ALGEBRAIC FOUNDATIONS TO SYMMETRY BREAKING IN QUANTUM FIELD THEORY

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Abstract. A novel Algebraic Topology approach to Supersymmetry (SUSY) and Symmetry Breaking in Quantum Field and Quantum Gravity theories is presented with a view to developing a wide range of physical applications, such as: nuclear fusion and other nuclear reactions in quantum chromodynamics, nonlinear physics at high energy densities, dynamic Jahn-Teller effects, superfluidity, high temperature superconductors, multiple scattering by molecular systems, molecular or atomic paracrystal structures, nanomaterials, ferromagnetism in glassy materials, spin glasses, quantum phase transitions and supergravity. This approach requires a unified conceptual framework that utilizes extended symmetries and quantum groupoid, algebroid and functorial representations of non–Abelian higher dimensional structures pertinent to quantized spacetime topology and state space geometry of quantum operator algebras.

1. Introduction

The theory of scattering by partially ordered, atomic or molecular, structures in terms of *paracrystals* and *lattice convolutions* was formulated in Hosemann and Bagchi (1962) using basic techniques of Fourier analysis and convolution products. A natural generalization of the resulting extended symmetries and their corresponding analytical version concerns a convolution algebra - a based theory that we will discuss in the context of a more general and original concept of a *convolution–algebroid of an extended symmetry groupoid of a paracrystal*, or indeed, of any molecular system with partially disordered/ordered structure. Further specific applications of the paracrystal theory to X-ray scattering, based on computer algorithms, programs and explicit numerical computations, were subsequently developed by the first author (Baianu, 1974) for one–dimensional paracrystals, partially ordered membrane lattices (Baianu, 1978) and other biological structures with partial structural disorder (Baianu, 1980). Such biological structures, 'quasi-crystals', and paracrystals, in general, provide rather interesting physical examples of such extended symmetries (cf Hindeleh and Hosemann, 1988). Further statistical analysis shows that a real paracrystal can be defined by a three dimensional convolution polynomial with an empirically derived * law (Hosemann et al. 1981).

Given these trends combining crystalline symmetries with (noncommutative) harmonic analysis (Mackey, 1992), we propose that the evolving mathematical concepts can be treated algebraically in terms of certain structured *groupoids* and their *C^*–convolution quantum*

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algebroids. As was shown in Baianu (1978), supported with specific examples, these systems of convolution can be expressed analytically, thus allowing the numerical computation of X-ray, or neutron, scattering by partially disordered membrane lattices via complex Fourier transforms of membrane structural models.

A salient, and well–fathomed concept from the mathematical perspective concerns that of a C^*-algebra of a (discrete) group (see e.g. Connes, 1994). The underlying vector space is that of complex valued functions with finite support, and the multiplication of the algebra is the fundamental *convolution product* which it is convenient for our purposes to write slightly differently from the common formula as

$$(f * g)(z) = \sum_{xy=z} f(x)g(y). \tag{1.1}$$

and *-operation

$$f^*(x) = \overline{f(x^{-1})} \ . \tag{1.2}$$

(The more usual expression of these formulas has a sum over the elements of the group.) For topological groups, where the underlying vector space consists of continuous complex valued functions, this product requires the availability of some structure of measure and of measurable functions, with the sum replaced by an integral. (Notice that this algebra has an identity, the function δ_1, which has value 1 on the identity 1 of the group, and has zero value elsewhere.)

On the other hand, post 1955, quantum theories adopted a new lease of life when von Neumann beautifully formulated QM in the mathematically rigorous context of Hilbert spaces. The basic definition of a von Neumann Algebra is outlined in the appendix. After recalling the concept of a quantum group in relationship to a (quantum) Hopf Algebra (see e.g. Majid, 1995), we shall proceed to relate these mainly algebraic concepts to symmetry and also consider their extensions in the context of local quantum physics and symmetry breaking. In this respect we can make 'inhomogeneity' comparisons: one the one hand, the example of paracrystals reveals thermodynamic disorder (entropy) within its own spacetime framework, whereas in spacetime itself, whatever the elected model, the inhomogeneity arises through (super) gravitational effects. More specifically, in the former case we have the technique of the Fourier transform (along with convolution and Haar measure), and in view of the latter, we may compare the resulting broken paracrystal symmetry with the supersymmetry prediction of weak gravitational fields (cf 'ghost' particles) along with the broken super symmetry of strong gravitational fields.

