CHAPTER III

THE EQUIVALENCE BETWEEN THE CATEGORY C OF CROSSED MODULES AND THE CATEGORY DA! OF SPECIAL DOUBLE ALGEBROIDS WITH CONNECTIONS

0. INTRODUCTION:

R.Brown and C.B.Spencer [B-S-1] have defined a functor (crossed modules over groups) ⇢ (double groupoids) , and they showed that this gives an equivalence between the category of crossed modules over groups and the category of special double groupoids with special connections and one vertex . The structure of connection on a double groupoid was shown in [B-H-1] to be equivalent to a structure of thin squares , and a convenient notation for thin squares was later developed and exploited by R.Brown [Br-2] . Also [S-1] proved an equivalence between 2-categories and double categories with connections . Thin structures on double categories were exploited in [S-W-1] . Finally , it was proved in [B-H-2] that crossed modules over groupoids are equivalent to double groupoids with connections ; indeed this is a special case of an equivalence between crossed complexes (over groupoids) and ω-groupoids .

Our programme is to prove results parallel to the above in the context of algebroids rather than groupoids ; that is we would like to prove that there exist an equivalence between ω-algebroids and crossed complexes (over algebroids) .

Rather than move to the general case immediately , we give in this chapter the case n = 2 , that is , for double algebroid . This will familiarise the reader with the techniques

involved . Also some of our lemmas for n = 2 will be applied the general case , and the complications of their proof makes it easier to give the case n = 2 when the notation is simpler than in general .

As explained in the Introduction , in this thesis we do not acheive the general result , but we do obtain a lot of information on the general situation and complete results for n = 2,3,4 .

1. THIN STRUCTURES AND CONNECTIONS:

We will use the example which was given in chapter 2 $ 3 in order to define the extra structure needed later (we should mention that the example of □B given before is analogous to the example of double category due to Ehresmann [Eh-1]) . But before that we start to define a special double algebroid .

Definition 3.1.1: Let D be a double R-algebroid . We say that D is a special double R-algebroid if $D_1 = D_2$.

Refering to the definition (2.1.9) , a morphism (Ψ_0,Ψ_1,Ψ_2) of double algebroids such that $\Psi_1 = \Psi_2$ is called a morphism of special double algebroids .

Suppose given a special double algebroid D . Then there will be squares of D with commuting boundary , that is , with edges given by

$$\begin{array}{ccc} & c & \\ a & \square & d \\ & b & \end{array}$$

and for which $ab = cd$. Examples of such squares are degenerate squares ;

$$\begin{array}{ccc} x & 1_x & x \\ a & \varepsilon_2 a & a \\ y & 1_y & y \end{array} \quad , \quad \begin{array}{ccc} x & a & y \\ 1_x & \varepsilon_1 a & 1_y \\ x & a & y \end{array} \quad .$$

Among the others there seems no way to distinguish any one from another . We therefore impose on D an additional structure of "thin" squares .

Definition 3.1.2: Let D be a special double algebroid . A thin structure on D is a morphism $\theta : \square D_1 \longrightarrow D$ of special double algebroids such that θ is the identity on D_1 . Hence $\underline{\partial}\theta\left(a \begin{smallmatrix} c \\ \\ b \end{smallmatrix} d\right) = \left(a \begin{smallmatrix} c \\ \\ b \end{smallmatrix} d\right)$. An element $\theta\left(a \begin{smallmatrix} c \\ \\ b \end{smallmatrix} d\right)$ is called thin , and is often written simply $\left(a \begin{smallmatrix} c \\ \\ b \end{smallmatrix} d\right)$, when θ is clear from the context .

Remark 3.1.3: Because θ is a morphism any composite of thin squares is thin ; any sum of thin squares is thin ; any scalar multiple of a thin square is thin . Thin squares should be thought of as generalisations of identity elements $\varepsilon_1 a$, $\varepsilon_2 a$ in a special double algebroid .

Instead of thin structures , one can use an alternative further structure on D , namely a connection (Γ , Γ') . This will be important later for generalisation to higher dimensions .

__Definition 3.1.4__: Let D be a special double algebroid . We define a __connection__ on D to be a pair of functions

Γ , Γ' : $D_2 \to D$ such that

(3.1.4)(i) for any $a \in D_2(x,y)$, then Γa , $\Gamma' a$ have edges given by

(Clearly these two squares are commutative) .

We assume the following axioms: for all $a,b \in D_2$ such that ab is defined

$$\left.\begin{aligned} \Gamma' a *_2 \Gamma a &= \varepsilon_1 a \\ \Gamma' a *_1 \Gamma a &= \varepsilon_2 a \end{aligned}\right\} \quad (3.1.4)(ii)$$

$$\left.\begin{aligned} \Gamma'(ab) &= (\Gamma' a *_1 \varepsilon_1 a) *_2 (\varepsilon_2 a *_1 \Gamma' a) = \Gamma' a *_2 (\varepsilon_2 a *_1 \Gamma' b) \\ \Gamma(ab) &= (\Gamma a *_1 \varepsilon_2 b) *_2 (\varepsilon_1 b *_1 \Gamma b) = (\Gamma a *_1 \varepsilon_2 b) *_2 \Gamma b \end{aligned}\right\} \quad (3.1.4)(iii)$$

(3.1.4)(iv) Let $\alpha, \beta, \gamma \in D$ have boundaries given by

$$\underline{\partial}\alpha = \left(a \;\begin{matrix} c \\ \\ b \end{matrix}\; d\right) , \quad \underline{\partial}\beta = \left(a_1 \;\begin{matrix} c \\ \\ b \end{matrix}\; d_1\right) , \quad \underline{\partial}\gamma = \left(a \;\begin{matrix} c_1 \\ \\ b_1 \end{matrix}\; d\right) ;$$

then we have

$$\Gamma'(a+a_1) *_2 (\alpha +_1 \beta) *_2 \Gamma(d+d_1) = (\Gamma' a *_2 \alpha *_2 \Gamma d) +_2$$
$$(\Gamma' a_1 *_2 \beta *_2 \Gamma d_1) .$$

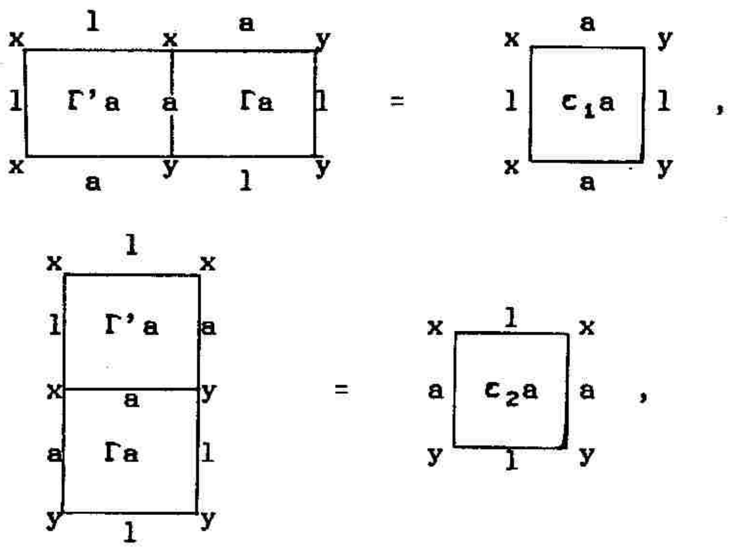
(3.1.4)(v) Let $r \in R$ and $\alpha \in D$ with boundary given by

$$\underline{\partial}\alpha = \left(a \;\begin{matrix} c \\ \\ b \end{matrix}\; d\right) ; \text{ then we have}$$

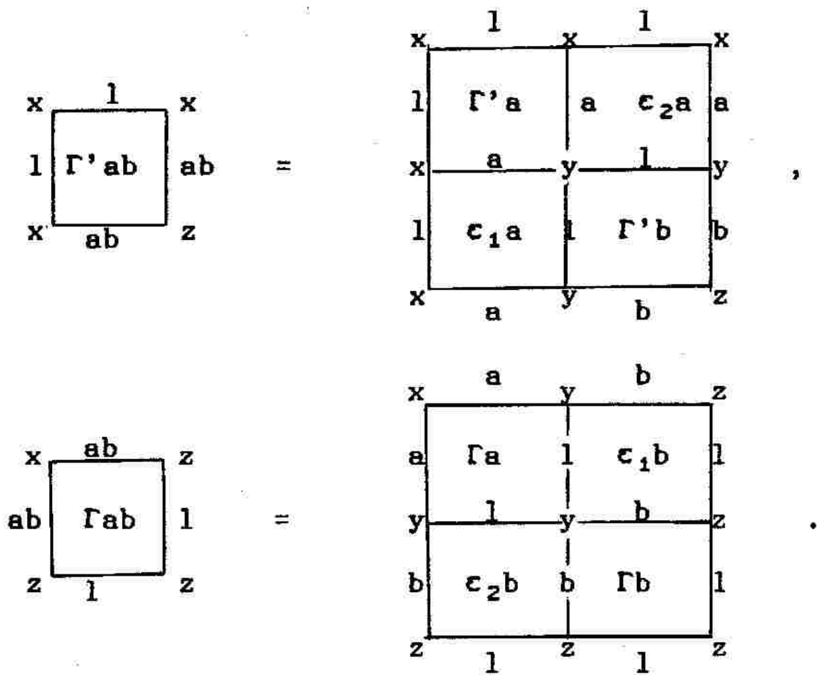
$$\Gamma' ra *_2 (r \cdot_1 \alpha) *_2 \Gamma rd = r \cdot_2 (\Gamma' a *_2 \alpha *_2 \Gamma d) .$$

These axioms make sense in terms of boundaries , as shown in the diagrams below :

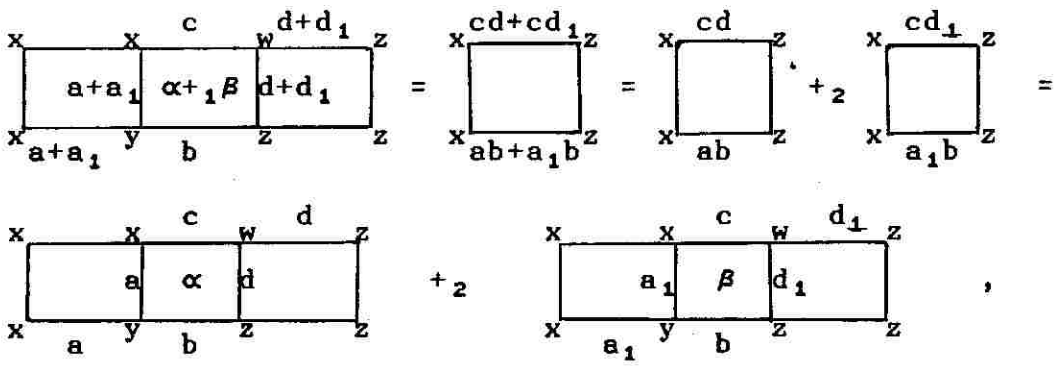
let $a: x \to y$, $b: y \to z$ for $x,y,z \in D_0$; then the axiom (3.1.4)(ii) can be pictured as



The axiom (3.1.4)(iii) is pictured as ;

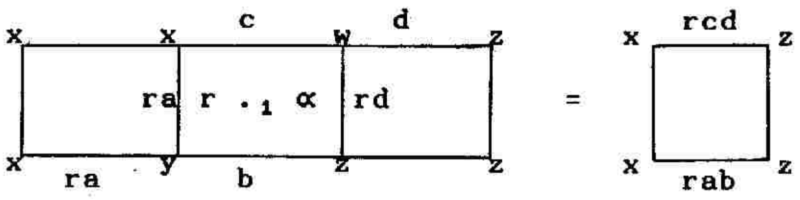


The axiom (3.1.4)(iv) , is pictured as



The axiom (3.1.4)(v) , is pictured as ;

the left hand side is ;



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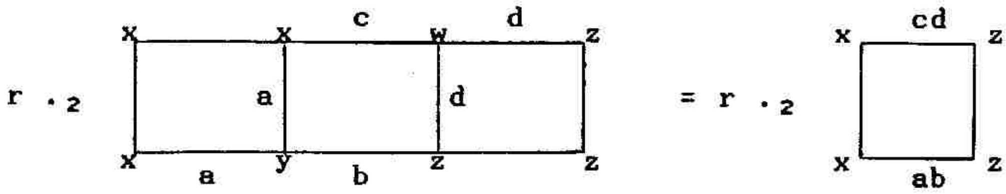
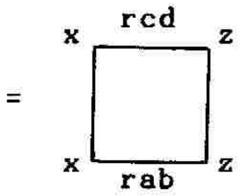
The other side is ;





. Thus the boundaries are equal .

<u>Remark 3.1.5</u>: The axioms (3.1.4)(i,ii,iii) are essentially the axioms for connection on a double category given in [S-1] . These axioms involve only the composition and not the additions or scalar multiplications of the algebroid structure . But the axioms (3.1.4)(iv,v) give relations between (Γ,Γ') and the additions and scalar multiplications . These axioms are equivalent to conditions on the folding operation given later in $3.2 and are not used until that section .

We go back to define a morphism between two special double algebroids with connections .

<u>Definition 3.1.6</u>: A morphism $\Psi : D \to E$ of special double algebroids with connections (Γ,Γ') , (Δ,Δ') is said to <u>preserve the connections</u> if and only if

$$\Delta\,\Psi_2 = \Psi_0\,\Gamma \ , \ \Delta'\,\Psi_2 = \Psi_0\,\Gamma' \ .$$

Such morphisms form the morphisms of the category of special double algebroids with connections , denoted by $\underline{DA}^!$.

We gave in proposition (2.2.1) a functor from double algebroids to crossed modules (over algebroids) , associating to D the crossed module (A,M,μ) with $A = D_2$ and M consisting of squares with boundary of the form $\left(1 \begin{smallmatrix} m \\ \\ 0 \end{smallmatrix} 1\right)$. We have a forgetful functor $\underline{DA}^!$ (special double algebroids with connection) ---→ (double algebroids) . The composite functor $\underline{DA}^!$ ---→ $\underline{C}$ (crossed modules) will be written as γ .

Notice that in a special double algebroid , a thin structure implies a connection satisfying (3.1.4)(i,ii,iii) where $\Gamma(a) = \theta\left(a \begin{smallmatrix} a \\ \\ 1 \end{smallmatrix} 1\right)$, $\Gamma'(a) = \theta\left(1 \begin{smallmatrix} 1 \\ \\ a \end{smallmatrix} a\right)$. To complete the equivalence between these two structures , we prove first that in a special double algebroid a thin structure may be recovered from a connection satisfying only (3.1.4) (i,ii,iii) . This result leads us to use connections instead of thin structures . The idea particularly in higher dimensions has been given in [B-H-1] in the double groupoid case , and partially in [S-1] , [S-W-1] , for double categories .

As mentioned above , the proof of the following theorem does not involve axioms (3.1.4)(iv,v) on connections .

<u>Theorem 3.1.7</u>: Let D be a special double algebroid with connection Γ , Γ' . Then there is a morphism of special double algebroids $\theta : \square D_1 \to D$, which is the identity on D_1 and such that $\Gamma a = \theta\left(a \begin{smallmatrix} a \\ \\ 1 \end{smallmatrix} 1\right)$, $\Gamma' a = \theta\left(1 \begin{smallmatrix} 1 \\ \\ b \end{smallmatrix} b\right)$.

Proof: For any a,b,c,d $\in D_1$ satisfying cd = ab , define functions θ_1 , θ_2 : $\square D_1 \longrightarrow D$ by

$$\theta_1\begin{pmatrix} & c & \\ a & & d \\ & b & \end{pmatrix} = (\varepsilon_1 c *_2 \Gamma' d) *_1 (\Gamma a *_2 \varepsilon_1 b) ,$$

$$\theta_2\begin{pmatrix} & c & \\ a & & d \\ & b & \end{pmatrix} = (\varepsilon_2 a *_1 \Gamma' b) *_2 (\Gamma c *_1 \varepsilon_2 d) .$$

The two definitions make sense in terms of boundaries ; Appendix I give diagrams for these definitions and for the proof of the next lemma .

