x, y)$ is an arbitrary smooth function of x and y and dy is the measure on V induced by the metric.

Introduce a bilinear flip σ :

$$\Omega^1(\mathcal{A}) \otimes_{\mathcal{A}} \Omega^1(\mathcal{A}) \xrightarrow{\sigma} \Omega^1(\mathcal{A}) \otimes_{\mathcal{A}} \Omega^1(\mathcal{A}) \quad (4.2)$$

We shall say that the metric is symmetric if

$$g \circ \sigma \propto g.$$

Many of the finite examples have unique metrics [158] as do some of the infinite ones [31]. Other definitions of a metric have been given, some of which are similar to that given above but which weaken the locality condition [32] and one [49] which defines a metric on the associated space of states.

5 Linear Connections

An important geometric problem is that of comparing vectors and forms defined at two different points of a manifold. The solution to this problem leads to the concepts of a connection and covariant derivative. We define a linear connection as a covariant derivative

$$\Omega^1(\mathcal{A}) \xrightarrow{D} \Omega^1(\mathcal{A}) \otimes_{\mathcal{A}} \Omega^1(\mathcal{A})$$

on the \mathcal{A} -bimodule $\Omega^1(\mathcal{A})$ with an extra right Leibniz rule

$$D(\xi f) = \sigma(\xi \otimes df) + (D\xi)f$$

defined using the flip σ introduced in (4.2). In ordinary geometry the map

$$D(dx^\lambda) = -\Gamma_{\mu\nu}^\lambda dx^\mu \otimes dx^\nu$$

defines the Christoffel symbols.

We define the torsion map

$$\Theta : \Omega^1(\mathcal{A}) \rightarrow \Omega^2(\mathcal{A})$$

by $\Theta = d - \pi \circ D$. It is left-linear. A short calculation yields

$$\Theta(\xi)f - \Theta(\xi f) = \pi \circ (1 + \sigma)(\xi \otimes df).$$

We shall impose the condition

$$\pi \circ (\sigma + 1) = 0 \quad (5.1)$$

on σ . It could also be considered as a condition on the product π . In fact in ordinary geometry it is the definition of π ; a 2-form can be considered as an antisymmetric tensor. Because of this condition the torsion is a bilinear map. Using σ a reality condition on the metric and the linear connection can be introduced [78]. In the commutative limit, when it exists, the commutator defines a Poisson structure, which normally would be expected to have an intimate relation with the linear connection. This relation has only been studied in very particular situations [149].

6 Gravity

The classical gravitational field is normally supposed to be described by a torsion-free, metric-compatible linear connection on a smooth manifold. One might suppose that it is possible to formulate a noncommutative theory of (classical/quantum) gravity by replacing the algebra of functions by a more general algebra and by choosing an appropriate differential calculus. It seems however difficult to introduce a satisfactory definition of local curvature and the corresponding curvature invariants [55, 68, 56]. One way of circumventing this problem is to consider classical gravity as an effective theory and the Einstein-Hilbert action as an induced action. We recall that the classical gravitational action is given by

$$S[g] = \mu_P^4 \Lambda_c + \mu_P^2 \int R.$$

In the noncommutative case there is a natural definition of the integral [50, 43, 45] but there does not seem to be a natural generalization of the Ricci scalar. One of the problems is the fact that the natural generalization of the curvature form is in general not right-linear in the noncommutative case. The Ricci scalar then will not be local. One way of circumventing these problems is to return to an old version of classical gravity known as induced gravity [185, 186]. The idea is to identify the gravitational action with the quantum corrections to a classical field in a curved background. If $\Delta[g]$ is the operator which describes the propagation of a given mode in presence of a metric g then one finds that, with a cut-off Λ , the effective action is given by

$$\Gamma[g] \propto \text{Tr} \log \Delta[g] \simeq \Lambda^4 \text{Vol}(V)[g] + \Lambda^2 S_1[g] + (\log \Lambda) S_2[g] + \dots$$

If one identifies $\Lambda = \mu_P$ then one finds that $S_1[g]$ is the Einstein-Hilbert action. A problem with this is that it can be only properly defined on a compact manifold with a metric of euclidean signature and Wick rotation on a curved space-time is a rather delicate if not dubious procedure. Another problem with this theory, as indeed with the gravitational field in general, is that it predicts an extremely large cosmological constant. The expression $\text{Tr} \log \Delta[g]$ has a natural generalization to the noncommutative case [111, 1, 34].

We have defined gravity using a linear connection, which required the full bimodule structure of the \mathcal{A} -module of 1-forms. One can argue that this was necessary to obtain a satisfactory definition of locality as well as a reality condition. It is possible to relax these requirements and define gravity as a Yang-Mills field [35, 135, 36, 81] or as a couple of left and right connections [55, 56]. If the algebra is commutative (but not an algebra of smooth functions) then to a certain extent all definitions coincide [136, 12].

7 Regularization

Using the diagram (1.4) we have argued that gravity regularizes propagators in quantum field theory through the formation of a noncommutative structure. Several explicit examples of this have been given in the literature [202, 148, 62, 124, 129, 39]. In particular an energy-momentum tensor constructed from regularized propagators [39] has been used as a source of a cosmological solution. The propagators appear as if they were derived from non-local theories on ordinary space-time [218, 175, 119, 205]. We required that the metric that we use be local in the sense that the map (4.1) is bilinear with respect to the algebra. One could say that the theory is as local as the algebra will permit. However, since the algebra is not an algebra of points this means that the theory appears to be non-local as an effective theory on a space-time manifold.