In recent years the techniques of Hopf symmetry and those of weak Hopf C*-algebras, or *quantum groupoids* as they alternatively are known (cf Böhm et al.,1999), provide important mechanisms for studying the broader relationships of the Wigner fusion rules algebra, $6j$–symmetry (Rehren, 1997) and the study of the noncommutative symmetries of subfactors within the Jones tower constructed from finite index depth 2 inclusion of factors, also from the viewpoint of related Galois correspondences (Nikshych and Vainerman, 2000). Quantum groupoids also figure prominently in the theory of dynamical deformations of quantum groups and the quantum Yang–Baxter equations (Etingof et al., 1999, 2001).

Motivated by these examples, we introduce through steps of generality, a framework for quantum symmetry breaking in terms of a *weak Hopf C*–algebroid with convolution* set in the context of *rigged Hilbert spaces* (Bohm and Gadella, 1989). Further with regard to a unified and global framework for symmetry breaking, we look towards *double groupoid* structures of (Brown and Spencer, 1976) and introduce the concepts of *quantum and graded Lie bialgebroids* which are expected to carry a similar C*–algebra convolution structure. The extension to *supersymmetry* leads then to superalgebra, superfield symmetries and their involvement in supergravity or Quantum Gravity theories for intense gravitational fields in fluctuating, quantized space-times. Our approach, although semi–expository, leads to a novel concept which exemplifies a certain *non–reductionist* viewpoint of the nature of physical space–time structure (Brown et al. 2007).

2. The weak Hopf C*–algebra and background to symmetry breaking

2.1. **Hopf algebras.** In this section we proceed through several stages of generality by relaxing the axioms for a Hopf algebra. The motivation starts by recalling the notion of a quantum group in relation to a Hopf algebra where the former is often realized as an automorphism group for a quantum space, that is, an object in a suitable category of generally noncommutative algebras. The most common guise of a quantum group is the dual of a noncommutative, nonassociative Hopf algebra. So we commence here establishing the concept of Hopf algebras as the fundamental building blocks following e.g. Chaician and Demichev (1996), Majid (1996). Firstly, a unital associative algebra consists of a linear space A together with two linear maps

$$
\begin{aligned}
&m : A \otimes A \longrightarrow A \ , \ (\textit{multiplication}) \\
&\eta : \mathbb{C} \longrightarrow A \ , \ (\textit{unity})
\end{aligned}
\tag{2.1}
$$

satisfying the conditions

$$
\begin{aligned}
m(m \otimes \mathbf{1}) &= m(\mathbf{1} \otimes m) \\
m(\mathbf{1} \otimes \eta) &= m(\eta \otimes \mathbf{1}) = \mathrm{id} \ .
\end{aligned}
\tag{2.2}
$$

This first condition can be seen in terms of a commuting diagram :

$$
\begin{array}{ccc}
A \otimes A \otimes A & \xrightarrow{m \otimes \mathrm{id}} & A \otimes A \\
{\scriptstyle \mathrm{id} \otimes m}\big\downarrow & & \big\downarrow{\scriptstyle m} \\
A \otimes A & \xrightarrow{m} & A
\end{array}
\tag{2.3}
$$

Next suppose we consider 'reversing the arrows', and take an algebra A equipped with a linear homorphisms $\Delta : A \longrightarrow A \otimes A$, satisfying, for $a, b \in A$:

$$
\begin{aligned}
\Delta(ab) &= \Delta(a)\Delta(b) \\
(\Delta \otimes \mathrm{id})\Delta &= (\mathrm{id} \otimes \Delta)\Delta \ .
\end{aligned}
\tag{2.4}
$$