Lemma 3.1.8: The two definitions θ_1 , θ_2 are equivalent , that is , $\theta_1 = \theta_2$.

Proof: Let a,b,c,d $\in D_1$ be such that cd = ab , then

$$\theta_1\begin{pmatrix} & c & \\ a & & d \\ & b & \end{pmatrix} = (\varepsilon_1 c *_2 \Gamma' d) *_1 (\Gamma a *_2 \varepsilon_1 b)$$

$= (\varepsilon_1 c *_2 \Gamma' d) *_1 \varepsilon_1 ab *_1 (\Gamma a *_2 \varepsilon_1 b)$ by the identity rule

$= (\varepsilon_1 c *_2 \Gamma' d) *_1 (\Gamma' ab *_2 \Gamma cd) *_1 (\Gamma a *_2 \varepsilon_1 b)$

by (3.1.4)(ii) and cd = ab

$= (\varepsilon_1 c *_2 \Gamma' d) *_1 \{[\ \Gamma' a *_2 (\varepsilon_2 a *_1 \Gamma' b)\ ] *_2 [(\Gamma c *_1 \varepsilon_2 d) *_2 \Gamma d]\} *_1 (\Gamma a *_2 \varepsilon_1 b)$ by (3.1.4)(iii)

$= (\varepsilon_1 c *_2 \Gamma' d) *_1 \{[\Gamma' a *_2 (\varepsilon_2 a *_1 \Gamma' b) *_2 (\Gamma c *_1 \varepsilon_2 d)] *_2 \Gamma d\} *_1 (\Gamma a *_2 \varepsilon_1 b)$ by associativity

$= \{\{\varepsilon_1 c *_1 [\Gamma' a *_2 (\varepsilon_2 a *_1 \Gamma' b) *_2 (\Gamma c *_1 \varepsilon_2 d)]\} *_2 (\Gamma' d *_1 \Gamma d) *_1 (\Gamma a *_2 \varepsilon_1 b)$ by (2.1.6)(ii)

$= [\varepsilon_1 c *_1 [\ \Gamma' a *_2 (\varepsilon_2 a *_1 \Gamma' b) *_2 (\Gamma c *_1 \varepsilon_2 d)]] *_1 (\Gamma a *_2 \varepsilon_1 b)$ by (3.1.4)(ii)

$= \varepsilon_1 c *_1 \{\{\Gamma' a *_2 [(\varepsilon_2 a *_1 \Gamma' b) *_2 (\Gamma c *_1 \varepsilon_2 d)]\} *_1 (\Gamma a *_2 \varepsilon_1 b)$ by associativity

$= \varepsilon_1 c *_1 \{(\Gamma' a *_1 \Gamma a) *_2 [[(\varepsilon_2 a *_1 \Gamma' b) *_2 (\Gamma c *_1 \varepsilon_2 d)] *_1 \varepsilon_1 b]\}\}$ by (2.1.6)(ii)

$= \varepsilon_1 c *_1 [(\varepsilon_2 a *_1 \Gamma' b) *_2 (\Gamma c *_1 \varepsilon_2 d)] *_1 \varepsilon_1 b$ by (3.1.4)(ii)

$= (\varepsilon_2 a *_1 \Gamma' b) *_2 (\Gamma c *_1 \varepsilon_2 d)$ by the identity rule

$= \theta_2\left(a \begin{smallmatrix} c \\ \\ b \end{smallmatrix} d\right)$. This is the complete proof of the lemma .

Now we continue to prove theorem (3.1.7) , that is , we prove θ satisfies the following :

i) $\theta\left(a \begin{smallmatrix} 1 \\ \\ 1 \end{smallmatrix} a\right) = \left(a \begin{smallmatrix} 1 \\ \\ 1 \end{smallmatrix} a\right)$, $\theta\left(1 \begin{smallmatrix} a \\ \\ a \end{smallmatrix} 1\right) = \left(1 \begin{smallmatrix} a \\ \\ a \end{smallmatrix} 1\right)$;

ii) $\theta\left(a \begin{smallmatrix} a \\ \\ 1 \end{smallmatrix} 1\right) = \left(a \begin{smallmatrix} a \\ \\ 1 \end{smallmatrix} 1\right)$, $\theta\left(1 \begin{smallmatrix} 1 \\ \\ a \end{smallmatrix} a\right) = \left(1 \begin{smallmatrix} 1 \\ \\ a \end{smallmatrix} a\right)$;

iii) $\theta\left(a \begin{smallmatrix} c \\ \\ b \end{smallmatrix} d\right) +_1 \theta\left(a_1 \begin{smallmatrix} c \\ \\ b \end{smallmatrix} d_1\right) = \theta\left(a+a_1 \begin{smallmatrix} c \\ \\ b \end{smallmatrix} d+d_1\right)$;

iv) $\theta\left(a \begin{smallmatrix} c \\ \\ b \end{smallmatrix} d\right) +_2 \theta\left(a \begin{smallmatrix} c_1 \\ \\ b_1 \end{smallmatrix} d\right) = \theta\left(a \begin{smallmatrix} c+c_1 \\ \\ b+b_1 \end{smallmatrix} d\right)$;

v) $r \cdot_1 \theta\left(a \begin{smallmatrix} c \\ \\ b \end{smallmatrix} d\right) = \theta\left(ra \begin{smallmatrix} c \\ \\ b \end{smallmatrix} rd\right)$;

vi) $r \cdot_2 \theta\left(a \begin{smallmatrix} c \\ \\ b \end{smallmatrix} d\right) = \theta\left(a \begin{smallmatrix} rc \\ \\ rb \end{smallmatrix} d\right)$;

vii) $\theta\left(a \begin{smallmatrix} c \\ \\ b \end{smallmatrix} d\right) *_2 \theta\left(d \begin{smallmatrix} c' \\ \\ b' \end{smallmatrix} e\right) = \theta\left(a \begin{smallmatrix} cc' \\ \\ bb' \end{smallmatrix} d\right)$;

viii) $\theta\left(a \begin{smallmatrix} c \\ \\ b \end{smallmatrix} d\right) *_1 \theta\left(a' \begin{smallmatrix} b \\ \\ e \end{smallmatrix} d'\right) = \theta\left(aa' \begin{smallmatrix} c \\ \\ e \end{smallmatrix} dd'\right)$.

The proof of (i) , (ii) are easy . To prove (iii) , we use the interchange law (2.1.6)(iii) , distributive law , (2.1.7) and $\theta = \theta_2$;

$\theta_2\left(a \begin{smallmatrix} c \\ \\ b \end{smallmatrix} d\right) +_1 \theta_2\left(a_1 \begin{smallmatrix} c \\ \\ b \end{smallmatrix} d_1\right) = [(\varepsilon_2 a *_1 \Gamma' b) *_2 (\Gamma c *_1 \varepsilon_2 d)] +_1$

$[(\varepsilon_2 a_1 *_1 \Gamma' b) *_2 (\Gamma c *_1 \varepsilon_2 d_1)]$

$= [(\varepsilon_2 a *_1 \Gamma' b) +_1 (\varepsilon_2 a_1 *_1 \Gamma' b)] *_2 [(\Gamma c *_1 \varepsilon_2 d) +_1$

$(\Gamma c *_1 \varepsilon_2 d_1)]$ by (2.1.6)(iii)

$= [(\varepsilon_2 a +_1 \varepsilon_2 a_1) *_1 \Gamma' b] *_2 [\Gamma c *_1 (\varepsilon_2 d +_1 \varepsilon_2 d_1)]$

by distributivity

$= (\varepsilon_2(a+a_1) *_1 \Gamma' b) *_2 (\Gamma c *_1 \varepsilon_2(d+d_1))$ by (2.1.7)

$= \theta_2(a+a_1 \underset{b}{\overset{c}{}} d+d_1)$.

To prove (iv) , we use (2.1.7),(2.1.6)(iv) , distributivity , and $\theta = \theta_1$;

$\theta_1(a \underset{b}{\overset{c}{}} d) +_2 \theta_1(a \underset{b_1}{\overset{c_1}{}} d) = [(\varepsilon_1 c *_2 \Gamma' d) *_1 (\Gamma a *_2 \varepsilon_1 b)] +_2$

$[(\varepsilon_1 c_1 *_2 \Gamma' d) *_1 (\Gamma a *_2 \varepsilon_1 b_1)]$

$= [(\varepsilon_1 c *_2 \Gamma' d) +_2 (\varepsilon_1 c_1 *_2 \Gamma' d)] *_1 [(\Gamma a *_2 \varepsilon_1 b) +_2$

$(\Gamma a *_2 \varepsilon_1 b_1)]$ by (2.1.6)(iv)

$= [(\varepsilon_1 c +_2 \varepsilon_1 c_1) *_2 \Gamma' d] *_1 [\Gamma a *_2 (\varepsilon_1 b +_2 \varepsilon_1 b_1)]$

by distributivity

$= (\varepsilon_1(c + c_1) *_2 \Gamma' d) *_1 (\Gamma a *_2 \varepsilon_1(b + b_1))$ by (2.1.7)

$= \theta_1(a \underset{b+b_1}{\overset{c+c_1}{}} d)$.

To prove (v) , we use the rule (2.1.5)(ii) and $\theta = \theta_2$;

$\theta_2(ra \underset{b}{\overset{c}{}} rd) = (\varepsilon_2 ra *_1 \Gamma' b) *_2 (\Gamma c *_1 \varepsilon_2 rd)$

$= (r \cdot_1 \varepsilon_2 a *_1 \Gamma' b) *_2 (\Gamma c *_1 r \cdot_1 \varepsilon_2 d)$

$= (r \cdot_1 (\varepsilon_2 a *_1 \Gamma' b)) *_2 (r \cdot_1 (\Gamma c *_1 \varepsilon_2 d))$

$= r \cdot_1 [(\varepsilon_2 a *_1 \Gamma' b) *_2 (\Gamma c *_1 \varepsilon_2 d)]$ by (2.1.5)(ii)

$= r \cdot_1 \theta(a \underset{b}{\overset{c}{}} d)$.

We can prove similarly that $\theta(a \underset{rb}{\overset{rc}{}} d) = r \cdot_2 \theta(a \underset{b}{\overset{c}{}} d)$

by using (2.1.5)(ii) and $\theta = \theta_1$.

For (vii) , we use the interchange law (2.1.6)(ii) , the identity rule , the associativity , (3.1.4)(ii) , the equality $cdd' = abd' = aa'e$ and $\theta = \theta_1$;

$$\theta\left(a \begin{matrix} cc' \\ \\ bb' \end{matrix} e\right) = (\varepsilon_1 cc' *_2 \Gamma' e) *_1 (\Gamma a *_2 \varepsilon_1 bb')$$

$$= (\varepsilon_1 c *_2 \varepsilon_1 c' *_2 \Gamma' e) *_1 \varepsilon_1 cdb' *_1 (\Gamma a *_2 \varepsilon_1 b *_2 \varepsilon_1 b')$$

by the identity rule

$$= (\varepsilon_1 c *_2 (\varepsilon_1 c' *_2 \Gamma' e)) *_1 (\varepsilon_1 c *_2 \varepsilon_1 d *_2 \varepsilon_1 b') *_1$$
$$((\Gamma a *_2 \varepsilon_1 b) *_2 \varepsilon_1 b') \quad \text{by the associativity}$$

$$= [\varepsilon_1 c *_2 (\varepsilon_1 c' *_2 \Gamma' e)] *_1 [(\varepsilon_1 c *_2 \Gamma' d) *_2 (\Gamma d *_2 \varepsilon_1 b')] *_1$$
$$[(\Gamma a *_2 \varepsilon_1 b) *_2 \varepsilon_1 b'] \quad \text{by (3.1.4)(ii)}$$

$$= [(\varepsilon_1 c *_1 (\varepsilon_1 c *_2 \Gamma' d)) *_2 ((\varepsilon_1 c' *_2 \Gamma' e) *_1 (\Gamma d *_2 \varepsilon_1 b'))]$$
$$*_1 [(\Gamma a *_2 \varepsilon_1 b) *_2 \varepsilon_1 b'] \quad \text{by (2.1.6)(ii)}$$

$$= [(\varepsilon_1 c *_2 \Gamma' d) *_1 (\Gamma a *_2 \varepsilon_1 b)] *_2 [(\varepsilon_1 c' *_2 \Gamma' e) *_1$$
$$(\Gamma d *_2 \varepsilon_1 b') *_1 \varepsilon_1 b'] \quad \text{by the identity rule and (2.1.6)(ii)}$$

$$= [(\varepsilon_1 c *_2 \Gamma' d) *_1 (\Gamma a *_2 \varepsilon_1 b)] *_2 [(\varepsilon_1 c' *_2 \Gamma' e) *_1$$
$$(\Gamma d *_2 \varepsilon_1 b')] \quad \text{by the identity rule}$$

$$= \theta_1\left(a \begin{matrix} c \\ \\ b \end{matrix} d\right) *_2 \theta_1\left(d \begin{matrix} c' \\ \\ b' \end{matrix} e\right) .$$

We can prove similarly that $\theta\left(a \begin{matrix} c \\ \\ b \end{matrix} d\right) *_1 \theta\left(a' \begin{matrix} b \\ \\ e \end{matrix} d'\right) =$

$\theta\left(aa' \begin{matrix} c \\ \\ e \end{matrix} dd'\right)$, by using (3.1.4)(ii) , the identity rule ,

the interchange law (2.1.6)(ii) , $cdd' = abd' = aa'e$ and $\theta = \theta_2$.

Then θ is a morphism . This is the complete proof of the theorem . □

We move now in the next section to construct a functor $\lambda : \underline{C} \dashrightarrow \underline{DA^!}$ by using a "folding" operation , whose definition involves the connections .

2. THE FOLDING OPERATION:

In this section , we introduce on squares of a special double algebroid with connections D an operation which has the effect of "folding" all edges of $\alpha \in D$ onto the edge $\partial_1^0 \alpha$.

This operation Φ transforms α into an element of the associated crossed module γD .

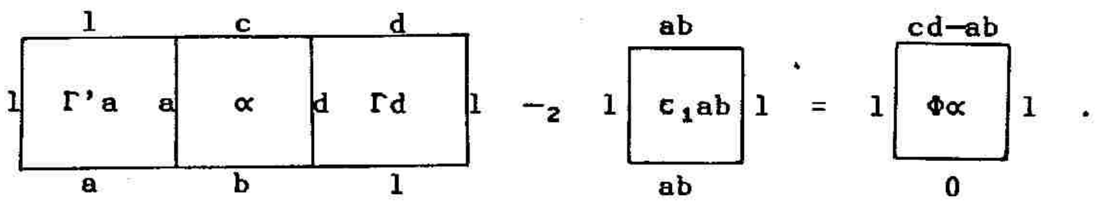
We define $\Phi : D \dashrightarrow D$ in the following way ; given $\alpha \in D$ with boundary edges in the form

,

we define

$$\Phi\alpha = (\Gamma' a *_2 \alpha *_2 \Gamma d) -_2 \varepsilon_1 ab \ .$$

It is easy to check that this composition and subtraction are defined . Simply , if α as above , then $\Phi\alpha$ has boundary in the form



Thus $\partial_1^0 \Phi\alpha = cd - ab$, $\partial_1^1 \Phi\alpha = 0$, $\partial_2^0 \Phi\alpha = 1$, $\partial_2^1 \Phi\alpha = 1$

and hence $\Phi\alpha \in \gamma D$.

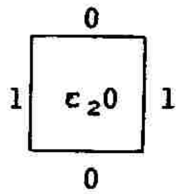
Proposition 3.2.1: $\Phi\alpha = \alpha$ if and only if α is in γD . In particular $\Phi^2\alpha = \Phi\alpha$ for all $\alpha \in D$.

