Towards Non Commutative Algebraic Topology

Ronnie Brown

University of Wales, Bangor

This is slightly edited version of the transparencies for a seminar at University College London, May 7, 2003. (Not all were used!)

References: http://www.bangor.ac.uk/∼mas010

http://www.bangor.ac.uk/∼mas010/fields-art3.pdf

Acknowledgements to: work of Henry Whitehead; many collaborators, particularly
Philip Higgins, Jean-Louis Loday, Tim Porter, Chris Wensley;
21 Bangor research students;
Alexander Grothendieck, for correspondence *à baton rompu* 1982-93 and Pursuing Stacks (1983, 600 pages).

Current support: Leverhulme Emeritus Fellowship 'crossed complexes and homotopy groupoids' to produce a book with now agreed title:

Title of this seminar!

Start (1965): My attempt to get a form of the Van Kampen Theorem (VKT) for the fundamental group which would also calculate the fundamental group of the circle S^1. Discovered Higgins' work on groupoids!

Motivation: expository and aesthetic, thinking about anomalies.

Trying to understand the algebraic structures underlying homotopy theory.

Homotopy and deformation underly notions of classification in many branches of mathematics.

Aim: explore the situation
(so can not be directed at other peoples' problems).

NOT mainstream algebraic topology –

we are digging a new and additional channel.

Overall plan is (avoiding the green bit)

geometry → underlying processes → algebra → number theory

algebra ↓ algorithms → computer implementation ↓ sums

Example: Extensions $A \rightarrowtail E \twoheadrightarrow T$ of groups A, T are determined by classes of factor systems

$$k^1 : T \to \text{Aut}(A), \quad k^2 : T \times T \to A.$$

But if $T = gp\langle x, y \mid x^3 y^{-2} \rangle$ is the Trefoil group, then T is infinite, so what can you do?

Our result : Extensions are determined by elements $a \in A,\ a_x, a_y \in \text{Aut}(A)$ such that $(a_x)^3(a_y)^{-2}$ is the inner automorphism determined by a.

Example: $f : P \to Q$ a morphism of groups. Form the cofibration sequence

$$BP \xrightarrow{Bf} BQ \to C(Bf).$$

Find π_2 and first k-invariant of $C(Bf)$.

E.g. $(P = C_3 \leqslant Q = S_4) \implies \pi_2(C(Bf)) \cong C_6$.

Use non commutative methods (and computers).

Major themes in 20th century mathematics:

- non commutativity
- higher dimensions
- K-theory
- local to global
- homology

(Atiyah, Bull LMS, 2002)

The VKT for π_1 is a classical example of a

non commutative local-to-global theorem.

So any tools which are developed for generalisations and for higher dimensional forms of it could be generally useful.

Non commutative algebraic topology conveniently combines all the above major themes, and has yielded substantial new calculations, new understanding, new prospects,
of which the last is possibly the most important.

Applications to concurrency (GETCO).

Recent EPSRC Grant on Higher dimensional algebra and differential geometry. Peter May's interest in higher categorical structures. Work with Tony Bak, Tim Porter on Tony's 'global actions'.

Why think of non commutative algebraic topology?

Back in history!

Topologists of the early 20th century knew well that:

1) Non commutative fundamental group $\pi_1(X, a)$ had applications in geometry, topology, analysis.

2) Commutative $H_n(X)$ were defined for all $n \geqslant 0$.

3) For connected X,

$$H_1(X) \cong \pi_1(X, a)^{ab}.$$

So they dreamed of

higher dimensional, non commutative versions of the fundamental group.

Gut feeling: dimensions > 1 need invariants which are

‘more non commutative’ than groups.

1932: ICM at Zürich:

Čech: submits a paper on higher homotopy groups

$$\pi_n(X, x);$$

Alexandroff, Hopf: prove commutativity for $n \geqslant 2$;
persuade Čech to withdraw his paper;
only a small paragraph appears in the Proceedings.