We call Δ a *comultiplication*, which is said to be *coaasociative* in so far that the diagram

$$\begin{array}{ccc} A \otimes A \otimes A & \xleftarrow{\Delta \otimes \mathrm{id}} & A \otimes A \\ {\scriptstyle \mathrm{id}\otimes\Delta}\uparrow & & \uparrow{\scriptstyle \Delta} \\ A \otimes A & \xleftarrow{\Delta} & A \end{array} \tag{2.5}$$

commutes. There is also a counterpart to η, the *counity* map $\varepsilon : A \longrightarrow \mathbb{C}$ satisfying

$$(\mathrm{id} \otimes \varepsilon) \circ \Delta = (\varepsilon \otimes \mathrm{id}) \circ \Delta = \mathrm{id}\ . \tag{2.6}$$

A *bialgebra* $(A, m, \Delta, \eta, \varepsilon)$ is a linear space A with maps $m, \Delta, \eta, \varepsilon$ satisfying the above properties.

Now to recover anything resembling a group structure, we must append such a bialgebra with an antihomomorphism $S : A \longrightarrow A$, satisfying $S(ab) = S(b)S(a)$, for $a, b \in A$. This map is defined implicitly via the property :

$$m(S \otimes \mathrm{id}) \circ \Delta = m(\mathrm{id} \otimes S) \circ \Delta = \eta \circ \varepsilon\ . \tag{2.7}$$

We call S the *antipode map.* A *Hopf algebra* is a bialgebra $(A, m, \eta, \Delta, \varepsilon)$ equipped with an antipode map S .

Commutative and noncommutative Hopf algebras form the backbone of quantum groups and are essential to the generalizations of symmetry. Indeed, in most respects a quantum group is identifiable with a Hopf algebra. When such algebras are associated to matrix groups there is considerable scope for representations on both finite and infinite dimensional Hilbert spaces.

2.2. **The weak Hopf Algebra.** In order to define a *weak Hopf algebra*, we can relax certain axioms for a Hopf algebras as follows :

(1) The comultiplication is not necessarily unit–preserving.

(2) The counit ε is not necessarily a homomorphism of algebras.

(3) The axioms for the antipode map $S : A \longrightarrow A$ with respect to the counit are as follows. For all $h \in H$,

$$\begin{aligned} m(\mathrm{id} \otimes S)\Delta(h) &= (\varepsilon \otimes \mathrm{id})(\Delta(1)(h \otimes 1)) \\ m(S \otimes \mathrm{id})\Delta(h) &= (\mathrm{id} \otimes \varepsilon)((1 \otimes h)\Delta(1)) \\ S(h) &= S(h_{(1)})h_{(2)}S(h_{(3)})\ . \end{aligned} \tag{2.8}$$

As frequently seen in the literature, a weak Hopf algebra is synonymous with a *quantum groupoid.* In our setting, a *Weak C*–Hopf algebra* is a weak *–Hopf algebra which admits a faithful *–representation on a Hilbert space. It is quite likely that other authors use the term ‘quantum groupoid’ in the sense of a weak C*–Hopf algebra. Eventually, the notion of a *weak C*–algebroid* will be main framework for the type of symmetry breaking we consider here. There are significant motivating examples concerning weak C*–Hopf algebras which deserve mentioning.

2.3. **Examples.**

(1) We refer here to Bais et al. (2002). Let G be a nonabelian group and $H \subset G$ a discrete subgroup. Let $F(H)$ denote the space of functions on H and $\mathbb{C}H$ the group algebra (which consists of the linear span of group elements with the group structure). *The quantum double* $D(H)$ (Drinfeld, 1987) is defined by

$$D(H) = F(H) \widetilde{\otimes} \mathbb{C}H \ , \tag{2.9}$$

where, for $x \in H$, the 'twisted tensor product' is specified by

$$\widetilde{\otimes} \mapsto \ (f_1 \otimes h_1)(f_2 \otimes h_2)(x) = f_1(x) f_2(h_1 x h_1^{-1}) \otimes h_1 h_2 \ . \tag{2.10}$$

The physical interpretation is often to take H as the 'electric gauge group' and $F(H)$ as the 'magnetic symmetry' generated by $\{f \otimes e\}$. In terms of the counit ε, the double $D(H)$ has a trivial representation given by $\varepsilon(f \otimes h) = f(e)$. We next look at certain features of this construction.