Proof: If $\Phi\alpha = \alpha$, then α has boundary edges given by

$(1 \; {}^{m}_{0} \; 1)$, for $m \in D_1 = D_2$ and then $\alpha \in \gamma D$ (by the construction given in chapter II) . The converse is clear . □

We now develop relations between Φ and the operations of the special double algebroid D .

First , let $0^2 = \varepsilon_2 0 \in D$, as in the diagram



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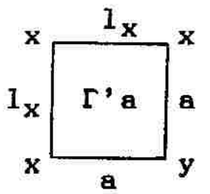
Proposition 3.2.2: Let $a \in D_1(x,y)$, then

i) $\Phi\Gamma' a = 0^2$, $\Phi\Gamma a = 0^2$,

ii) $\Phi\varepsilon_1 a = 0^2$, $\Phi\varepsilon_2 a = 0^2$.

Proof: i) Since $a \in D_1(x,y)$, then $\Gamma' a$ has boundary in the form



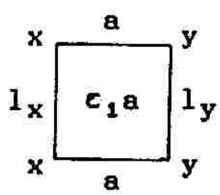
and then $\Phi\Gamma' a = (\Gamma' 1_x *_2 \Gamma' a *_2 \Gamma a) -_2 \varepsilon_1 a$

$= \varepsilon_1 a -_2 \varepsilon_1 a$ by (3.1.4)(ii)

$= 0^2$.

We can prove similarly that $\Phi\Gamma a = 0^2$.

ii) Since $a \in D_1(x,y)$, then $\varepsilon_1 a$ has boundary edges given by



and then $\Phi\varepsilon_1 a = (\Gamma' 1_x *_2 \varepsilon_1 a *_2 \Gamma 1_y) -_2 \varepsilon_1 a = 0^2$. Similarly we can prove that $\Phi\varepsilon_2 a = 0^2$ by (3.1.4)(ii) . □

The following proposition is the main technical work required for the proof of the equivalence of categories given in the next sections .

<u>Proposition 3.2.3</u>: Let $\alpha, \beta \in D$ and $r \in R$, then the following hold whenever each left-hand side is defined :

i) $\Phi(\alpha +_1 \beta) = \Phi\alpha +_2 \Phi\beta$,

ii) $\Phi(\alpha +_2 \beta) = \Phi\alpha +_2 \Phi\beta$,

iii) $\Phi(\alpha *_1 \beta) = (\Phi\alpha *_2 \varepsilon_1\partial_2^1\beta) +_2 (\varepsilon_1\partial_2^0\alpha *_2 \Phi\beta)$,

iv) $\Phi(\alpha *_2 \beta) = (\varepsilon_1\partial_1^0\alpha *_2 \Phi\beta) +_2 (\Phi\alpha *_2 \varepsilon_1\partial_1^1\beta)$,

v) $\Phi(r \cdot_1 \alpha) = r \cdot_2 \Phi\alpha$, $\Phi(r \cdot_2 \alpha) = r \cdot_2 \Phi\alpha$.

(Appendix II gives diagrams for the proof of the above proposition) .

<u>Proof</u>: i) If α, β have boundaries given by

$$\underline{\partial}\alpha = \left(a \;\begin{matrix} c \\ \\ b \end{matrix}\; d\right) , \quad \underline{\partial}\beta = \left(a_1 \;\begin{matrix} c \\ \\ b \end{matrix}\; d_1\right) ,$$

then $\Phi(\alpha +_1 \beta) = [\Gamma'(a+a_1) *_2 (\alpha +_1 \beta) *_2 \Gamma(d+d_1)] -_2 \varepsilon_1(a+a_1)b$

$= [(\Gamma' a *_2 \alpha *_2 \Gamma d) +_2 (\Gamma' a_1 *_2 \beta *_2 \Gamma d_1)] -_2 [\varepsilon_1 ab +_2 \varepsilon_1 a_1 b]$

by (3.1.4)(iv)

$= [(\Gamma' a *_2 \alpha *_2 \Gamma d) -_2 \varepsilon_1 ab] +_2 [(\Gamma' a_1 *_2 \beta *_2 \Gamma d_1) -_2 \varepsilon_1 a_1 b]$

$= \Phi\alpha +_2 \Phi\beta$.

ii) This follows from the algebroid rules for $+_2$, $*_2$.

iii) If α , β having boundaries given by

$\underline{\partial}\alpha = \left(a \;\begin{matrix} c \\ \\ b \end{matrix}\; d\right)$, $\underline{\partial}\beta = \left(a' \;\begin{matrix} b \\ \\ e \end{matrix}\; d'\right)$, then $\alpha *_1 \beta$ has boundary edges in the form $\underline{\partial}(\alpha *_1 \beta) = (aa' \overset{c}{\underset{e}{}} dd')$.

Then $\Phi(\alpha *_1 \beta) = (\Gamma' aa' *_2 (\alpha *_1 \beta) *_2 \Gamma dd') -_2 \varepsilon_1 aa'e$

$= \{[\Gamma' a *_1 (\varepsilon_1 a *_2 \Gamma' a')] *_2 (\alpha *_1 \beta) *_2 [(\Gamma d *_2 \varepsilon_1 d') *_1$

$\Gamma d]\} -_2 \varepsilon_1 aa'e$ by (3.1.4)(ii)

$= \{[\Gamma' a *_2 \alpha *_2 (\Gamma d *_2 \varepsilon_1 d')] *_1 [(\varepsilon_1 a *_2 \Gamma' a') *_2 \beta *_2 \Gamma d']\}$

$-_2 \varepsilon_1 aa'e$ by (2.1.6)(ii) and the associativity

$= \{[(\Gamma' a *_2 \alpha *_2 \Gamma d) *_2 \varepsilon_1 d'] *_1 [\varepsilon_1 a *_2 (\Gamma' a' *_2 \beta *_2 \Gamma d')\} -_2$

$(\varepsilon_1 aa'e *_1 \varepsilon_1 aa'e)$ by the associativity and the identity rule

$= \{[(\Gamma' a *_2 \alpha *_2 \Gamma d) *_2 \varepsilon_1 d'] -_2 \varepsilon_1 aa'e\} *_1 \{[\varepsilon_1 a *_2 (\Gamma' a' *_2$

$\beta *_2 \Gamma d')] -_2 \varepsilon_1 aa'e\}$ by (2.1.6)(iv)

$= \{[(\Gamma' a *_2 \alpha *_2 \Gamma d) *_2 \varepsilon_1 d'] -_2 \varepsilon_1 abd' +_2 \varepsilon_1(abd' - aa'e)\} *_2$

$\{\varepsilon_1 a *_2 [(\Gamma' a' *_2 \beta *_2 \Gamma d') -_2 \varepsilon_1 a'e]\}$ by distributivity

$=\{\{[(\Gamma' a *_2 \alpha *_2 \Gamma d) -_2 \varepsilon_1 ab\ ] *_2 \varepsilon_1 d'\} +_2 \varepsilon_1(abd' - aa'e)\} *_2$

$\{\varepsilon_1 a *_2 \Phi\beta\}$ by distributivity

$= [(\Phi\alpha *_2 \varepsilon_1 d') +_2 \varepsilon_1(abd' - aa'e)] *_2 [\varepsilon_1 0 +_2 (\varepsilon_1 a *_2 \Phi\alpha)]$

by the identity rule

$= [(\Phi\alpha *_2 \varepsilon_1 d') *_2 \varepsilon_1 0] +_2 [\varepsilon_1(abd' - aa'e) *_2 (\varepsilon_1 a *_2 \Phi\beta)]$

by (2.1.6)(iv)

$= (\Phi\alpha *_2 \varepsilon_1 d') +_2 (\varepsilon_1 a *_2 \Phi\beta)$ by the identity rule

$= (\Phi\alpha *_2 \varepsilon_1 \partial_2^1 \beta) +_2 (\varepsilon_1 \partial_2^0 \alpha *_2 \Phi\beta)$.

iv) If α , β have boundaries given by

$$\underline{\partial}\alpha = (a \overset{c}{\underset{b}{}} d) , \quad \underline{\partial}\beta = (d \overset{c'}{\underset{b'}{}} e) ,$$

then $\alpha *_2 \beta$ has boundary edges in the form $(a \begin{smallmatrix} cc' \\ bb' \end{smallmatrix} e)$.

Now we compute $\Phi(\alpha *_2 \beta) = [\Gamma' a *_2 (\alpha *_2 \beta) *_2 \Gamma e] -_2 \varepsilon_1 abb'$

$=[(\Gamma' a *_2 \alpha) *_2 \varepsilon_2 d *_2 (\beta *_2 \Gamma e)] -_2 \varepsilon_1 abb'$

by associativity and the identity rule

$= [(\Gamma'a *_2 \alpha) *_2 (\Gamma'd *_1 \Gamma d) *_2 (\beta *_2 \Gamma e)] -_2 \varepsilon_1 abb'$

by (3.1.4)(i)

$= \{[\varepsilon_1 c *_1 (\Gamma'a *_2 \alpha)] *_2 (\Gamma'd *_1 \Gamma d) *_2 [(\beta *_2 \Gamma e) *_1 \varepsilon_1 b']\}$
$-_2 \varepsilon_1 abb'$ by the identity rule

$= \{\{(\varepsilon_1 c *_2 \Gamma'd) *_1 [(\Gamma'a *_2 \alpha) *_2 \Gamma d]\} *_2 [(\beta *_2 \Gamma e) *_1 \varepsilon_1 b']\} -_2 \varepsilon_1 abb'$ by (2.1.6)(ii)

$= \{[\varepsilon_1 c *_2 \Gamma'd *_2 \beta *_2 \Gamma e] *_1 [(\Gamma'a *_2 \alpha *_2 \Gamma d *_2 \varepsilon_1 b']\} -_2 (\varepsilon_1 abb' *_1 \varepsilon_1 abb')$ by (2.1.6)(ii) and associativity

$= \{[(\varepsilon_1 c *_2 \Gamma'd *_2 \beta *_2 \Gamma e) -_2 \varepsilon_1 cdd'] +_2 \varepsilon_1(cdb' - abb')\}$
$*_1 \{[(\Gamma'a *_2 \alpha *_2 \Gamma d *_2 \varepsilon_1 b') -_2 \varepsilon_1 abb']\}$ by (2.1.6)(iv)

$= \{\{\varepsilon_1 c *_2 [(\Gamma'd *_2 \beta *_2 \Gamma e) -_2 \varepsilon_1 db']\} +_2 \varepsilon_1(cdb' - abb')\} *_1$
$\{(\Gamma'a *_2 \alpha *_2 \Gamma d) -_2 \varepsilon_1 ab) *_2 \varepsilon_1 b'\}$ by the identity rule

$= [(\varepsilon_1 c *_2 \Phi\beta) +_2 \varepsilon_1(cdb' - abb')] *_1 [\Phi\alpha *_2 \varepsilon_1 b']$

$= [(\varepsilon_1 c *_2 \Phi\beta) +_2 \varepsilon_1(cdb' - abb')] *_1 [\varepsilon_1 0 +_2 (\Phi\alpha *_2 \varepsilon_1 b')]$

$= [(\varepsilon_1 c *_2 \Phi\beta) *_1 \varepsilon_1 0] +_2 [\varepsilon_1(cdb' - abb') *_1 (\Phi\alpha *_2 \varepsilon_1 b')]$

by (2.1.6)(iv)

$= (\varepsilon_1 c *_2 \Phi\beta) +_2 (\Phi\alpha *_2 \varepsilon_1 b')$ by the identity rule

$= (\varepsilon_1 \partial_1^0 \alpha *_2 \Phi\beta) +_2 (\Phi\alpha *_2 \varepsilon_1 \partial_1^1 \beta)$.

It is clear that (v) is satisfied by using (3.1.4)(v) for the first rule , and the algebroid laws for the second . This completes the proof of the proposition . □

We are ready now to construct a functor say λ from the category $\underline{C}$ of crossed modules (over algebroids) to the category $\underline{DA}^!$ of special double algebroids with connections .

3. THE FUNCTOR $\lambda : C \longrightarrow DA^{!}$:

In this section , we construct a special double algebroid with connections from a crossed module (over an algebroid) by using the folding operation .

Let (A,M,μ) be a crossed module (over an algebroid) , and let $D_0 = A_0$ (the set of objects) , $D_1 = D_2 = A$ (the algebroid of arrows of A) . Since $\square D_1 = \square A$ is a special double algebroid with thin structure , then the folding operation Φ applies to it and so for $\underline{a} \in \square A$ with boundary edges $(a_1 \, {}^{a_3}_{a_2} \, a_4)$, we have

$\partial_1^0 \Phi \underline{a} = a_3 a_4 - a_1 a_2$. We let D be given by

$D = \{(\underline{a},\varsigma) : \underline{a} \in \square D_1 , \varsigma \in M \text{ such that } \mu\varsigma = \partial_1^0 \Phi \underline{a}\}$. Thus we can define the maps ε_j , ∂_1^i , ∂_2^i , Γ , Γ' (for $j = 1,2$ and $i = 0,1$) in the following way :

if $a_1 \in D_1$, define $\varepsilon_j a_1 = (\varepsilon_j a_1, 0)$, where ε_i is defined by (2.1.7) . Clearly $\varepsilon_j a_1 \in D$ (since $\Phi \, \varepsilon_j a_1 = 0^2$) . Also define ∂_1^i , $\partial_2^i : D \longrightarrow D_1 = D_2$ by : if $(\underline{a},\varsigma) \in D$, then the boundary edges of $(\underline{a},\varsigma)$ are to be those of $\underline{a}$,i.e. $\underline{\partial}(\underline{a},\varsigma) = \underline{\partial} \, \underline{a}$.

Define a thin structure $\theta : \square D_1 \longrightarrow D$ by $\theta(\underline{a}) = (\underline{a}, 0)$ (here $0 \in M(\partial_1^0 \partial_2^0 \underline{a} , \partial_1^1 \partial_2^1 \underline{a})$.

We define now some algebraic structure on elements of D . First we define two additions ; namely $+_1$, $+_2$.

For $+_1$, let $(\underline{a},\varsigma)$,$(\underline{b},\eta) \in D$ with $\partial_1^i \underline{a} = \partial_1^i \underline{b}$; then we define $(\underline{a},\varsigma) +_1 (\underline{b},\eta) = (\underline{a} +_1 \underline{b}, \varsigma + \eta)$.

For $+_2$, let $(\underline{a},\varsigma),(\underline{b},\eta) \in D$ with $\partial_2^j \underline{a} = \partial_2^j \underline{b}$; we define

$(\underline{a},\zeta) +_2 (\underline{b},\eta) = (\underline{a} +_2 \underline{b} , \zeta + \eta)$.