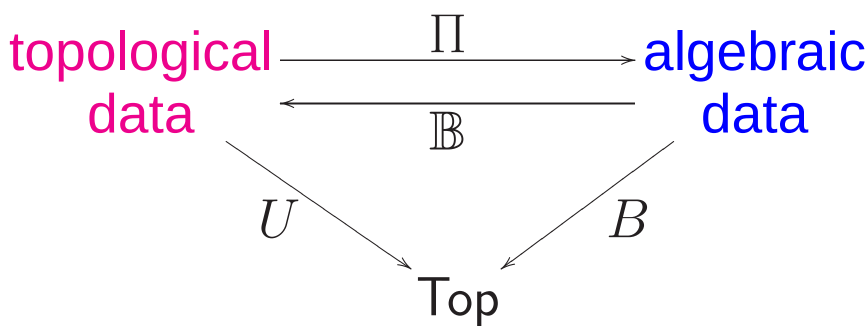
Reason for commutativity (in modern terms):

group objects in groups are commutative groups.

$\pi_2(X, x)$, even considered as a $\pi_1(X, x)$-module, is only a pale shadow of the 2-type of X.

What is going on?

Overall philosophy: look for
algebraic models of homotopy types



(*)

1) U is a forgetful functor and $B = U \circ \mathbb{B}$;

2) Π is defined homotopically;

3) (local to global, allowing calculation!):
Π preserves certain colimits;

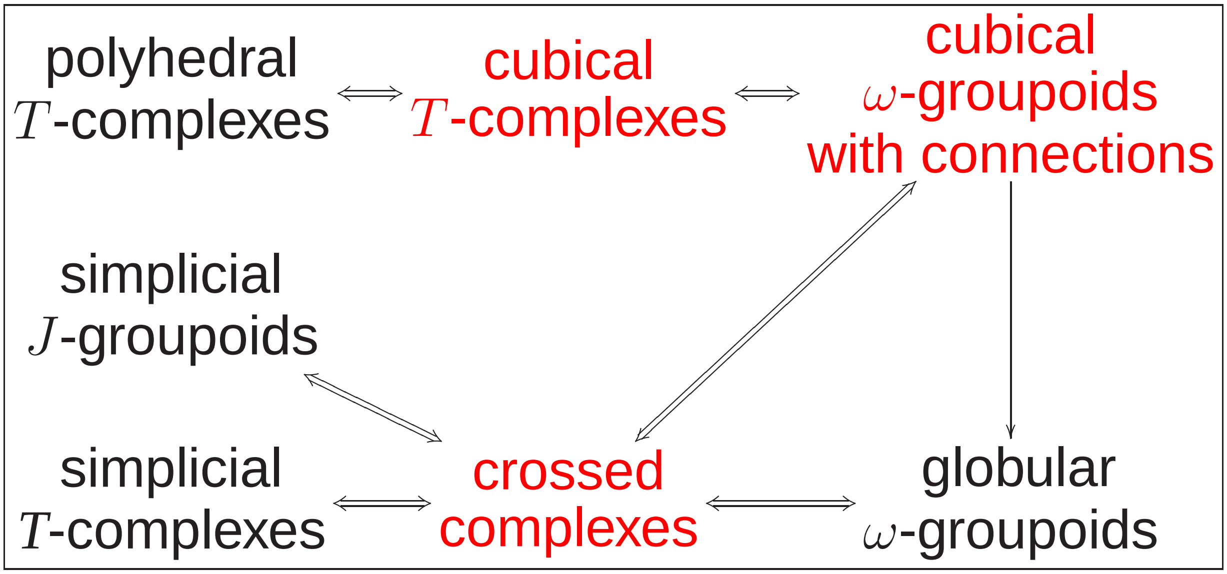
3) (algebra models the geometry)
$\Pi \circ \mathbb{B} \simeq 1$;

4) (capture homotopical information):
$\exists$ natural transformation $1 \to \mathbb{B} \circ \Pi$ with good properties.

Some examples:

topological data	algebraic data
spaces with base point	groups
spaces with a set of base points	groupoids
based pairs	crossed modules
filtered spaces	crossed complexes
n-cubes of spaces	cat^n-groups

So on the blue side we have various generalisations of groups. Here are some more!



