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"Chevalley Supergroups"

preprint arXiv:0808.0785v1 [math.RA] *(2008)*

INTRODUCTION

In his work of 1955, on the later called *Chevalley groups*, Chevalley provided an easy to build, combinatorial construction of simple algebraic groups over any field, for every possible simple algebraic group. In particular, this gave a new proof of the existence theorem for simple algebraic groups, but also found new examples of simple abstract groups, which had escaped, up to then, to the inspection of the group theorists.

Later on, Chevalley's original construction was suitably modified, so to provide a construction of split simple algebraic group schemes over $\mathbb{Z}$, defined in functorial terms.

In this paper we want to adapt this philosophy to the setup of supergeometry, so to give a construction of simple algebraic supergroups over an arbitrary field. In order to explain our work, we first shortly revisit the whole classical construction.

Let $\mathfrak{g}$ be a finite dimensional semisimple Lie algebra over an algebraically closed field $\mathbb{K}$ (e.g. $\mathbb{K} = \mathbb{C}$). Fix in $\mathfrak{g}$ a Cartan subalgebra; then a root system is defined, and $\mathfrak{g}$ splits into weight spaces indexed by the roots and by zero. Also, $\mathfrak{g}$ has a special basis, called *Chevalley* basis, for which the structure constants are integers, satisfying special conditions in terms of the root systems. This defines an integral form of $\mathfrak{g}$, called *Chevalley Lie algebra*.

In the universal enveloping algebra of $\mathfrak{g}$, there exists a $\mathbb{Z}$–integral form, called *Kostant algebra*, with a special "PBW-like" basis of ordered monomials, whose factors are divided powers of weight vectors and binomial coefficients of Cartan generators, corresponding to elements of the Chevalley basis in the Chevalley Lie algebra.

If V is a faithful semisimple $\mathfrak{g}$–module, there exists a $\mathbb{Z}$–lattice $M \subseteq V$, which is stable by the action of the Kostant algebra. Hence the Kostant algebra acts on the vector space $V_{\Bbbk} := \Bbbk \otimes_{\mathbb{Z}} M$ for any field $\Bbbk$. Moreover there exists an integral form $\mathfrak{g}_V$ of $\mathfrak{g}$ leaving the lattice invariant and depending only on the representation V and not on the choice of the lattice.

For any root vector X of $\mathfrak{g}$, we take the exponential $\exp(tX) \in \mathrm{GL}(V_{\Bbbk})$, $t \in \Bbbk$ (as X acts as nilpotent, the expression makes sense). The subgroup of $\mathrm{GL}(V_{\Bbbk})$ generated by all the $\exp(tX)$, for all roots and all t, is the *Chevalley group* $G_V(\Bbbk)$, as introduced by Chevalley.

We want to extend Chevalley's construction to the supergeometric setting.

In supergeometry the best way to introduce supergroups is via their functor of points. Unlikely the classical setting, the points over a field of a supergroup tell us very little of the supergroup itself. In fact such points miss the odd coordinates and describe only the classical part of the supergroup. In other words, over a field we cannot see anything beyond classical geometry.

The functor of points approach realizes an affine supergroup as a representable group-valued functor from the category of superalgebras (salg) to the category of sets (sets), where superalgebras are simply $\mathbb{Z}_2$–graded algebras. So the supergeometric properties are encoded by the category we start from, while our functors will be valued in the ordinary categories of sets, groups, Lie algebras and so on, as usually happens in supergeometry.

As in classical geometry, this approach is very geometric and very elegant, however it hides a great difficulty: we must show that any such functor is *representable* before we rightfully can call it a supergroup.

Our initial data are the Lie superalgebras, which are direct sums of finitely many classical Lie superalgebras: in our construction they play the role of semisimple Lie algebras in Chevalley's setting.

We start by proving some basic results on such Lie superalgebras, like the existence of *Chevalley bases*, and a PBW-like theorem for the Kostant $\mathbb{Z}$–form of the universal enveloping superalgebra.

Next we take a faithful representation V of a Lie superalgebra $\mathfrak{g}$ with a Chevalley basis and we show that there exists a lattice M in V fixed by the Kostant superalgebra and also by a certain integral form $\mathfrak{g}_V$ of $\mathfrak{g}$, which again depends on V only. We then define a group-valued functor G_V, from the category of commutative superalgebras to the category of sets, as follows. For any commutative superalgebra A, $G_V(A)$ is the subgroup of $\mathrm{GL}(V(A))$ — the general linear supergroup on V — generated by the one-parameter unipotent subgroups acting on M. In the supergeometric setting, one must carefully define the one-parameter subgroups, which come with three possible superdimensions: $1|0$, $0|1$ and $1|1$. This also will enter in our discussion.

We prove a wealth of properties of the groups $G_V(A)$. In particular, we find a factorization $G_V(A) = G_0(A)\, G_1(A) \cong G_0(A) \times G_1(A)$ which is a group-theoretical counterpart of the $\mathbb{Z}_2$–splitting $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$. Indeed, $G_0(A) := G(A_0)$ is (roughly) given by the classical Chevalley group attached to $\mathfrak{g}_0$ and V, while $G_1(A)$ has to be thought as the exponential of $\mathfrak{g}_1 \otimes A_1$. Moreover, we show that $G_1 : A \mapsto G_1(A)$ is a representable functor.

Despite the analogy with Chevalley construction, G_V is not a representable functor, hence it is not an algebraic supergroup. This is a phenomenon already observed at the classical level: one-parameter subgroups, defined via their functor of points, do not generate Chevalley groups over an arbitrary commutative ring. Hence we need to consider the sheafification $\mathbf{G}_V$ of the functor G_V, which coincides with G_V on local algebras (we provide at the end an appendix with a brief treatment of sheafification of functors and a summary of the properties we need). In particular, $\mathbf{G}_V$ inherits the factorization $\mathbf{G}_V = \mathbf{G}_0\, \mathbf{G}_1 \cong \mathbf{G}_0 \times \mathbf{G}_1$, with $\mathbf{G}_1 = G_1$ and $\mathbf{G}_0$ being the classical Chevalley group-scheme associated to $\mathfrak{g}_0$ and V. As we prove that $\mathbf{G}_1$ is an affine superspace, it is also representable; as $\mathbf{G}_0$ is representable too, the above factorization implies that $\mathbf{G}_V$ is representable as well, and so we rightfully take it to be, by definition, our "Chevalley supergroup".

We finish the paper with some results about Chevalley supergroups. In particular, the functoriality in V of our construction, and the fact that, over a field $\Bbbk$, the Lie superalgebra $\mathrm{Lie}(\mathbf{G}_V)$ of a Chevalley supergroup associated to a Lie superalgebra $\mathfrak{g}$ and a simple faithful $\mathfrak{g}$–module V is just $\Bbbk \otimes \mathfrak{g}_V$.

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