We define two scaler multiplications: let $(\underline{a},\zeta) \in D$ and $r \in R$; then $r \cdot_1 (\underline{a},\zeta) = (r \cdot_1 \underline{a} , r.\zeta)$, $r \cdot_2 (\underline{a},\zeta) = (r \cdot_2 \underline{a} , r.\zeta)$. Note that these definitions make sense . Thus we have

$$\partial_1^0\Phi(\underline{a} +_1 \underline{b}) = \partial_1^0\Phi\underline{a} +_2 \partial_1^0\Phi\underline{b} = \mu\zeta + \mu\eta = \mu(\zeta + \eta) ,$$

$$\partial_1^0\Phi(r \cdot_1 \underline{a}) = \partial_1^0(r \cdot_2 \Phi\underline{a}) = r . \partial_1^0\Phi\underline{a} = r . \mu\zeta = \mu(r . \zeta) .$$

Next , we define two compositions :

let $(\underline{a},\zeta)$, $(\underline{b},\eta) \in D$ with $\partial_1^1\underline{a} = \partial_1^0\underline{b}$; then we define

$$(\underline{a},\zeta) *_1 (\underline{b},\eta) = (\underline{a} *_1 \underline{b}, \zeta^{\partial_2^1\underline{b}} + {}^{\partial_2^0\underline{a}}\eta)$$

. If $(\underline{a},\zeta),(\underline{b},\eta) \in D$ with $\partial_2^1\underline{a} = \partial_2^0\underline{b}$, then we define

$$(\underline{a},\zeta) *_2 (\underline{b},\eta) = (\underline{a} *_2 \underline{b} , {}^{\partial_1^0\underline{a}}\eta + \zeta^{\partial_1^1\underline{b}}) .$$

We must verify the appropriate boundary condition , we have

$$\partial_1^0\Phi(\underline{a} *_1 \underline{b}) = \partial_1^0[(\Phi\underline{a} *_2 \varepsilon_1\partial_2^1\underline{b}) + (\varepsilon_1\partial_2^0\underline{a} *_2 \Phi\underline{b})$$

$$= (\partial_1^0\Phi\underline{a} *_2 \partial_1^0\varepsilon_1\partial_2^1\underline{b}) +_2 (\partial_1^0\varepsilon_1\partial_2^0\underline{a} *_2 \partial_1^0\Phi\underline{b})$$

$$= (\mu\zeta *_2 \partial_2^1\underline{b}) +_2 (\partial_2^0\underline{a} *_2 \mu\eta) = \mu(\zeta^{\partial_2^1\underline{b}} + {}^{\partial_2^0\underline{a}}\eta)$$

by (1.3.1)(iii) and (1.3.2)(i) .

Thus we are ready to give the main result of this section .

<u>Proposition 3.3.1</u>: The above structure is a special double algebroid with connections .

<u>Proof</u>: First , we want to verify that $(+_1 , *_1 , \cdot_1)$ and $(+_2, *_2, \cdot_2)$ each give an algebroid structure , that is , $*_1$, $*_2$ are R-bilinear morphisms and satisfy the associative condition . It is clear that $*_1$ is an R-bilinear morphism .

Thus $(+_1, *_1, \cdot_1)$ an R-algebroid if $*_1$ satisfies associativity .

Let $(\underline{a},\zeta),(\underline{b},\eta),(\underline{c},\xi) \in D$. Then

$$[(\underline{a},\zeta) *_1 (\underline{b},\eta)] *_1 (\underline{c},\xi) = [\underline{a} *_1 \underline{b} \, , \, \zeta^{\partial_2^1 \underline{b}} + {}^{\partial_2^0 \underline{a}}\eta] *_1 (\underline{c},\xi)$$

$$= [(\underline{a} *_1 \underline{b}) *_1 \underline{c} \, , \, (\zeta^{\partial_2^1 \underline{b}} + {}^{\partial_2^0 \underline{a}}\eta)^{\partial_2^1 \underline{c}} + {}^{\partial_2^0(\underline{a} *_1 \underline{b})}\xi] \, .$$

On the other hand ;

$$(\underline{a},\zeta) *_1 [(\underline{b},\eta) *_1 (\underline{c},\xi)] = (\underline{a},\zeta) *_1 [(\underline{b} *_1 \underline{c}) \, , \, \eta^{\partial_2^1 \underline{c}} + {}^{\partial_2^0 \underline{b}}\xi]$$

$$=[\underline{a} *_1 (\underline{b} *_1 \underline{c}) \, , \, \zeta^{\partial_2^1(\underline{b} *_1 \underline{c})} + {}^{\partial_2^0 \underline{a}}(\eta^{\partial_2^1 \underline{c}} + {}^{\partial_2^0 \underline{b}}\xi)] \, .$$

Clearly $(\underline{a} *_1 \underline{b}) *_1 \underline{c} = \underline{a} *_1 (\underline{b} *_1 \underline{c})$. To prove that

$$(\zeta^{\partial_2^1 \underline{b}} + {}^{\partial_2^0 \underline{a}}\eta)^{\partial_2^1 \underline{c}} + {}^{\partial_2^0(\underline{a} *_1 \underline{b})}\xi = \zeta^{\partial_2^1(\underline{b} *_1 \underline{c})} + {}^{\partial_2^0 \underline{a}}(\eta^{\partial_2^1 \underline{c}} + {}^{\partial_2^0 \underline{b}}\xi)$$

, we start with the right hand side ;

$$= \zeta^{\partial_2^1(\underline{b} *_1 \underline{c})} + ({}^{\partial_2^0 \underline{a}}\eta)^{\partial_2^1 \underline{c}} + {}^{\partial_2^0 \underline{a}}({}^{\partial_2^0 \underline{b}}\xi) \quad \text{by (1.3.1)(i,iii)}$$

$$= (\zeta^{\partial_2^1 \underline{b}})^{\partial_2^1 \underline{c}} + ({}^{\partial_2^0 \underline{a}}\eta)^{\partial_2^1 \underline{c}} + {}^{\partial_2^0 \underline{a}}({}^{\partial_2^0 \underline{b}}\xi) \quad \text{by (1.3.1)(i,iii)}$$

$$= (\zeta^{\partial_2^1 \underline{b}} + {}^{\partial_2^0 \underline{a}}\eta)^{\partial_2^1 \underline{c}} + {}^{\partial_2^0(\underline{a} *_1 \underline{b})}\xi \quad \text{by (1.3.1)(i,iii)}$$

= left hand side . Then $[(\underline{a},\zeta) *_1 (\underline{b},\eta)] *_1 (\underline{c},\xi) = (\underline{a},\zeta) *_1 [(\underline{b},\eta) *_1 (\underline{c},\xi)]$. The verification of the associativity with respect to $*_2$ is similar to that of $*_1$. Thus $(+_2 , *_2 , \cdot_2)$ is an R-algebroid . So we get algebroid structures for each of these two kind of operations .

Next , we want to verify the relations between these operations , and the rules for connections .

For the rules (2.1.3), (2.1.4) the proofs are obvious, since $D_1 = D_2$. Now we verify the rule (2.1.5)(i-iii).

Let $(\underline{a},\zeta),(\underline{b},\eta) \in D$, then $(\underline{a},\zeta) +_2 (\underline{b},\eta) = (\underline{a} +_2 \underline{b}, \zeta+\eta)$ and hence $r \cdot_1 (\underline{a} +_2 \underline{b}, \zeta+\eta) = (r \cdot_1 (\underline{a} +_2 \underline{b}), r \cdot_1 (\zeta+\eta))$

$= [(r \cdot_1 \underline{a} +_2 r \cdot_1 \underline{b}), ((r . \zeta) + (r . \eta))]$

$= ((r \cdot_1 \underline{a}), (r . \zeta)) +_2 ((r \cdot_1 \underline{b}), (r . \eta))$

$= r \cdot_1 (\underline{a}, \zeta) +_2 r \cdot_1 (\underline{b}, \eta)$.

We prove similarly that

$r \cdot_2 [(\underline{a}, \zeta) +_1 (\underline{b}, \eta)] = r \cdot_2 (\underline{a}, \zeta) +_1 r \cdot_2 (\underline{b}, \eta)$,

if $(\underline{a}, \zeta) +_1 (\underline{b}, \eta)$ is defined. Thus the rule (2.1.5)(i) is satisfied.

For (2.1.5)(ii), suppose given $(\underline{a},\zeta)$, $(\underline{b},\eta)$ such that $(\underline{a},\zeta) *_2 (\underline{b},\eta)$ is defined. Then

$$r \cdot_1 [(\underline{a},\zeta) *_2 (\underline{b},\eta)] = r \cdot_1 [(\underline{a} *_2 \underline{b}), \; {}^{\partial_1^0\underline{a}}\eta + \zeta^{\partial_1^1\underline{b}}]$$

$$= (r \cdot_1 (\underline{a} *_2 \underline{b}), r . ({}^{\partial_1^0\underline{a}}\eta + \zeta^{\partial_1^1\underline{b}}))$$

$$=((r \cdot_1 \underline{a} *_2 r \cdot_1 \underline{b}),(r . {}^{\partial_1^0\underline{a}}\eta) + (r . \zeta^{\partial_1^1\underline{b}})) \text{ by (1.3.1)(iv)}$$

$$=[((r \cdot_1 \underline{a})*_2(r \cdot_1 \underline{b})),({}^{\partial_1^0(r \cdot_1 \underline{a})}(r . \eta)+_2(r . \zeta)^{\partial_1^1(r \cdot_1 \underline{b})}$$

by bilinearity

$= (r \cdot_1 \underline{a}, r . \zeta) *_2 (r \cdot_1 \underline{b}, r . \eta)$

$= (r \cdot_1 (\underline{a},\zeta)) *_2 (r \cdot_1 (\underline{b},\eta))$. Similarly for the second part of (2.1.5)(ii).

Finally, for (2.1.5)(iii), given $(\underline{a},\zeta) \in D$, then

$r \cdot_2 [s \cdot_1 (\underline{a},\zeta)] = r \cdot_2 [s \cdot_1 \underline{a}, s . \zeta]$

$= [r \cdot_2 (s \cdot_1 \underline{a}), r . (s . \zeta)] = [s \cdot_1 (r \cdot_2 \underline{a}), s . (r . \zeta)]$

$= s \cdot_1 [r \cdot_2 \underline{a}, r . \zeta] = s \cdot_1 (r \cdot_2 (\underline{a},\zeta))$.

Next , we want to verify the interchange laws (2.1.6)(i–iv) .

For (2.1.6)(i) , let $(\underline{a},\zeta),(\underline{b},\eta),(\underline{c},\xi),(\underline{d},\psi) \in D$ such that $(\underline{a},\zeta) +_1 (\underline{b},\eta)$, $(\underline{a},\zeta) +_2 (\underline{c},\xi)$, $(\underline{b},\eta) +_2 (\underline{d},\psi)$, $(\underline{c},\xi) +_1 (\underline{d},\psi)$ are defined , then

$[(\underline{a},\zeta) +_1 (\underline{b},\eta)] +_2 [(\underline{c},\xi) +_1 (\underline{d},\psi)] = (\underline{a} +_1 \underline{b} \ , \ \zeta + \eta) +_2 (\underline{c} +_1 \underline{d} \ , \ \xi + \psi)$

$= [(\underline{a} +_1 \underline{b}) +_2 (\underline{c} +_1 \underline{d}) \ , \ (\zeta + \eta) + (\xi + \psi)]$

$= [(\underline{a} +_2 \underline{c}) +_1 (\underline{b} +_2 \underline{d}) \ , \ (\zeta + \xi) + (\eta + \psi)]$

$= (\underline{a} +_2 \underline{c} \ , \ \zeta + \xi) +_1 (\underline{b} +_2 \underline{d} \ , \ \eta + \psi)$

$=((\underline{a},\zeta) +_2 (\underline{c},\xi)) +_1 ((\underline{b},\eta) +_2 (\underline{d},\psi))$.

For (2.1.6)(ii) , let $(\underline{a},\zeta),(\underline{b},\eta),(\underline{c},\xi),(\underline{d},\psi) \in D$ such that $(\underline{a},\zeta) *_1 (\underline{b},\eta)$, $(\underline{a},\zeta) *_2 (\underline{c},\xi)$, $(\underline{b},\eta) *_2 (\underline{d},\psi)$, $(\underline{c},\xi) *_1 (\underline{d},\psi)$ are defined , then

$((\underline{a},\zeta) *_1 (\underline{b},\eta)) *_2 ((\underline{c},\xi) *_1 (\underline{d},\psi)) =$

$(\underline{a} *_1 \underline{b} \ , \ \zeta^{\partial^1_2 \underline{b}} + {}^{\partial^0_2 \underline{a}}\eta) *_2 (\underline{c} *_1 \underline{d} \ , \ \xi^{\partial^1_2 \underline{d}} + {}^{\partial^0_2 \underline{c}}\psi) =$

$[(\underline{a} *_1 \underline{b}) *_2 (\underline{c} *_1 \underline{d}) \ , \ {}^{\partial^0_1(\underline{a} *_1 \underline{b})}(\xi^{\partial^1_2 \underline{d}} + {}^{\partial^0_2 \underline{c}}\psi) + (\zeta^{\partial^1_2 \underline{b}} + {}^{\partial^0_2 \underline{a}}\eta)^{\partial^1_1(\underline{c} *_1 \underline{d})} \ ]$

$= [(\underline{a} *_2 \underline{c}) *_1 (\underline{b} *_2 \underline{d}) \ , \ {}^{\partial^0_1 \underline{a}}(\xi^{\partial^1_2 \underline{d}} + {}^{\partial^0_2 \underline{c}}\psi) + (\zeta^{\partial^1_2 \underline{b}} + {}^{\partial^0_2 \underline{a}}\eta)^{\partial^1_1 \underline{d}} \ ]$

$= [(\underline{a} *_2 \underline{c}) *_1 (\underline{b} *_2 \underline{d}) \ , \ ({}^{\partial^0_1 \underline{a}}\xi)^{\partial^1_2 \underline{d}} + {}^{\partial^0_1 \underline{a}\partial^0_2 \underline{c}}\psi + \zeta^{\partial^1_2 \underline{b}\partial^1_1 \underline{d}} + {}^{\partial^0_2 \underline{a}}(\eta^{\partial^1_1 \underline{d}}) \ ]$ by (1.3.1)(i,iii) .

On the other hand ;

$[(\underline{a},\zeta) *_2 (\underline{c},\xi)] *_1 [(\underline{b},\eta) *_2 (\underline{d},\psi)] =$

$$[\underline{a} *_2 \underline{c} , \ {}^{\partial^0_1\underline{a}}\xi + \zeta^{\partial^1_1\underline{c}}] *_1 [\underline{b} *_2 \underline{d} , \ {}^{\partial^0_1\underline{b}}\psi + \eta^{\partial^1_1\underline{d}}] =$$

$$[(\underline{a} *_2 \underline{c}) *_1 (\underline{b} *_2 \underline{d}) , ({}^{\partial^0_1\underline{a}}\xi + \zeta^{\partial^1_1\underline{c}})^{\partial^1_2(\underline{b} *_2 \underline{d})} + {}^{\partial^0_2(\underline{a} *_2 \underline{c})}({}^{\partial^0_1\underline{b}}\psi + \eta^{\partial^1_1\underline{d}})] =$$

$$[(\underline{a} *_2 \underline{c}) *_1 (\underline{b} *_2 \underline{d}) ,({}^{\partial^0_1\underline{a}}\xi)^{\partial^1_2\underline{d}} + (\zeta^{\partial^1_1\underline{c}})^{\partial^1_2\underline{d}} + {}^{\partial^0_2\underline{a}}({}^{\partial^0_1\underline{b}}\psi) + {}^{\partial^0_2\underline{a}}(\eta^{\partial^1_1\underline{d}})] \text{ by (1.3.1)(i,iii) .}$$

In order for these to be equal , we need ;

$$\zeta^{\partial^1_1\underline{c}\partial^1_2\underline{d}} + {}^{\partial^0_2\underline{a}\partial^0_1\underline{b}}\psi = {}^{\partial^0_1\underline{a}\partial^0_2\underline{c}}\psi + \zeta^{\partial^1_2\underline{b}\partial^1_1\underline{d}}$$

i.e. $\zeta^{\partial^1_1\underline{c}\partial^1_2\underline{d}} - \zeta^{\partial^1_2\underline{b}\partial^1_1\underline{d}} = {}^{\partial^0_1\underline{a}\partial^0_2\underline{c}}\psi - {}^{\partial^0_2\underline{a}\partial^0_1\underline{b}}\psi$

i.e. $\zeta^{\partial^1_1\underline{c}\partial^1_2\underline{d} - \partial^1_2\underline{b}\partial^1_1\underline{d}} = {}^{\partial^0_1\underline{a}\partial^0_2\underline{c} - \partial^0_2\underline{a}\partial^0_1\underline{b}}\psi$

i.e. $\zeta^{\partial^0_1\underline{d}\partial^1_2\underline{d} - \partial^0_2\underline{d}\partial^1_1\underline{d}} = {}^{\partial^0_1\underline{a}\partial^1_2\underline{a} - \partial^0_2\underline{a}\partial^1_1\underline{a}}\psi$

i.e. $\zeta^{\Phi\underline{d}} = {}^{\Phi\underline{a}}\psi$.