The equivalence of the red structures is required for the proof of the Brown-Higgins GVKT.

Features of groupoids:
structure in dimension 0 and 1;
composition operation is partially defined;
allows the combination of groups and graphs, or groups and space.

Higher dimensional algebra (for me) is the study and application of algebraic structures whose domains of the operations are given by geometric conditions.
This allows for a vast range of new algebraic structures related to geometry.

Why so many structures?

More compact convex sets in dimension 2 than dimension 1!

The algebra has to express and cope with structures defined by different geometries.

Easiest example: Cubes.

Cubical methods are used in order to express the intuitions of

1) Multiple compositions
(algebraic inverses to subdivisions);

2) Defining a commutative cube.

3) Proving a multiple composition of commutative cubes is commutative (Stokes' Theorem?!).

4) Construction and properties of higher homotopy groupoids.

5) Homotopies and tensor products:

$$I^m \times I^n \cong I^{m+n}$$

Category FTop of filtered spaces:

$$X_* : X_0 \subseteq X_1 \subseteq X_2 \subseteq \cdots \subseteq X_\infty$$

of subspaces of X_∞.

Homotopical quotient:

$$p : R(X_*) \longrightarrow \varrho(X_*) = R(X_*)/\equiv$$

where

$$R(X_*)_n = \mathsf{FTop}(I_*^n, X_*),$$

$I_*^n = n$-cube with its skeletal filtration,
$\equiv$: homotopy through filtered maps rel vertices of I^n.
Then $R(X_*)$, $\varrho(X_*)$ are cubical sets with connections.
(Connections are extra ‘degeneracy’ operations.)

But $R(X_*)$ has standard partial compositions:
for $i = 1, \ldots, n$, if $a, b \in R(X_*)_n$ and $\partial_i^+ a = \partial_i^- b$ we can define $a \circ_i b \in R(X_*)_n$ by

$$(a \circ_i b)(t_1, \ldots, t_n) = \begin{cases} a(\ldots, 2t_i, \ldots) & t_i \leqslant \frac{1}{2}, \\ b(\ldots, 2t_i - 1, \ldots) & t_i \geqslant \frac{1}{2}. \end{cases}$$

Major result 1): The standard compositions on $R(X_*)$ are inherited by $\varrho(X_*)$ to make it the fundamental cubical ω-groupoid of X_*.

This is quite difficult to prove, and is non trivial even in dimension 2. The result is precise in that there is just enough filtration room to prove it.

Major result 2): The quotient map $p : R(X_*) \to \varrho(X_*)$ is a Kan fibration of cubical sets.

This result is almost unbelievable. Its proof has to give a systematic method of deforming a cube with the right faces 'up to homotopy' into a cube with exactly the right faces, using the given homotopies.

Here is an application of 2) which is essential in many proofs.

Theorem: Lifting multiple compositions

Let $[\alpha_{(r)}]$ be a multiple composition in $\varrho_n(X_*)$. Then representatives $a_{(r)}$ of the $\alpha_{(r)}$ may be chosen so that the composition $a_{(r)}$ is well defined in $R_n(X_*)$.

Explanation: To say that $[\alpha_{(r)}]$ is well defined says representatives $a_{(r)}$ agree with neighbours up to homotopy, and these homotopies are arbitrary. All these homotopies have to be used to obtain the representatives which actually agree with their neighbours.

This is an example of why setting up higher homotopy groupoids is not straightforward.

Proof: The multiple composition $[\alpha_{(r)}]$ determines a cubical map

$$A : K \longrightarrow \varrho(X_*)$$

where the cubical set K corresponds to a subdivision of the geometric cube.

Consider the diagram

$$\begin{array}{ccc} * & \longrightarrow & R(X_*)\,. \\ \big\downarrow & \overset{A'}{\nearrow} & \big\downarrow p \\ K & \underset{A}{\longrightarrow} & \varrho(X_*) \end{array}$$

Then K collapses to $*$, written $K \searrow *$.
By the fibration result,
A lifts to A', which represents $a_{(r)}$, as required.