8 Kaluza-Klein theory

We mentioned in the Introduction that one of the first, obvious applications of noncommutative geometry is as an alternative hidden structure of Kaluza-Klein theory. This means that one leaves space-time as it is and one modifies only the extra dimensions; one replaces their algebra of functions by a noncommutative algebra, usually of finite dimension to avoid the infinite tower of massive states of traditional Kaluza-Klein theory. Because of this restriction and because the extra dimensions are purely algebraic in nature the length scale associated with them can be arbitrary [153], indeed as large as the Compton wave length of a typical massive particle.

The algebra of Kaluza-Klein theory is therefore, for example, a product algebra of the form

$$\mathcal{A} = \mathcal{C}(V) \otimes M_n.$$

Normally V would be chosen to be a manifold of dimension four, but since much of the formalism is identical to that of the M (atrix)-theory of D -branes [14, 83, 51]. For the simple models with a matrix extension one can use as gravitational action the Einstein-Hilbert action in ‘dimension’ $4+d$, including possibly Gauss-Bonnet terms [147, 153, 152, 154, 121]. For a more detailed review we refer to a lecture [155] at the 5th Hellenic school in Corfu.

9 Quantum groups and spaces

The set of smooth functions on a manifold is an algebra. This means that from any function of two variables one can construct a function of one by multiplication. If the manifold happens to be a Lie group then there is another operation which to any function of one variable constructs a function of two.

This is called co-multiplication and is usually written Δ :

$$(\Delta f)(g_1, g_2) = f(g_1 g_2).$$

It satisfies a set of consistency conditions with the product. Since the expression ‘noncommutative group’ designates something else the noncommutative version of an algebra of smooth functions on a Lie group has been called a ‘quantum group’. It is neither ‘quantum’ nor ‘group’. The first example was found by Kulish & Reshetikhin [133] and by Sklyanin [199]. A systematic description was first made by Woronowicz [213], by Jimbo [109], Manin [167, 168] and Drinfeld [65]. The Lie group $SO(n)$ acts on the space \mathbb{R}^n ; the Lie group $SU(n)$ acts on \mathbb{C}^n . The ‘quantum’ versions $SO_q(n)$ and $SU_q(n)$ of these groups act on the ‘quantum spaces’ \mathbb{R}_q^n and \mathbb{C}_q^n . These latter are noncommutative algebras with special covariance properties. The first differential calculus on a quantum space was constructed by Wess & Zumino [211]. There is an immense literature on quantum groups and spaces, from the algebraic as well as geometric point of view. We have included some of it in the bibliography. We mention in particular the collection of articles edited by Doebner & Hennig [61] and Kulish [130] and the introductory text by Kassel [113].

10 Mathematics

At a more sophisticated level one would have to add a topology to the algebra. Since we have identified the generators as hermitian operators on a Hilbert space, the most obvious structure would be that of a von Neumann algebra. We refer to Connes [45] for a description of these algebras within the context of noncommutative geometry. A large part of the interest of mathematicians in noncommutative geometry has been concerned with the generalization of topological invariants [42, 55, 170] to the noncommutative case. It was indeed this which lead Connes to develop cyclic cohomology. Connes [50, 43] has also developed and extended the notion of a Dixmier trace on certain types of algebras as a possible generalization of the notion of an integral. The representation theory of quantum groups is an active field of current interest since the pioneering work of Woronowicz [216]. For a recent survey we refer to the book by Klymuk & Schmüdgen [125]. Another interesting problem is the relation between differential calculi covariant under the (co-)action of quantum groups and those constructed using the spectral-triple formalism of Connes. Although it has been known for some time [68, 59, 86, 41, 79] that many if not all of the covariant calculi have formal Dirac ‘operators’ it is only recently that mathematicians have considered [190, 191] to what extent these ‘operators’ can be actually represented as real operators on a Hilbert space and to what extent they satisfy the spectral-triple conditions.

11 String Theory

Last, but not least, is the possible relation of noncommutative geometry to string theory. We have mentioned that since noncommutative geometry is pointless a field theory on it will be divergence-free. In particular monopole configurations will have finite energy, provided of course that the geometry in which they are constructed can be approximated by a noncommutative geometry, since the point on which they are localized has been replaced by a volume of fuzz, This is one characteristic that it shares with string theory. Certain monopole solutions in string theory have finite energy [85] since the point in space (a D -brane) on which they are localized has been replaced by a throat to another ‘adjacent’ D -brane.

In noncommutative geometry the string is replaced by a certain finite number of elementary volumes of ‘fuzz’, each of which can contain one quantum mode. Because of the nontrivial commutation relations the ‘line’ $\delta q^\mu = q^{\mu'} - q^\mu$ joining two points $q^{\mu'}$ and q^μ is quantized and can be characterized [39] by a certain number of creation operators a_j each of which creates a longitudinal displacement. They would correspond to the rigid longitudinal vibrational modes of the string. Since it requires no energy to separate two points the string tension would be zero. This has not much in common with traditional string theory.

We mentioned in the previous section that noncommutative Kaluza-Klein theory has much in common with the M (atrix) theory of D -branes. What is lacking is a satisfactory supersymmetric extension. Finally we mention that there have been speculations that string theory might give rise naturally to space-time uncertainty relations [137] and that it might also give rise [108] to a noncommutative theory of gravity. More specifically there have been attempts [64, 63, 192] to relate a noncommutative structure of space-time to the quantization of the open string in the presence of a non-vanishing B -field.

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What follows constitutes in no way a complete bibliography of noncommutative geometry. It is strongly biased in favour of the author’s personal interests and the few subjects which were touched upon in the text.

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