The last equation follows from the crossed module rule (1.3.2)(ii) , since both sides are $\zeta * \psi$.

For (2.1.6)(iii) , let $(\underline{a},\zeta),(\underline{b},\eta),(\underline{c},\xi),(\underline{d},\psi) \in D$ such that $(\underline{a},\zeta) *_2 (\underline{b},\eta)$, $(\underline{c},\xi) *_2 (\underline{d},\psi)$, $(\underline{a},\zeta) +_1 (\underline{c},\xi)$, $(\underline{b},\eta) +_1 (\underline{d},\psi)$ are defined , then

$[(\underline{a},\zeta) *_2 (\underline{b},\eta)] +_1 [(\underline{c},\xi) *_2 (\underline{d},\psi)]$

$= [\underline{a} *_2 \underline{b} , \ {}^{\partial^0_1\underline{a}}\eta + \zeta^{\partial^1_1\underline{b}}] +_1 [\underline{c} *_2 \underline{d} , \ {}^{\partial^0_1\underline{c}}\psi + \xi^{\partial^1_1\underline{d}}]$

$= [(\underline{a} *_2 \underline{b}) + (\underline{c} *_2 \underline{d}) , ({}^{\partial^0_1\underline{a}}\eta + \zeta^{\partial^1_1\underline{b}}) + ({}^{\partial^0_1\underline{c}}\psi + \xi^{\partial^1_1\underline{d}})]$

$$= [(\underline{a} +_1 \underline{c}) *_2 (\underline{b} +_1 \underline{d}) , ({}^{\partial^0_1 \underline{a}}\eta + {}^{\partial^0_1 \underline{c}}\psi) + (\varsigma^{\partial^1_1 \underline{b}} + \xi^{\partial^1_1 \underline{d}})]$$

$$= [(\underline{a} +_1 \underline{c}) *_2 (\underline{b} +_1 \underline{d}) , {}^{\partial^0_1 \underline{a}}(\eta + \psi) + (\varsigma + \xi)^{\partial^1_1 \underline{d}}]$$

by (1.3.1)(ii) and the above hypothesis

$$= [(\underline{a} +_1 \underline{c}) *_2 (\underline{b} +_1 \underline{d}) , {}^{\partial^0_1(\underline{a} +_1 \underline{c})}(\eta + \psi) + (\varsigma + \xi)^{\partial^1_1(\underline{b} +_1 \underline{d})}]$$

$$= [\underline{a} +_1 \underline{c} , \varsigma + \xi] *_2 [\underline{b} +_1 \underline{d} , \eta + \psi] .$$

We can use a similar argument to verify (2.1.6)(iv) .

It is clear that ε_j satisfy the rule (2.1.7) , and θ satisfy the conditions (3.1.4)(i-v) . This completes the proof . □

Thus any crossed module (over an algebroid) gives a special double algebroid with connections . If (A,M,μ) , (A',M',μ') are two crossed modules (over algebroids) and $(\alpha,\beta) : (A,M,\mu) \longrightarrow (A',M',\mu')$ is a morphism , then there exist a morphism $(\lambda\alpha, \varepsilon\beta, I) : (D,D_1,D_2,D_0) \longrightarrow (D',D_1',D_2',D_0')$ (since $A = D_1 = D_2$, $A' = D_1' = D2'$, $A_0 = D_0$, $A_0' = D_0'$ and D,D' have been constructed from (A,M) , (A',M') respectively) .

Thus there exist a functor λ from the category of crossed modules (over algebroids) $\underline{C}$ to the category of special double algebroids with connections $\underline{DA^!}$, that is we have a functor

$$\lambda : \underline{C} \longrightarrow \underline{DA^!} .$$

4. THE EQUIVALENCE OF CATEGORIES:

In this section , we want to prove the main result , which is the equivalence of the two categories $\underline{C}$, $\underline{DA^!}$.

Theorem 3.4.1: The functors γ , λ defined previously form an adjoint equivalence

$$\gamma : \underline{DA^!} \rightleftarrows \underline{C} : \lambda \ .$$

<u>Proof</u>: First , we want to prove that $\gamma\lambda$ is naturally equivalent to the identity , that is , $\gamma\lambda \cong 1$.

Let (A,M,μ) be an object of $\underline{C}$ and let $(A',M',\mu') = \gamma\lambda(A,M,\mu)$. Then $A_0 = A_0'$ and $A = A'$. It is clear that M' is defined on the same set of objects A_0 . Define a map $g:M \to M'$ by

$g(m) = (m;\ 1 \ {}^{\mu m}_{0}\ 1)$, and let $I:A \to A'$ be the identity map .

We want now to prove that $(I,g):(A,M,\mu) \to (A',M',\mu')$ is a crossed module morphism , that is $I\mu = \mu' g$ and g preserves the actions . Clearly I,g are algebroid morphisms and $\mu' g = I\mu$. So it is enough to show that (I,g) preserves the actions .

Take $m:x \to y \in M(x,y)$ and let $b:y \to z \in A(y,z)$. Thus

$$g(m^b) = (m^b;1 \ {}^{(\mu m)b}_{0}\ 1) \qquad \text{by (1.3.2)(i)}$$

$$= (m;1 \ {}^{\mu m}_{0}\ 1) *_2 \varepsilon_1 b = (m;1 \ {}^{\mu m}_{0}\ 1)^b = g(m)^b \ .$$

We prove similarly that $g({}^b m) = {}^b g(m)$.

We define now a map $(I,f):(A',M',\mu') \to (A,M,\mu)$ such that (I,g) , (I,f) are inverse to each other . Let $I:A' \to A$ be the identity map and define $f:M' \to M$ by

$f(m;1 \ {}^{\mu m}_{0}\ 1) = m$.

Clearly I,f are algebroid morphisms and $\mu f = \mu'$. Thus (I,f) is a crossed module morphism if it preserves the action , that is ,

let $(m;1 \ {}^{\mu m}_{0}\ 1) \in M'$ and $b \in A'$. Then

$$f[(m;1 \begin{matrix}\mu m\\ \\ 0\end{matrix} 1)^b] = f[(m;1 \begin{matrix}\mu m\\ \\ 0\end{matrix} 1) *_2 \varepsilon_1 b] = f[(m^b;1 \begin{matrix}(\mu m)b\\ \\ 0\end{matrix} 1)]$$

$$= f(m^b;1 \begin{matrix}\mu(m^b)\\ \\ 0\end{matrix} 1) = m^b = [f(m;1 \begin{matrix}\mu m\\ \\ 0\end{matrix} 1)]^b .$$

It is clear that $(I,g),(I,f)$ are inverse to each other . Therefore $\gamma\lambda$ is naturally equivalent to the identity .

Second , we want to show that $\lambda\gamma$ is naturally equivalent to the identity , that is , $1 \cong \lambda\gamma$.

Let D be an object of $\underline{DA}^!$ and let $E = \lambda\gamma(D)$. Then $D_0 = E_0$, $D_1 = D_2 = E_1 = E_2$. We define $\eta : D \longrightarrow E$ to be the identity on D_0 and $D_1 = D_2$ and on D as follows :

let $\alpha \in D$, define $\eta(\alpha) = (\underline{\partial}\alpha , \Phi\alpha)$. First we prove ;

<u>Lemma 3.4.1</u>: The map η is a morphism of double R-algebroid with connections (Γ,Γ') .

<u>Proof</u>: It suffices to prove that η preserves $+_1$, $+_2$, $*_1$, $*_2$, $\cdot_1$, $\cdot_2$, and the connection Γ , Γ' .

For $+_1$, let $\alpha,\beta \in D$ such that $\alpha +_1 \beta$ is defined , then

$$\eta(\alpha +_1 \beta) = [\underline{\partial}(\alpha +_1 \beta) , \Phi(\alpha +_1 \beta)] = (\underline{\partial}\alpha +_1 \underline{\partial}\beta , \Phi\alpha + \Phi\beta)$$

(since $\Phi a, \Phi\beta \in \gamma(D)$)

$$= (\underline{\partial}\alpha , \Phi\alpha) +_1 (\underline{\partial}\beta , \Phi\beta) = \eta\alpha +_1 \eta\beta .$$

We can prove similarly that $\eta(\alpha +_2 \beta) = \eta\alpha +_2 \eta\beta$, if $\alpha +_2 \beta$ is defined .

For $*_1$, let $\alpha,\beta \in D$ such that α,β have boundaries in the form $(a \begin{matrix}c\\ \\ b\end{matrix} d)$, $(a' \begin{matrix}b\\ \\ e\end{matrix} d')$ respectively , then

$$\eta(\alpha *_1 \beta) = [\underline{\partial}(\alpha *_1 \beta) , \Phi(\alpha *_1 \beta)]$$

$$= (\underline{\partial}\alpha *_1 \underline{\partial}\beta , (\Phi a)^{d'} + {}^{a}(\Phi\beta)) \qquad \text{by (3.2.3)(iii)} .$$

On the other hand ;

$$\eta\alpha *_1 \eta\beta = (\underline{\partial}\alpha , \Phi\alpha) *_1 (\underline{\partial}\beta , \Phi\beta)$$

$= (\underline{\partial}\alpha *_1 \underline{\partial}\beta \, , \, (\Phi\alpha)^{d'} + {}^{a}(\Phi\beta)) = \eta(\alpha *_1 \beta)$.

We prove similarly that $\eta(\alpha *_2 \beta) = \eta\alpha *_2 \eta\beta$, if $\alpha *_2 \beta$ is defined .

For $\cdot_1$, let $\alpha \in D$ and $r \in R$, then

$\eta(r \cdot_1 \alpha) = (\underline{\partial}(r \cdot_1 \alpha) \, , \, \Phi(r \cdot_1 \alpha)) = (r \cdot_1 \underline{\partial}\alpha \, , \, r \cdot_2 \Phi\alpha)$

by (3.2.3)(v)

$= (r \cdot_1 \underline{\partial}\alpha \, , \, r \cdot \Phi\alpha)$ (since $\Phi\alpha \in \gamma D$)

$= r \cdot_1 (\underline{\partial}\alpha \, , \, \Phi\alpha) = r \cdot_1 \eta\alpha$. Similarly for $\cdot_2$, we get

$\eta(r \cdot_2 \alpha) = r \cdot_2 \eta\alpha$.

Finally , for the connection Γ, Γ' , let $a \in D_1 = D_2$, so $\Gamma a \in D$ and then $\eta(\Gamma a) = (\underline{\partial}\Gamma a \, , \, \Phi\Gamma a) = (\underline{\partial}\Gamma a \, , \, 0^2)$ by (3.2.2)(i) $= \Gamma a$. Similarly for Γ' . This is the complete proof of the lemma . □

We continue now to prove the theorem . First , we define $\eta' : E \longrightarrow D$ to be the identity on E_0 and $E_1 = E_2$ and on E by the formulae :

$$\eta'(\alpha, \varsigma) = \left(1 \begin{matrix} c \\ \\ cd \end{matrix} d\right) *_1 [\Phi\alpha +_2 \varepsilon_1 ab] *_1 \left(a \begin{matrix} ab \\ \\ b \end{matrix} 1\right) \text{ as shown below:}$$

$$\begin{array}{c} c \\ 1 \; \boxed{\; t_1 \;} \; d \\ cd \end{array} *_1 \left(\begin{array}{c} cd-ab \\ 1 \; \boxed{\; \Phi\alpha \;} \; 1 \\ 0 \end{array} +_2 \begin{array}{c} ab \\ 1 \; \boxed{\; \varepsilon_1 ab \;} \; 1 \\ ab \end{array} \right) *_1 \begin{array}{c} ab \\ a \; \boxed{\; t_2 \;} \; 1 \\ b \end{array} ,$$

whenever (α, ς) has boundary edges of the form $\left(a \begin{matrix} c \\ \\ b \end{matrix} d\right)$ and

t_1, t_2 are abbreviations for the thin elements with boundaries

$\left(1 \begin{matrix} c \\ \\ cd \end{matrix} d\right) , \left(a \begin{matrix} ab \\ \\ b \end{matrix} 1\right)$.

<u>Lemma 3.4.3</u>: The maps η , η' are inverse to each other , that is , (i) $\eta\eta' = 1$ (ii) $\eta'\eta = 1$.

<u>Proof</u>: (i) Let $(\alpha, \varsigma) \in E$, with $\partial_1^0 \Phi\alpha = \mu\varsigma$ and α has boundary

edges given by $\left(a \;\begin{smallmatrix} c \\ \\ b \end{smallmatrix}\; d\right)$, then

$$\eta\eta'(\alpha,\varsigma) = \eta\left[\left(1 \;\begin{smallmatrix} c \\ \\ cd \end{smallmatrix}\; d\right) *_1 (\Phi\alpha +_2 \varepsilon_1 ab) *_1 \left(a \;\begin{smallmatrix} ab \\ \\ b \end{smallmatrix}\; 1\right)\right]$$

$$= \eta(\alpha) = (\underline{\partial}\alpha \; , \; \Phi\alpha) \; .$$

It is clear that α , $\underline{\partial}\alpha$ has the same boundary $\left(a \;\begin{smallmatrix} c \\ \\ b \end{smallmatrix}\; d\right)$, and

$\mu\,\Phi\alpha = \mu\varsigma$. Thus $\eta\eta'(\alpha,\varsigma) = (\alpha,\varsigma)$.

(ii) Let $\alpha \in D$, where α has boundary $\left(a \;\begin{smallmatrix} c \\ \\ b \end{smallmatrix}\; d\right)$, so

$\eta'\eta(\alpha) = \eta'(\underline{\partial}\alpha \; , \; \Phi\alpha) = \alpha$ (since $\underline{\partial}\alpha$, α have the same boundaries) . This is the complete proof of lemma (3.4.3) . □

This completes the proof that $\eta : D \dashrightarrow E$ is an isomorphism . The naturality of η is clear . So we have proved the natural equivalence $1 \cong \lambda\gamma$. □

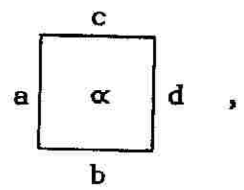
We move on to give a property of these objects by using the above theorem .

5. REFLECTION:

In this section we use the above theorem to show that every object in $\underline{DA^!}$ has a nice property called "reflection" : in a special double algebroid with connection the two algebroid structures are isomorphic .

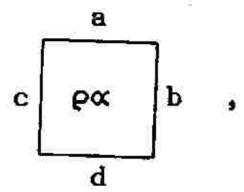
This property has been given in the double groupoid case in [B-2] under the name "rotation" . Reflections in double categories with connection have also studied in [S-1],[S-W-] .