Major result 3): If $X_\infty = U \cup_W V$ with U, V, W open, and the induced filtrations U_*, V_*, W_* are connected then

C) X_* is connected;

I) The following diagram

$$\begin{array}{ccc} \varrho(W_*) & \longrightarrow & \varrho(V_*) \\ \downarrow & & \downarrow \\ \varrho(U_*) & \longrightarrow & \varrho(X_*) \end{array}$$

is a pushout of cubical ω-groupoids with connection.

Proof Outline: Verify the universal property with regard to maps to G. Take $a \in\in \varrho(X_*)_n$. Subdivide $a = [a_{(r)}]$ so that each $a_{(r)}$ lies in U or V. Use connectivity to deform $a_{(r)}$ to $a'_{(r)} \in R(Y_*), Y = U, V, W$ such that $a' = [a'_{(r)}]$ is defined. Map the pieces over to G and recombine. Analogy with email.

You have to prove independence of choices. This needs a technology of commutative cubes.

Applications: Translate to crossed complexes.

Down to earth and explain crossed modules

JHC Whitehead in 1939-50 abstracted properties of

$$\partial : \pi_2(X, X_1, a) \to \pi_1(X_1, a) \qquad \text{(*)}$$

to define a Crossed Module:
morphism of groups
$\mu : M \to P$ and action $M \times P \to M,\ (m, p) \mapsto m^p$
of the group P on the group M such that:
CM1) $\mu(m^p) = p^{-1}(\mu m)p$ **CM2)** $n^{-1}mn = m^{\mu n}$
for all $m, n \in M, p \in P$.

Now a key concept in non commutative algebraic topology and homological algebra.

Simple consequences of the axioms:
- $\operatorname{Im}\mu$ is normal in P
- $\operatorname{Ker}\mu$ is central in M and is acted on trivially by $\operatorname{Im}\mu$, so that $\operatorname{Ker}\mu$ inherits an action of $M/\operatorname{Im}\mu$.

Standard algebraic examples:

(i) normal inclusion $M \lhd P$;

(ii) inner automorphism map $\chi : M \to \text{Aut } M$;

(iii) the zero map $0 : M \to P$ where M is a P-module;

(iv) an epimorphism $M \to P$ with kernel contained in the centre of M.

Theorem (Mac Lane-Whitehead, 1950 *Crossed modules classify all connected weak based homotopy 2-types.*

Crossed modules as candidates for 2-dimensional groups?

1974 (published 1978): Brown and Higgins proved that the functor

$$\Pi_2 : \text{(based pairs of spaces)} \to \text{(crossed modules)}$$

preserves certain colimits. This allows totally new 2-dimensional homotopical calculations. One can compute with crossed modules in a similar, but more complicated, manner to that for groups.

Recent work with Chris Wensley uses symbolic computation to do more sums.

The aim of these new calculations is to prove (i.e. test) the power of the machinery.

Grothendieck's aim in Pursuing Stacks was Non Abelian Homological Algebra.

The real aim is an extension of method, in the belief that methods last longer than theorems.

Next show examples of a new concept and calculations.

Induced crossed modules (Brown-Higgins, 1978).

$f : P \to Q$ a group morphism.

$$\text{crossed } P\text{-module} \left\{ \begin{array}{ccc} M & \longrightarrow & f_*(M) \\ {\scriptstyle\mu}\downarrow & & \downarrow{\scriptstyle\partial} \\ P & \xrightarrow{f} & Q \end{array} \right\} \text{crossed } Q\text{-module}$$

$$f_* : \text{crossed } P\text{-modules} \to \text{crossed } Q\text{-modules}$$

Universal property: left adjoint to pullback by f.

Construction: generated by symbols

$$m^q, \; m \in M, q \in Q$$

with $\partial(m^q) = q^{-1}(fm)q$ and rules

$$(m^p)^q = m^{(fp)q}, \text{ CM2 for } \partial.$$

Example of a ‘change of base’ construction.