For the purpose of braiding relations there is an R matrix, $R \in D(H) \otimes D(H)$, leading to the operator

$$\mathcal{R} \equiv \sigma \cdot (\Pi^A_\alpha \otimes \Pi^B_\beta)(R) \ , \tag{2.11}$$

in terms of the Clebsch–Gordan series $\Pi^A_\alpha \otimes \Pi^B_\beta \cong N^{AB\gamma}_{\alpha\beta C} \ \Pi^C_\gamma$, and where σ denotes a flip operator. The operator $\mathcal{R}^2$ is sometimes called the *monodromy* or *Aharanov–Bohm phase factor*. In the case of a condensate in a state $|v\rangle$ in the carrier space of some representation Π^A_α . One considers the maximal Hopf subalgebra T of a Hopf algebra A for which $|v\rangle$ is T–invariant; specifically :

$$\Pi^A_\alpha(P) \ |v\rangle = \varepsilon(P)|v\rangle \ , \ \forall P \in T \ . \tag{2.12}$$

(2) For the second example, consider $A = F(H)$. The algebra of functions on H can be broken to the algebra of functions on H/K, that is, to $F(H/K)$, where K is normal in H, that is, $HKH^{-1} = K$. Next, consider $A = D(H)$. On breaking a purely electric condensate $|v\rangle$, the magnetic symmetry remains unbroken, but the electric symmetry $\mathbb{C}H$ is broken to $\mathbb{C}N_v$, with $N_v \subset H$, the stabilizer of $|v\rangle$. From this we obtain $T = F(H) \widetilde{\otimes} \mathbb{C}N_v$.

(3) In Nikshych and Vainerman (2000) quantum groupoids (as weak C*–Hopf algebras, see below) were studied in relationship to the noncommutative symmetries of depth 2 von Neumann subfactors. If

$$A \subset B \subset B_1 \subset B_2 \subset \dots \tag{2.13}$$

is the Jones extension induced by a finite index depth 2 inclusion $A \subset B$ of II_1 factors, then $Q = A' \cap B_2$ admits a quantum groupoid structure and acts on B_1, so that $B = B_1^Q$ and $B_2 = B_1 \rtimes Q$. Similarly, in Rehren (1997) 'paragroups' (derived from weak C*–Hopf algebras) comprise (quantum) groupoids of equivalence classes such as associated with 6j–symmetry groups (relative to a fusion rules algebra). They correspond to type II von Neumann algebras in quantum mechanics, and arise as symmetries where the local subfactors (in the sense of containment of observables within fields) have depth 2 in the Jones extension. Related is how a von Neumann algebra N, such as of finite index depth 2, sits inside a weak Hopf algebra formed as the crossed product $N \rtimes A$ (Böhm et al. 1999).

(4) In Mack and Schomerus (1992) using a more general notion of the Drinfeld construction, develop the notion of a *quasi triangular quasi–Hopf algebra* (QTQHA) is developed with the aim of studying a range of essential symmetries with special properties, such the quantum group algebra $\mathrm{U}_q(\mathrm{sl}_2)$ with $|q| = 1$. If $q^p = 1$, then it is shown that a QTQHA is canonically associated with $\mathrm{U}_q(\mathrm{sl}_2)$. Such QTQHAs are claimed as the true symmetries of minimal conformal field theories.