For each object $(D,\Gamma,\Gamma') \in \underline{DA^!}$, there is a reflection $\rho : D \dashrightarrow D$ such that on edges ρ behaves as follows : let α be a square in D , pictured as



,

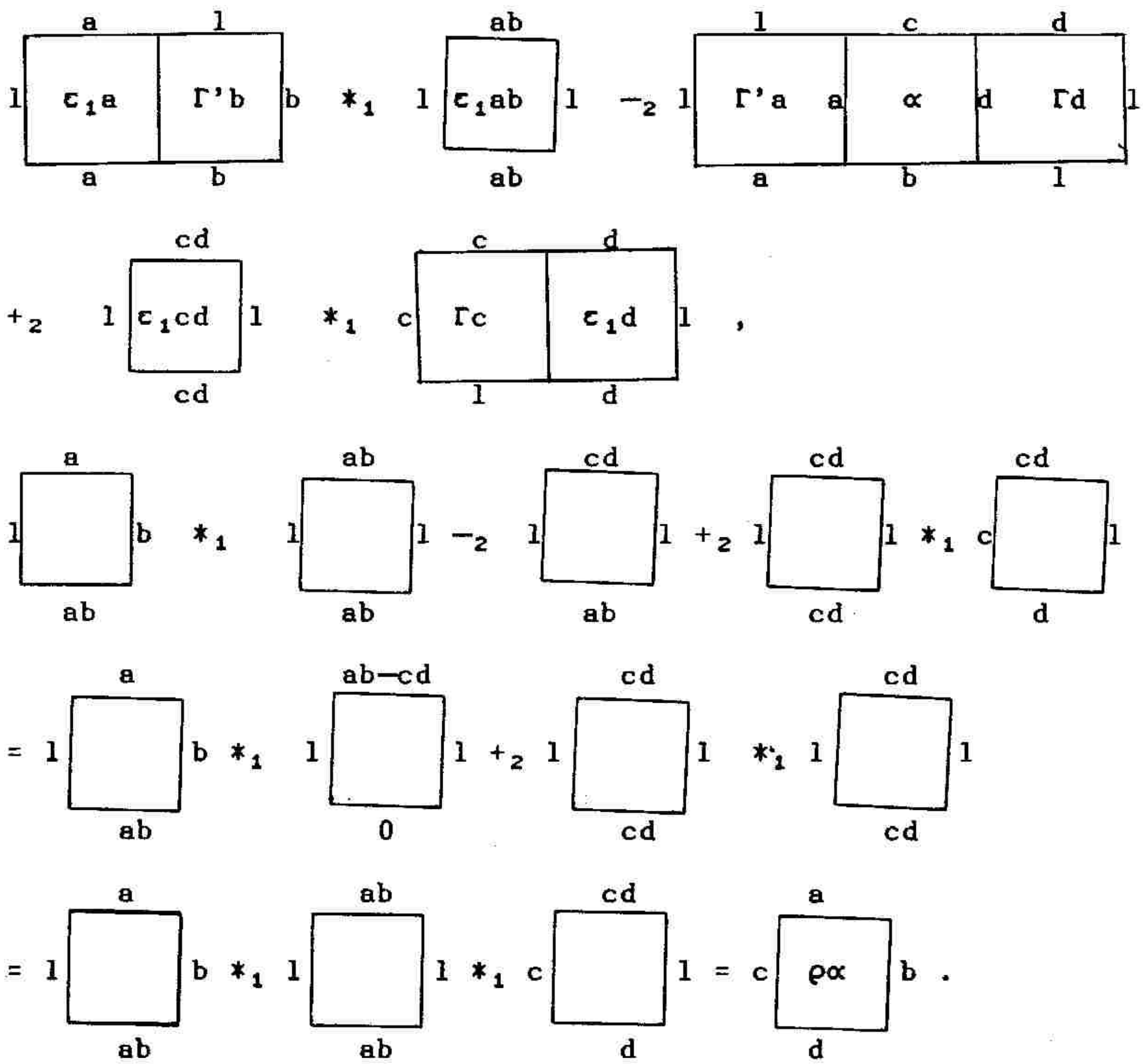
then $\rho\alpha$ is a square in the form



,

and $\rho\alpha$ is defined by

$\rho\alpha = (\varepsilon_1 a *_2 \Gamma' b) *_1 [(\varepsilon_1 ab -_2 (\Gamma' a *_2 \alpha *_2 \Gamma d)) +_2 \varepsilon_1 cd] *_1$ $(\Gamma c *_2 \varepsilon_1 d)$, as shown diagrammatically ;



__Theorem 3.5.1__: The reflection ρ satisfies

i) $\rho(\Gamma a) = \Gamma a$, $\rho(\Gamma' a) = \Gamma' a$, $\rho(\varepsilon_1 a) = \varepsilon_1 a$, $\rho(\varepsilon_2 a) = \varepsilon_2 a$, for $a \in D_2$ or D_1 .

ii) $\rho(\alpha +_1 \beta) = \rho\alpha +_2 \rho\beta$, $\rho(\gamma +_2 \delta) = \rho\gamma +_1 \rho\delta$, whenever $\alpha +_1 \beta$, $\gamma +_2 \delta$ are defined .

iii) $\rho(\alpha *_1 \beta) = \rho\alpha *_2 \rho\beta$, $\rho(\gamma *_2 \delta) = \rho\gamma *_1 \rho\delta$, whenever $\alpha *_1 \beta$, $\gamma *_2 \delta$ are defined .

iv) $\rho^2 = id$.

v) $\rho(r \cdot_1 \alpha) = r \cdot_2 \rho\alpha$, $\rho(r \cdot_2 \alpha) = r \cdot_1 \rho\alpha$, where $r \in R$.

__Proof__: By theorem (3.4.1) , we may assume that D is the double algebroid arising from a crossed module $\mu: M \longrightarrow A$. So if $\alpha \in D$, we may write $\alpha = \left(m;a \begin{matrix} c \\ \\ b \end{matrix} d\right)$, where $m \in M$, $a,b,c,d \in A$ and $\mu m = cd - ab$. We calculate now $\rho(\alpha)$ as follows :

$$\rho(\alpha) = \left(0;1 \begin{matrix} a \\ \\ ab \end{matrix} b\right) *_1 \left[\left(\left(0;1 \begin{matrix} ab \\ \\ ab \end{matrix} 1\right) -_2 \left(m;1 \begin{matrix} cd \\ \\ ab \end{matrix} 1\right)\right) +_2 \left(0;1 \begin{matrix} cd \\ \\ cd \end{matrix} 1\right)\right] *_1 \left(0;c \begin{matrix} cd \\ \\ d \end{matrix} 1\right)$$

$$= \left(0;1 \begin{matrix} a \\ \\ ab \end{matrix} b\right) *_1 \left[\left(\left(0;1 \begin{matrix} ab \\ \\ ab \end{matrix} 1\right) +_2 \left(-m;1 \begin{matrix} -cd \\ \\ -ab \end{matrix} 1\right)\right) +_2 \left(0;1 \begin{matrix} cd \\ \\ cd \end{matrix} 1\right)\right] *_1 \left(0;c \begin{matrix} cd \\ \\ d \end{matrix} 1\right)$$

$$=\left(0;1 \begin{matrix} a \\ \\ ab \end{matrix} b\right) *_1 \left(-m;1 \begin{matrix} ab \\ \\ cd \end{matrix} 1\right) *_1 \left(0;c \begin{matrix} cd \\ \\ d \end{matrix} 1\right) = \left(-m;c \begin{matrix} a \\ \\ d \end{matrix} b\right) .$$

Now we verify the relations (i-v) .

i) $\rho(\Gamma a) = \rho\left(0;a \begin{matrix} a \\ \\ 1 \end{matrix} 1\right) = \left(0;a \begin{matrix} a \\ \\ 1 \end{matrix} 1\right) = \Gamma a$ and by similar way for $\Gamma' a$, $\varepsilon_1 a$, $\varepsilon_2 a$.

ii) Let $\alpha, \beta \in D$ with boundaries $(a \ {}^{c}_{b} \ d)$, $(a_1 \ {}^{c}_{b} \ d_1)$, then

$\rho(\alpha +_1 \beta) = (-(m+m_1);\ c \ {}^{a+a_1}_{d+d_1} \ b)$. On the other hand ;

$\rho(\alpha) +_2 \rho(\beta) = (-m;c \ {}^{a}_{d} \ b) +_2 (-m_1;c \ {}^{a_1}_{d_1} \ b) = (-(m+m_1);c \ {}^{a+a_1}_{d+d_1} \ b)$

$= \rho(\alpha +_1 \beta)$. Thus $\rho(\alpha +_1 \beta) = \rho\alpha +_2 \rho\beta$. Also we prove similarly that $\rho(\gamma +_2 \delta) = \rho\gamma +_1 \rho\delta$.

iii) Let $\alpha, \beta \in D$ with boundaries $(a \ {}^{c}_{b} \ d)$, $(a' \ {}^{b}_{e} \ d')$, then

$\rho(\alpha *_1 \beta) = (-(mm');c \ {}^{aa'}_{dd'} \ e)$. On the other hand ;

$\rho(\alpha) *_2 \rho(\beta) = (-m;c \ {}^{a}_{d} \ b) *_2 (-m';b \ {}^{a'}_{d'} \ e) = (-(mm');c \ {}^{aa'}_{dd'} \ e)$.

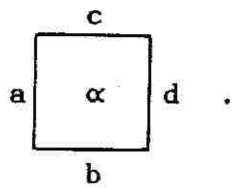
Thus $\rho(\alpha *_1 \beta) = \rho\alpha *_2 \rho\beta$. Similarly for $\rho(\gamma *_2 \delta) = \rho\gamma *_1 \rho\delta$. The calculation of (iv),(v) are easy to verify . Therefore ρ satisfies the relations (i-v) . □

APPENDIX I

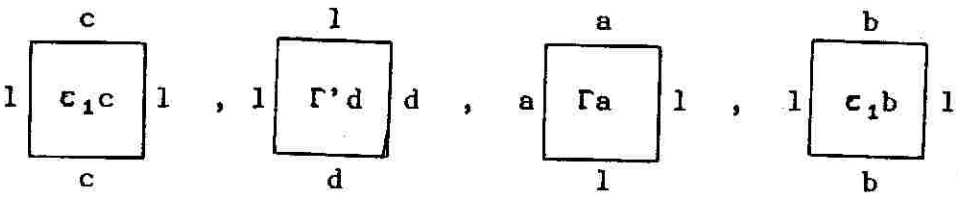
Verification of Theorem (3.1.7) and Lemma (3.1.8):

i) The definition of θ_1 , θ_2 :

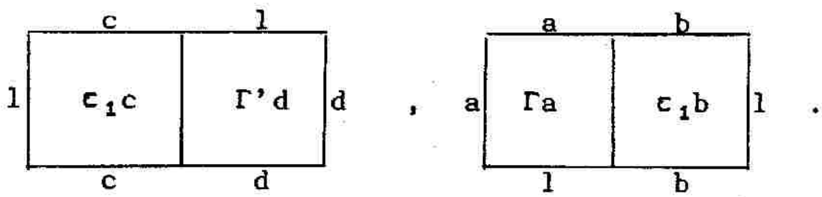
Let $a,b,c,d \in D_1$ with $cd = ab$ and α has boundary given by



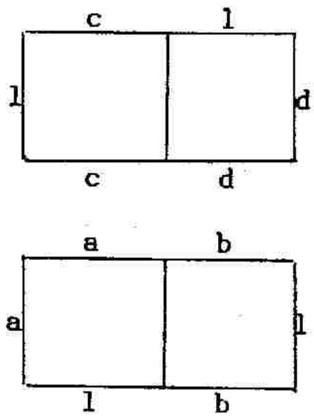
Since $\theta_1\left(a \begin{smallmatrix} c \\ \\ b \end{smallmatrix} d\right) = (\varepsilon_1 c *_2 \Gamma' d) *_1 (\Gamma a *_2 \varepsilon_1 b)$ and $\varepsilon_1 c$, $\Gamma' d$, Γa , $\varepsilon_1 b$ have boundaries given by



and then $(\varepsilon_1 c *_2 \Gamma' d)$, $(\Gamma a *_2 \varepsilon_1 b)$ have boundaries in the form



Thus $(\varepsilon_1 c *_2 \Gamma' d) *_1 (\Gamma a *_2 \varepsilon_1 b)$ is defined (since $cd = ab$) ; namely



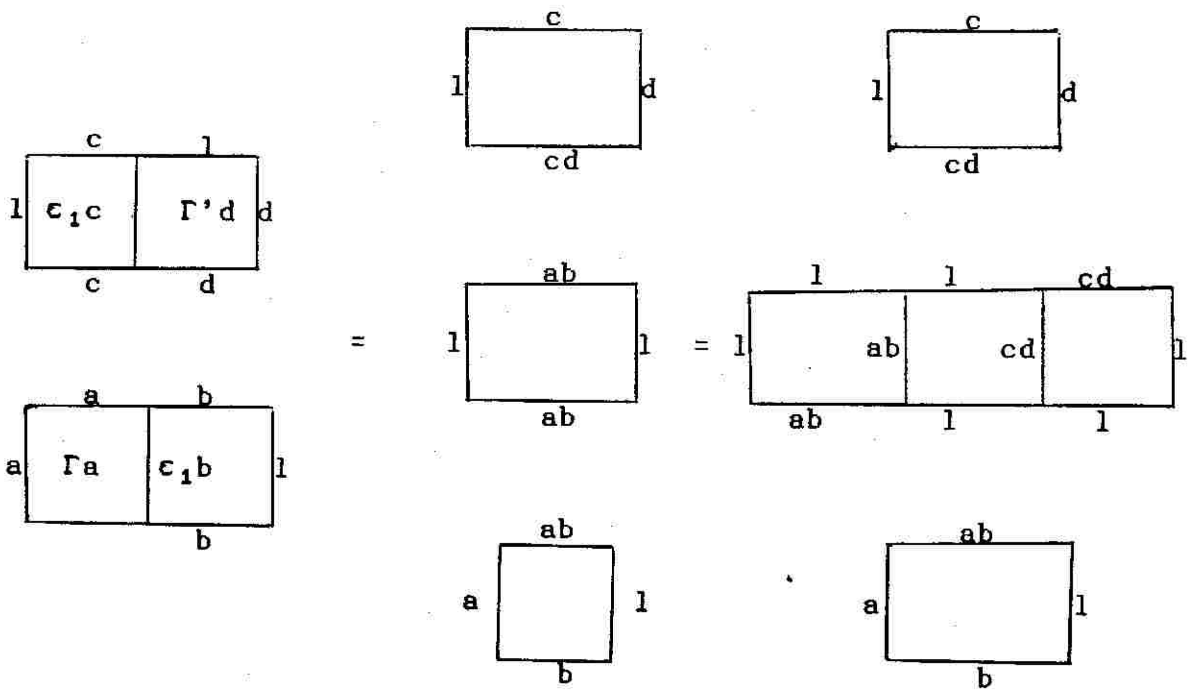
.

Similarly for the definition of θ_2 .

