Facts About the Octacube Sculpture

written by Adrian Ocneanu, October 2005

Location

The sculpture is on public display in the lobby of the Mathematics Department in the McAllister building, between the HUB and the Old Main in the central part of the Pennsylvania State University campus.

Mapping

The sculpture represents a three dimensional map of the surface of a four dimensional regular solid. Adrian Ocneanu developed and copyrighted the map, called windowed radial stereographic projection. The projection is the first good method for representing four dimensional solids as it shows the 2d walls of the 3d rooms, not only their 1d scaffolding. The edge and corner angles between walls and are all equal, preserving the 4d symmetry of the model.

The Earth is 3-dimensional and we can make a map of its surface on a 2 dimensional sheet of paper. In a similar way in one dimension higher, the sculpture represents a map in our 3-dimensional space of the surface of a 4-dimensional body. On a map of the Earth the countries are 2d polygons, in the sculpture they are 3d rooms, so the sculpture is similar to an apartment building. Windows are cut into the walls before the projection, making the inner rooms of the sculpture visible.

On a map of the 3d Earth the angles between roads are maintained, while the scale of the map varies in order to stretch the surface of the sphere on a sheet - a map with these properties is called conformal in mathematics. A good conformal map is the stereographic projection, in which a light bulb is placed at the north pole and the shadow of the globe surface is taken on a sheet underneath the south pole. The southern hemisphere remains small in the middle of the map while the northern hemisphere, close to the projection pole, becomes big and surrounds the center of the map. If the initial object is not spherical, it is first radially projected on the surface of a sphere surrounding it, with a light bulb placed in the center of the sphere.

The surface of the 4-dimensional solid represented by the sculpture is first projected radially on a sphere and then stereographically in our 3d world which plays the role of a flat map. The stereographic projection is conformal in any dimension, and explains why the angles between walls and at the corners are constant throughout the sculpture. The inner part of the sculpture, image of the southern hemisphere, is small while the outer part imaging the northern hemisphere near the projection pole is large. A vertex of the octacube is actually located on the projection pole, so the walls surrounding it are cut half way to the pole and their large shadows give the outer legs of the sculpture extending toward infinity.

As a consequence of the symmetry preserved in the sculpture, the reflection in the flat faces of the front of the sculpture matches the parts behind, thus giving the stainless steel the feeling of transparency.

Name

The sculpture is titled "Octacube" as its 24 vertices consist of the 16 vertices of the 4 dimensional cube, also known as hypercube or tesseract, together with the centers of its 8 rooms, which form the vertices of the 4d octahedron.

Construction
Adrian Ocneanu wrote the software which gave the cutting instructions for 1/8" stainless steel sheets to the machines in the Penn State Engineering Shop. The machinists then worked for almost a year to bend into spherical shapes and weld the 96 triangular pieces which meet in a giant puzzle, 12 at each of the 23 vertices of the sculpture - the 24th vertex is at infinity. The sculpture measures 6 x 6 x 6 ft and weighs 1200 lbs.

Financing
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Mathematical structures
The sculpture was chosen because it encodes more mathematical structures in different branches of mathematics and physics than any other 4 dimensional object. It starts as a map of the 3d surface of the 4th of the 6 four dimensional regular solids, called the octacube, which has 24 vertices, 96 edges, 96 triangular faces and 24 octahedral 3d rooms.

As the angles between any two walls are 120º, and the walls are spherical or flat, the rooms of the sculpture could be in principle realized as soap bubbles, and are described mathematically as minimal surfaces.

The vertices of the sculpture are the centers of the 24 spheres which can surround a sphere in 4 dimensions, in a very tight pattern possible only in dimensions 2,4, 8 and 24. Five of these spheres intersect the sculpture base on circles which are engraved in the granite surface of the base. Sphere packings are important in number theory and cryptography.

The nodes and mid 3d rooms of the sculpture encode the rotational symmetries of a usual cube. Each point V encodes a rotation of angle alpha around the axis OV with the length OV equal to the tangent of alpha/4. The vertices vertices of a cube are divided according to parity into 2 tetrahedra. Rotations of the cube which send each of these tetrahedra into itself encode the vertices of the octacube. Rotations of the cube which switch the two tetrahedra encode the mid rooms of the octacube.

The vertices of the sculpture encode also a crystallographic structure called a root system of type D4 which describes how to build a higher dimensional symmetry group from 24 two dimensional components located at the vertices of the sculpture, in a branch of mathematics called Lie groups and algebras. Vertices are arranged 6 on each of the 12 circles and 4 lines of the sculpture, with each segment of a circle or line representing an angle of 60º of a circle in 4d, and with vertices not on the same circle being perpendicular to each other. Using in addition to the 24 vertices of the sculpture also part or all of the 24 mid room points, one obtains the same way the crystallographic root systems of type B4, C4 and F4.

Finally the symmetry structure called inversion in the unit sphere between the inner half and the outer half of the sculpture can be used to explain the relation between spin 1/2 and spin 1 structures in mathematical physics.