Example 1) Let $f : P \to Q$ be a morphism of groups, inducing a cofibration sequence

$$BP \to BQ \to C(Bf).$$

Algebraic description of the 2-type of $C(Bf)$ as an induced crossed module $f_*(P \to P)$, so we can calculate specific examples.

1) (Brown, Wensley, 1995) M, P, Q finite $\implies f_*(M)$ finite. Hence computations of homotopy 2-type of $B(C(Bf))$ when $\mu = 1_P : P \to P$ and $f : P \triangleleft Q$; more generally of a homotopy pushout

$$\begin{array}{ccc} BP & \longrightarrow & BQ \\ \downarrow & & \downarrow \\ B(M \to P) & \longrightarrow & X \end{array}$$

2) $\mu = 1 : F(R) \to F(R)$, $\omega' : F(R) \to Q$ defined by $\omega : R \to Q$. Then

$$\partial : C(\omega) = \omega'_*(F(R)) \to Q$$

is the free crossed Q-module on ω. (Defined directly by Whitehead).

Corollary is a major result:

Theorem W (1949)

$$\pi_2(X_1 \cup \{e^2_r\}_{r \in R}, a) \to \pi_1(X_1, a)$$

is isomorphic to the free crossed $\pi_1(X_1, a)$*-module on the classes of the attaching maps of the 2-cells.*

This is important for relating combinatorial group theory and 2-dimensional topology. (Identities amon relations.)

Some Computer Calculations (C.D. Wensley using GAP): $[m, n]$ is the nth group of order m in GAP.

$M \lhd P$; $f : P \leqslant S_4$. Calculate $f_* M$.
Set $C_2 = \langle (1,2) \rangle$, $C_2' = \langle (1,2)(3,4) \rangle$, $C_2^2 = \langle (1,2), (3,4) \rangle$.

M	P	$f_* M$	$\ker \partial$	Aut$(f_* M)$
C_2	C_2	$GL(2,3)$	C_2	$S_4 C_2$
C_3	C_3	$C_3\ SL(2,3)$	C_6	$[144, 183]$
C_3	S_3	$SL(2,3)$	C_2	S_4
S_3	S_3	$GL(2,3)$	C_2	$S_4 C_2$
C_2'	C_2'	$[128, ?]$	$C_4 C_2^3$	
C_2'	C_2^2, C_4	H_8^+	C_4	$S_4 C_2$
C_2'	D_8	C_2^3	C_2	$SL(3,2)$
C_2^2	C_2^2	$S_4 C_2$	C_2	$S_4 C_2$
C_2^2	D_8	S_4	I	S_4
C_4	C_4	$[96, 219]$	C_4	$[96, 227]$
C_4	D_8	S_4	I	S_4
D_8	D_8	$S_4 C_2$	C_2	$S_4 C_2$

$\ker \partial \cong \pi_2(C(Bf))$.

Need the non commutative structure to find this.

Hard to determine the first k-invariant in $H^3(\text{Coker}\partial, \ker \partial)$.

Geometric significance of the table?

Conclusion:

Key inputs: VKT for the fundamental groupoid $\pi_1(X, X_0)$ on a set X_0 of base points (RB: 1967).

CLAIM: all of 1-dimensional homotopy theory is better presented using groupoids rather than groups.

Substantiated in books by Brown (1968) and Higgins (1971). Ignored by most topologists!

Hint as to higher dimensional prospects:

(Group objects in groupoids) $\Leftrightarrow$ (crossed modules).
(Grothendieck school, 1960s).
Generalising:

(congruences on a group) $\Leftrightarrow$ (normal subgroups).

Further outlook: Generalise this to other algebraic structures than groups.

See work of Fröhlich, Lue, Tim Porter.

Groupoids in Galois Theories (Grothendieck, Magid, Janelidze).

So look for higher homotopy groupoids.

And applications of groupoids, multiple groupoids, and higher categorical structures in mathematics and science.

Hence the term higher dimensional algebra (RB, 1987). Web search shows many applications.

Pursuing Stacks has been a strong international influence.

I gave an invited talk in Delhi in February to an International Conference on Theoretical Neurobiology!

It is still early days!