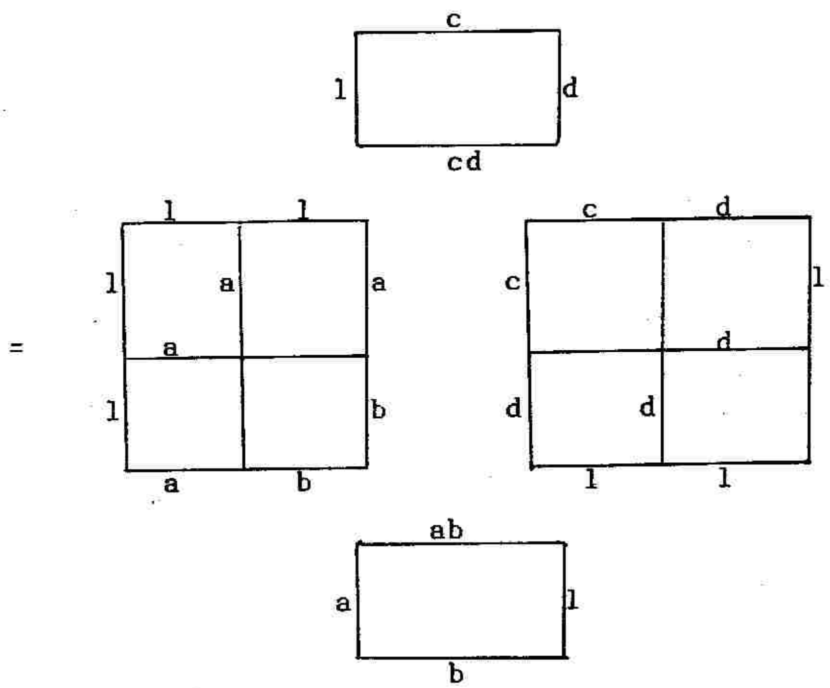
ii) <u>Lemma 3.1.8</u>:

$$\theta_1\left(a \begin{smallmatrix} c \\ \\ b \end{smallmatrix} d\right) = (\varepsilon_1 c *_2 \Gamma' d) *_1 (\Gamma a *_2 \varepsilon_1 b) \text{ which is}$$

diagrammatically given by



(since cd = ab) ,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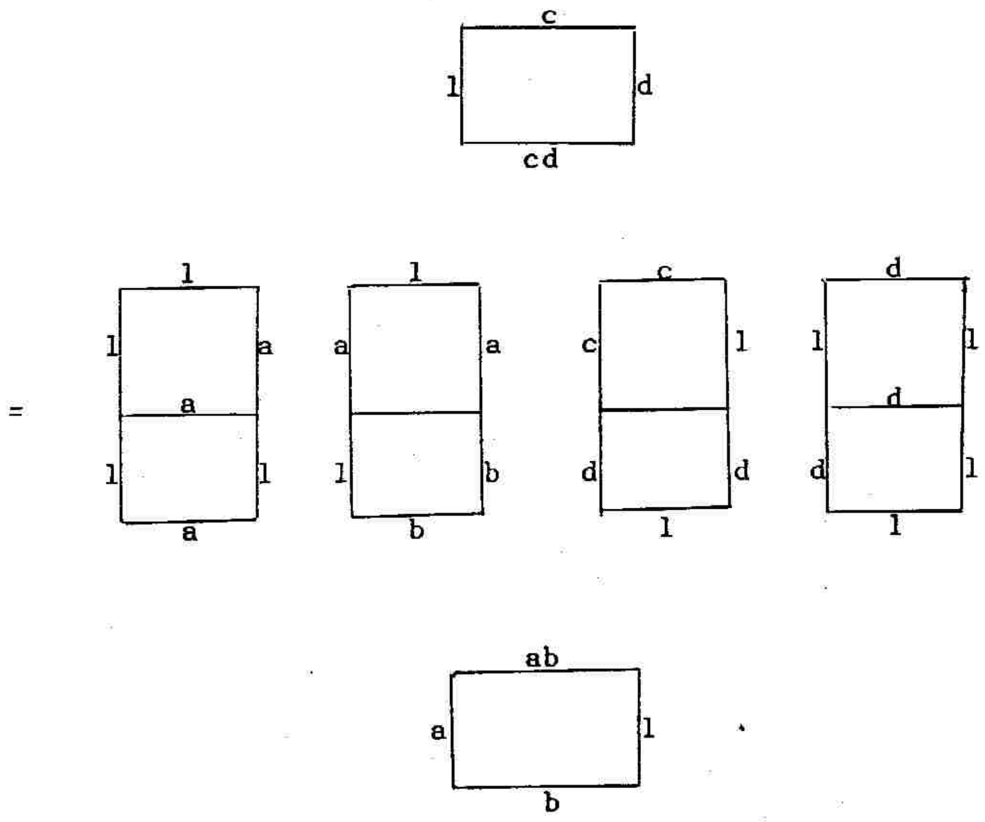
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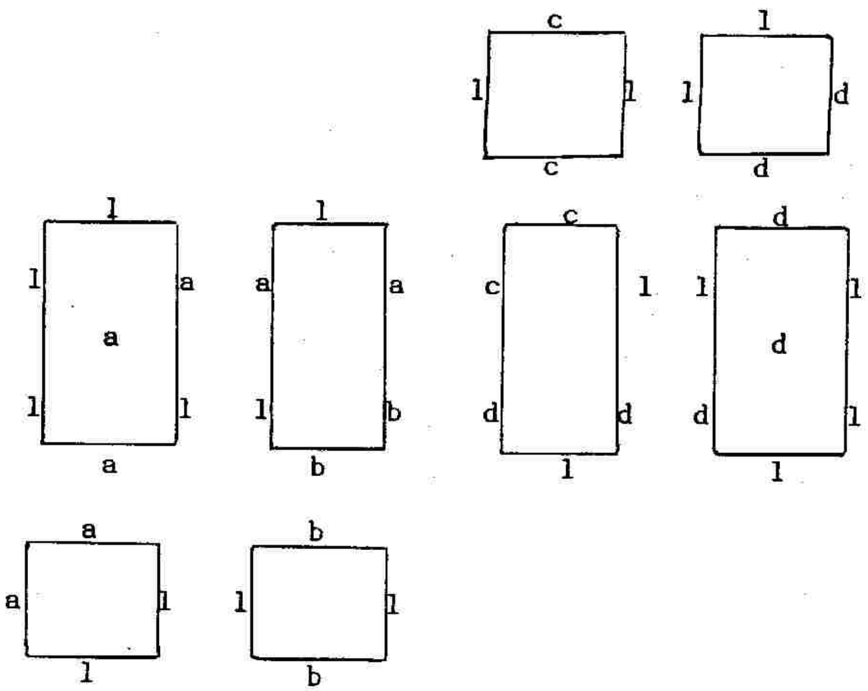

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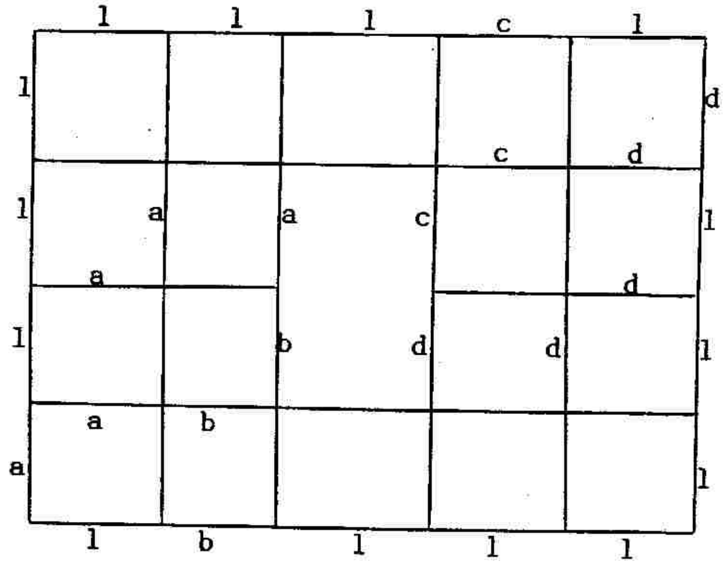

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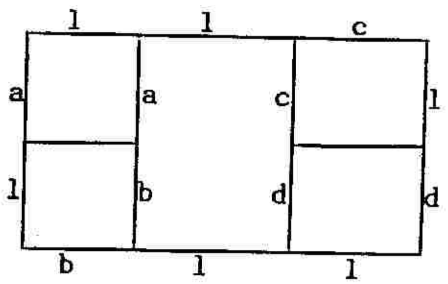
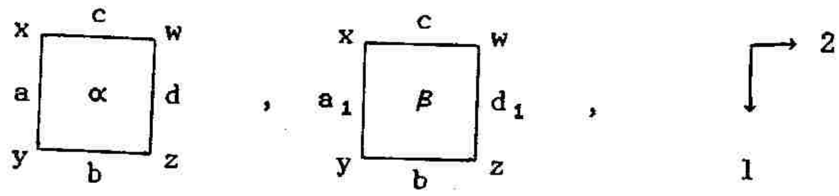


$= (\varepsilon_2 a *_1 \Gamma' b) *_2 (\Gamma c *_1 \varepsilon_2 d)$.

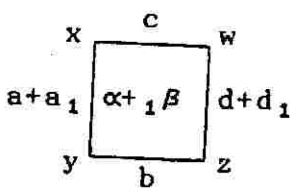
APPENDIX II

Verification of proposition (3.2.3) diagrammatically:

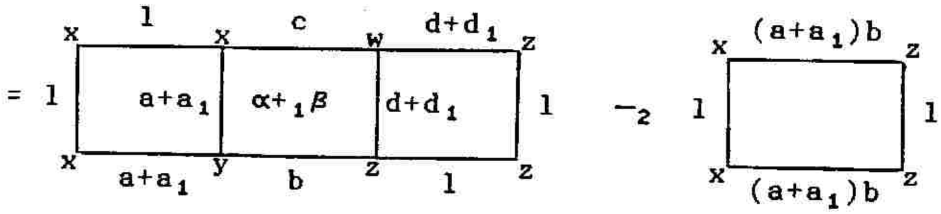
i) Let $\alpha, \beta \in D$ be given by

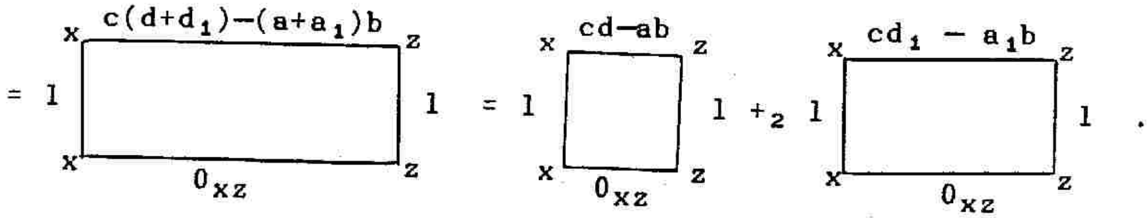


thus $\alpha +_1 \beta$ is in the form



$\Phi(\alpha +_1 \beta) = [\Gamma'(a+a_1) *_2 (\alpha +_1 \beta) *_2 \Gamma(d+d_1)] -_2 \varepsilon_1(a+a_1)b$,

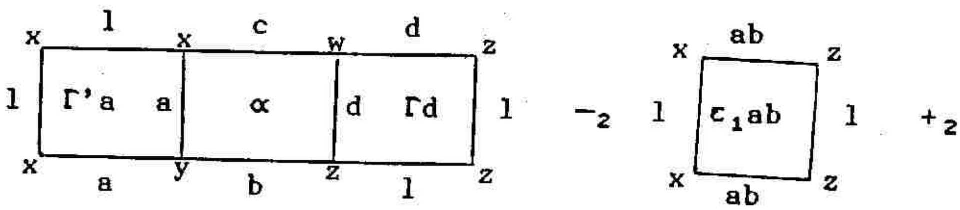


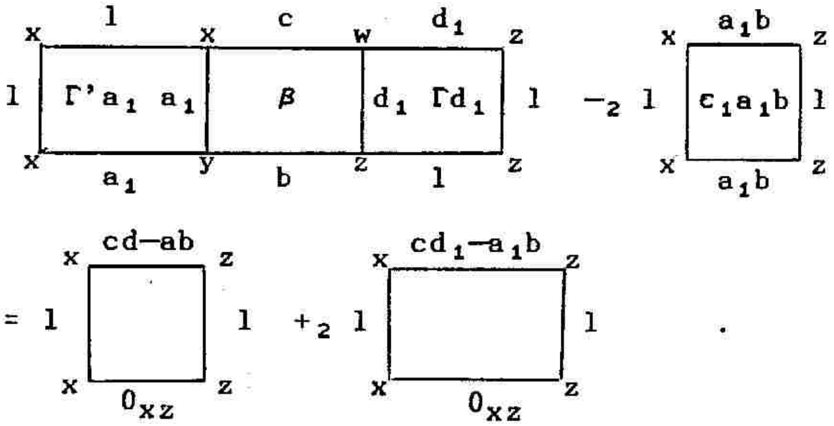


On the other hand ;

$\Phi\alpha +_2 \Phi\beta = [(\Gamma' a *_2 \alpha *_2 \Gamma d) -_2 \varepsilon_1 ab] +_2$

$[(\Gamma' a_1 *_2 \beta *_2 \Gamma d_1) -_2 \varepsilon_1 a_1 b]$, is in the form

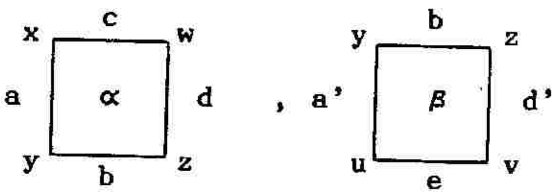




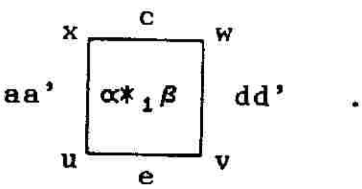
ii) Similarly for $\Phi(\alpha +_2 \beta) = \Phi\alpha +_2 \Phi\beta$.

iii) For $\Phi(\alpha *_1 \beta) = (\Phi\alpha *_2 \varepsilon_1\partial_2^1\beta) +_2 (\varepsilon_1\partial_1^0\alpha *_2 \Phi\beta)$.

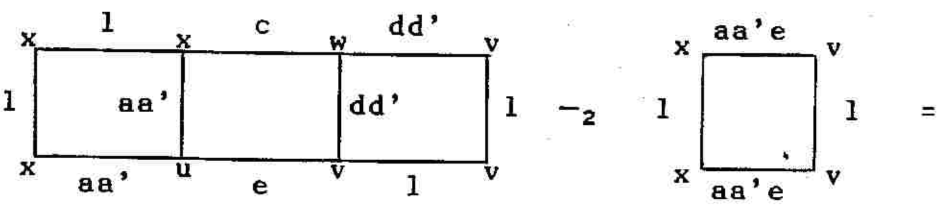
Let α , β be given by

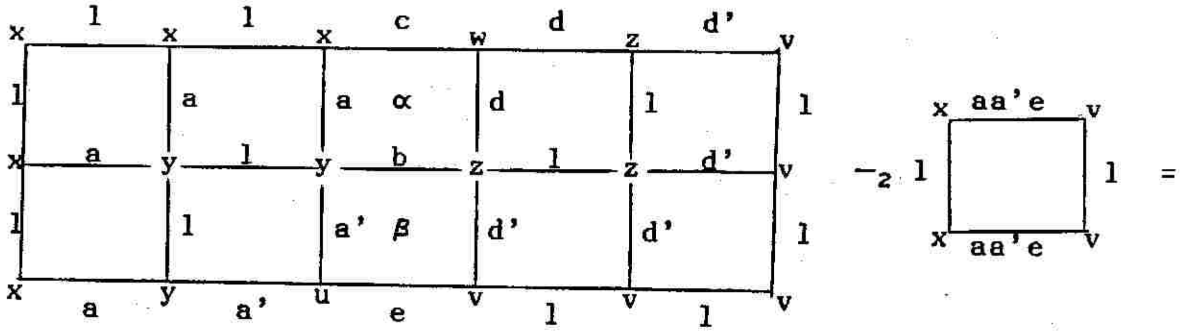


and so $\alpha *_1 \beta$ is in the form



Now $\Phi(\alpha *_1 \beta) = (\Gamma'aa' *_2 (\alpha *_1 \beta) *_2 \Gamma dd') -_2 \varepsilon_1 aa'e$, which is diagrammatically pictured as