3. Quantum Groupoids and the Groupoid C*–Algebra

3.1. **Quantum compact groupoids.** Let G be a (topological) groupoid. We denote by $C_c(\mathsf{G})$ the space of smooth complex–valued functions with compact support on G . In particular, for all $f, g \in C_c(\mathsf{G})$, the function defined via convolution

$$(f \ * \ g)(\gamma) = \int_{\gamma_1 \circ \gamma_2 = \gamma} f(\gamma_1) g(\gamma_2) \ , \tag{3.1}$$

is again an element of $C_c(\mathsf{G})$, where the convolution product defines the composition law on $C_c(\mathsf{G})$. We can turn $C_c(\mathsf{G})$ into a *–algebra once we have defined the involution $*$, and this is done by specifying $f^*(\gamma) = \overline{f(\gamma^{-1})}$. We recall following Landsman (1998) that a *representation* of a groupoid G, consists of a family (or field) of Hilbert spaces $\{\mathcal{H}_x\}_{x \in X}$ indexed by $X = \mathrm{Ob}\ \mathsf{G}$, along with a collection of maps $\{U(\gamma)\}_{\gamma \in \mathsf{G}}$, satisfying:

1. $U(\gamma) : \mathcal{H}_{s(\gamma)} \longrightarrow \mathcal{H}_{r(\gamma)}$, is unitary.
2. $U(\gamma_1 \gamma_2) = U(\gamma_1) U(\gamma_2)$, whenever $(\gamma_1, \gamma_2) \in \mathsf{G}^{(2)}$.
3. $U(\gamma^{-1}) = U(\gamma)^*$, for all $\gamma \in \mathsf{G}$.

Suppose now G is a Lie groupoid. Then the isotropy group G_x is a Lie group, and for a (left or right) Haar measure μ_x on G_x, we can consider the Hilbert spaces $\mathcal{H}_x = L^2(\mathsf{G}_x, \mu_x)$ as exemplifying the above sense of a representation. Putting aside some technical details which can be found in Connes (1994), Landsman (2006), the overall idea is to define an operator of Hilbert spaces

$$\pi_x(f) : L^2(\mathsf{G}_x, \mu_x) \longrightarrow L^2(\mathsf{G}_x, \mu_x) \ , \tag{3.2}$$

given by

$$(\pi_x(f)\xi)(\gamma) = \int f(\gamma_1) \xi(\gamma_1^{-1}\gamma) \ d\mu_x \ , \tag{3.3}$$

for all $\gamma \in \mathsf{G}_x$, and $\xi \in \mathcal{H}_x$. For each $x \in X = \mathrm{Ob}\ \mathsf{G}$, π_x defines an involutive representation $\pi_x : C_c(\mathsf{G}) \longrightarrow \mathcal{H}_x$. We can define a norm on $C_c(\mathsf{G})$ given by

$$\|f\| = \sup_{x \in X} \|\pi_x(f)\| \ , \tag{3.4}$$

whereby the completion of $C_c(\mathsf{G})$ in this norm, defines *the reduced* C^**–algebra* $C^*_r(\mathsf{G})$ *of* G. It is perhaps the most commonly used C*–algebra for Lie groupoids (groups) in noncommutative geometry.

Compact quantum groupoids were introduced in Landsman (1998) as a simultaneous generalization of a compact groupoid and a quantum group. Since the construction is relevant to that which we propose, it deserves some exposition before we step to the next level of generality. Firstly, let $\mathfrak{A}$ and $\mathfrak{B}$ denote C*–algebras equipped with a *–homomorphism $\eta_s : \mathfrak{B} \longrightarrow \mathfrak{A}$, and a *–antihomomorphism $\eta_t : \mathfrak{B} \longrightarrow \mathfrak{A}$ whose images in $\mathfrak{A}$ commute. A noncommutative Haar measure is defined as a completely positive map $P : \mathfrak{A} \longrightarrow \mathfrak{B}$ which satisfies $P(A\eta_s(B)) = P(A)B$. Alternatively, the composition $\mathcal{E} = \eta_s \circ P : \mathfrak{A} \longrightarrow \eta_s(B) \subset \mathfrak{A}$ is a faithful conditional expectation.