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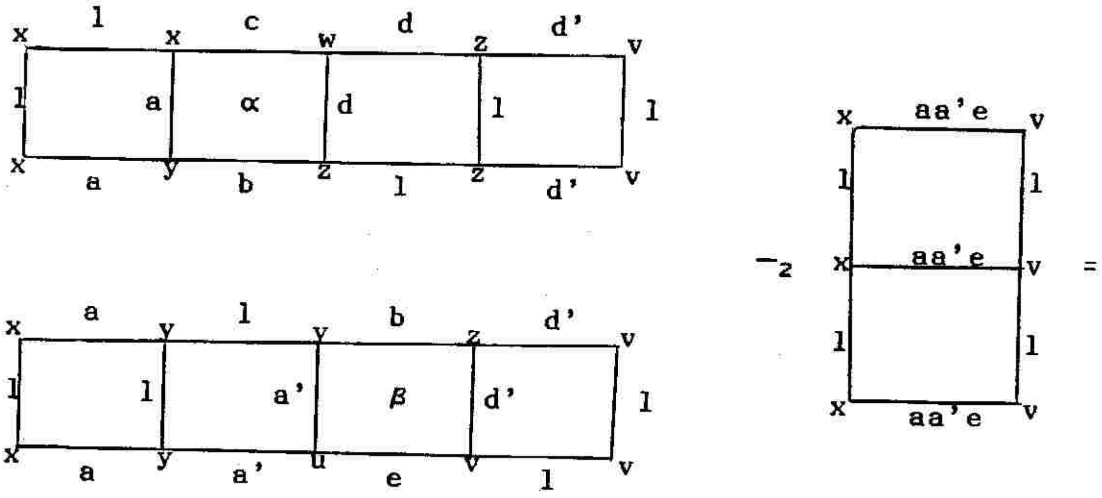
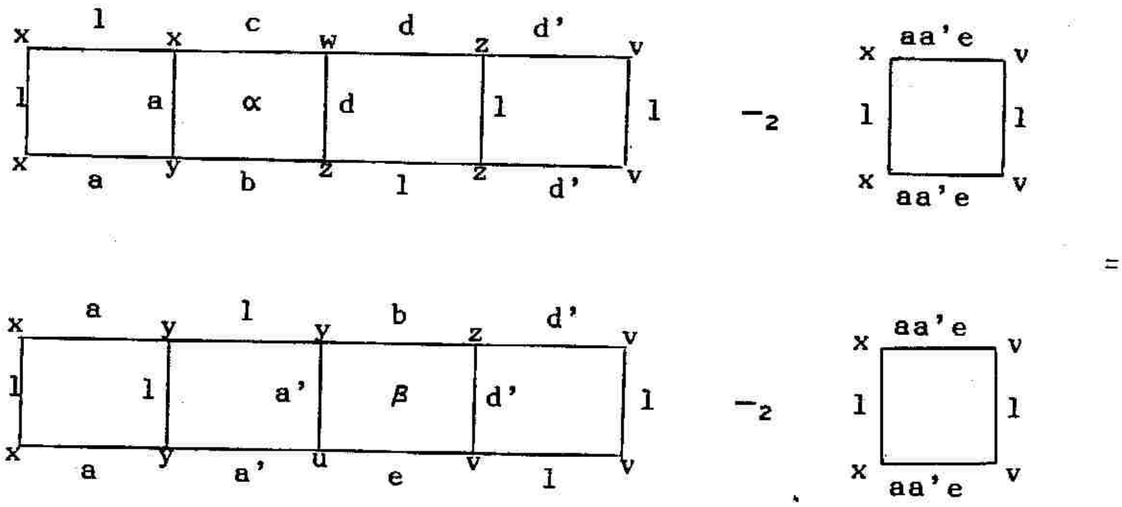
1 a a α d 1 1
x a y 1 y b z 1 z d' v
1 1 a' β d' d' 1
x a y a' u e v 1 v 1 v
−2
x aa'e v
1 1
x aa'e v
=


x 1 x c w d z d' v
1 a α d 1 1
x a y b z 1 z d' v
x a y 1 y b z d' v
1 1 a' β d' 1
x a y a' u e v 1 v
−2
x aa'e v
1 1
x aa'e v
1 1
x aa'e v
=


x 1 x c w d z d' v
1 a α d 1 1
x a y b z 1 z d' v
−2
x aa'e v
1 1
x aa'e v
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1 1 a' β d' 1
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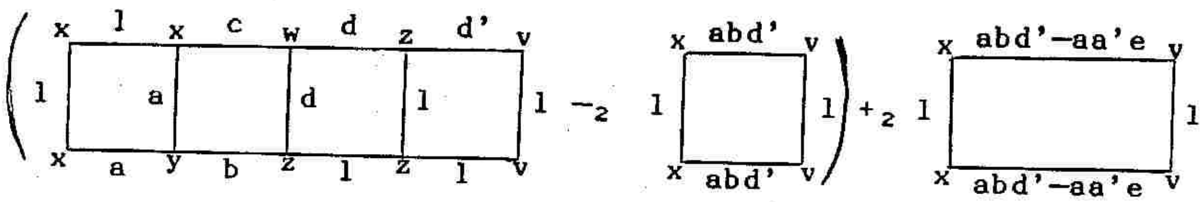
−2
x aa'e v
1 1
x aa'e v



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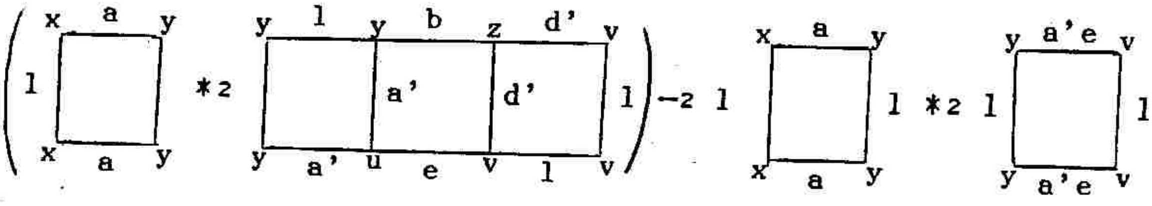

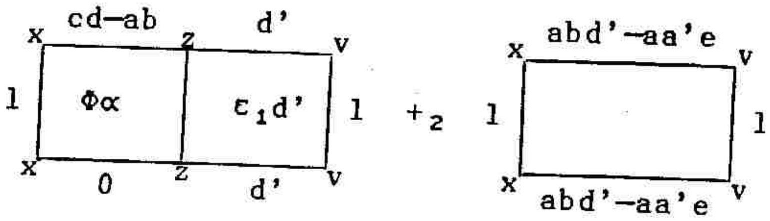

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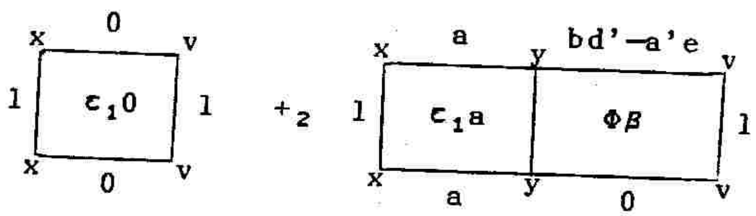
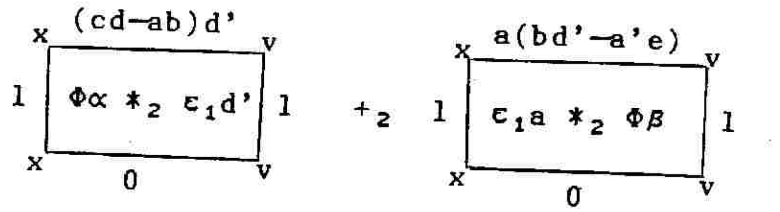
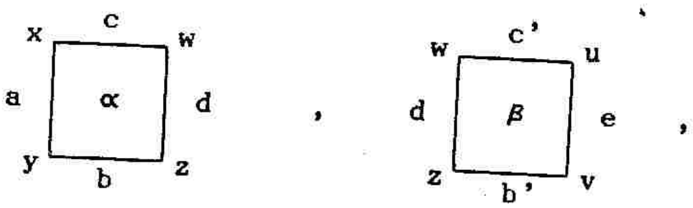


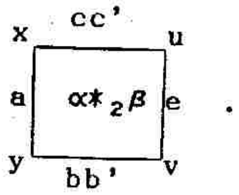
=  .

iv) For $\Phi(\alpha *_2 \beta) = (\varepsilon_1\partial_1^0\alpha *_2 \Phi\beta) +_2 (\Phi\alpha *_2 \varepsilon_1\partial_1^1\beta)$.

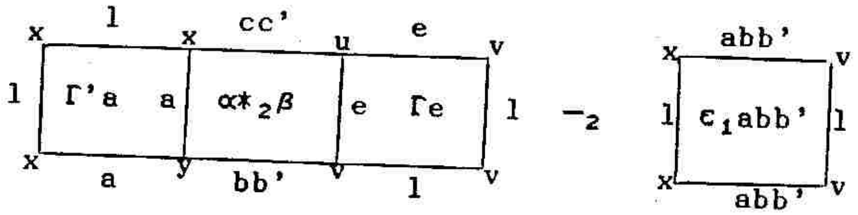
Let α, β be given by

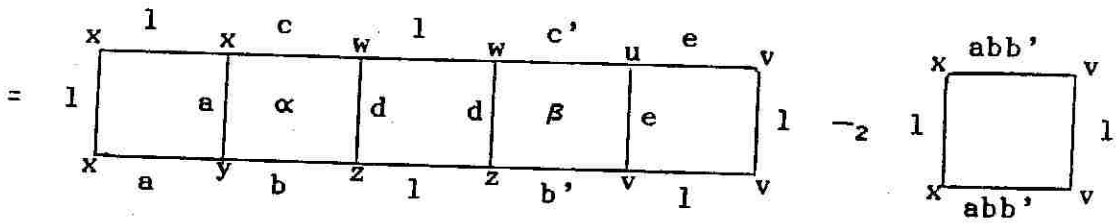


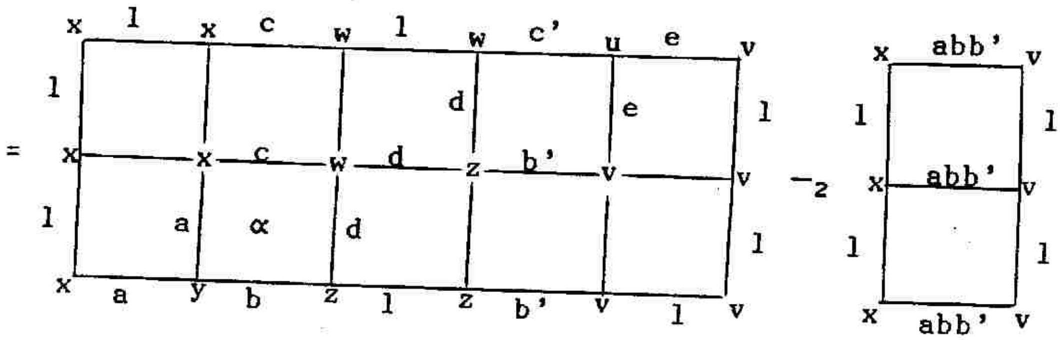
then $\alpha *_2 \beta$ is in the form

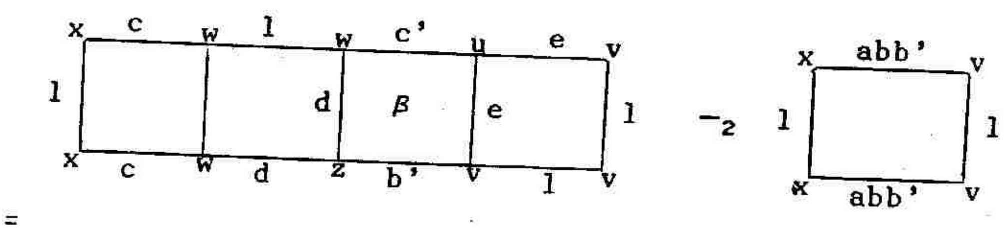


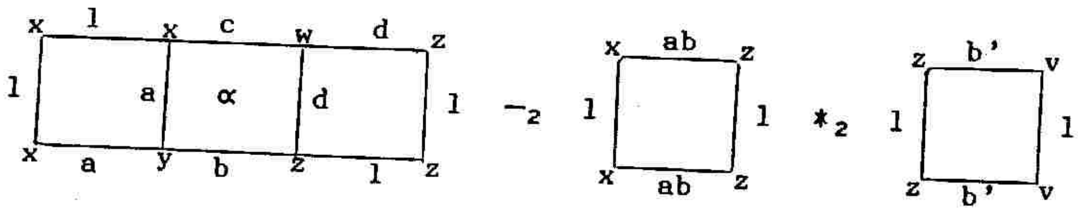
Then $\Phi(\alpha *_2 \beta) = (\Gamma' a *_2 (\alpha *_2 \beta) *_2 \Gamma d) -_2 \varepsilon_1 abb'$ which is in the form

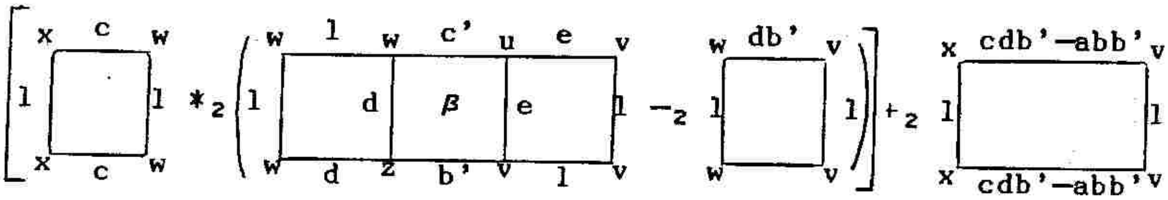












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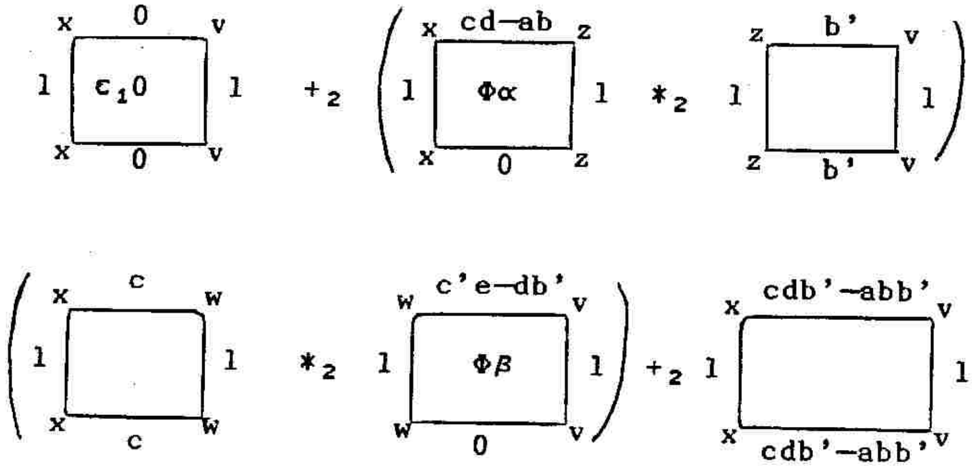

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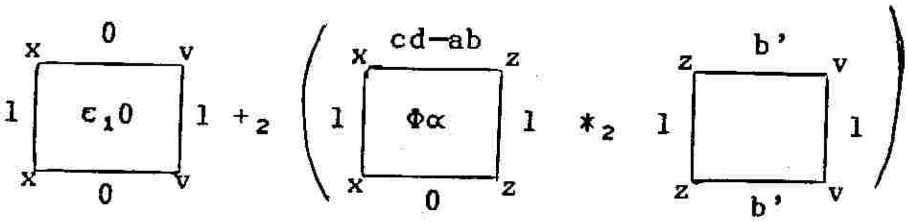

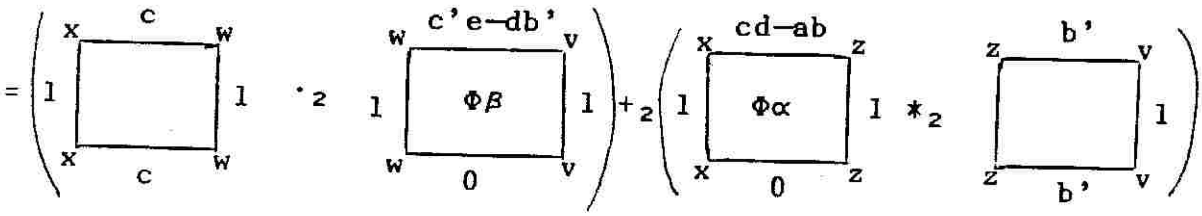

$= (\varepsilon_1 c *_2 \Phi\beta) +_2 (\Phi\alpha *_2 \varepsilon_1 b')$.

v) The rules $\Phi(r \cdot_1 \alpha) = r \cdot_2 \Phi\alpha$ and $\Phi(r \cdot_2 \alpha) = r \cdot_2 \Phi\alpha$ are clear .