The next step requires a little familiarity with the theory of Hilbert modules (see e.g. Lance, 1995). We define a left $\mathfrak{B}$–action λ and a right $\mathfrak{B}$–action ρ on $\mathfrak{A}$ by $\lambda(B)A = A\eta_t(B)$ and $\rho(B)A = A\eta_s(B)$. For the sake of localization of the intended Hilbert module, we implant a $\mathfrak{B}$–valued inner product on $\mathfrak{A}$ given by $\langle A, C\rangle_{\mathfrak{B}} = P(A^*C)$. Since P is faithful, we fit a new norm on $\mathfrak{A}$ given by $\|A\|^2 = \|P(A^*A)\|_{\mathfrak{B}}$. The completion of $\mathfrak{A}$ in this new norm is denoted by $\mathfrak{A}^-$ leading then to a Hilbert module over $\mathfrak{B}$.

The tensor product $\mathfrak{A}^- \otimes_{\mathfrak{B}} \mathfrak{A}^-$ can be shown to be a Hilbert bimodule over $\mathfrak{B}$, which for $i = 1, 2$, leads to *–homorphisms $\varphi^i : \mathfrak{A} \longrightarrow \mathcal{L}_{\mathfrak{B}}(\mathfrak{A}^- \otimes \mathfrak{A}^-)$. Next is to define the (unital) C*–algebra $\mathfrak{A} \otimes_{\mathfrak{B}} \mathfrak{A}$ as the C*–algebra contained in $\mathcal{L}_{\mathfrak{B}}(\mathfrak{A}^- \otimes \mathfrak{A}^-)$ that is generated by $\varphi^1(\mathfrak{A})$ and $\varphi^2(\mathfrak{A})$. The last stage of the recipe for defining a compact quantum groupoid entails considering a certain coproduct operation $\Delta : \mathfrak{A} \longrightarrow \mathfrak{A} \otimes_{\mathfrak{B}} \mathfrak{A}$, together with a coinverse $Q : \mathfrak{A} \longrightarrow \mathfrak{A}$ that it is both an algebra and bimodule antihomomorphism. Finally, the following axiomatic relationships are observed :

$$\begin{aligned} (\mathrm{id} \otimes_{\mathfrak{B}} \Delta) \circ \Delta &= (\Delta \otimes_{\mathfrak{B}} \mathrm{id}) \circ \Delta \\ (\mathrm{id} \otimes_{\mathfrak{B}} P) \circ \Delta &= P \\ \tau \circ (\Delta \otimes_{\mathfrak{B}} Q) \circ \Delta &= \Delta \circ Q \end{aligned} \tag{3.5}$$

where τ is a flip map : $\tau(a \otimes b) = (b \otimes a)$.

4. The weak C*–Hopf algebroid and symmetries

4.1. **The Algebroid concept.** By an *algebroid structure* A we shall specifically mean also a ring, or more generally an algebra, but *with several objects* (instead of a single object), in the sense of Mitchell (1965). More precisely, let us assume for instance that we are given a commutative ring R with identity. Then an R-category, or an R*-algebroid*, will be defined as a category enriched in the monoidal category of R-modules, with respect to the monoidal structure of tensor product. This means simply that for all objects b, c of A, the set $A(b, c)$ is given the structure of an R-module, and composition $A(b, c) \times A(c, d) \to A(b, d)$ is R–bilinear, or is a morphism of R-modules $A(b, c) \otimes_R A(c, d) \to A(b, d)$.

If G is a groupoid (or, more generally, a category) then we can construct an *R-algebroid* $R\mathsf{G}$ as follows. The object set of $R\mathsf{G}$ is the same as that of G and $R\mathsf{G}(b,c)$ is the free R-module on the set $\mathsf{G}(b,c)$, with composition given by the usual bilinear rule, extending the composition of G.

Alternatively, we can define $\bar{R}\mathsf{G}(b,c)$ to be the set of functions $\mathsf{G}(b,c) \to R$ with finite support, and then we define the *convolution product* can be defined:

$$(f * g)(z) = \sum\{(fx)(gy) \mid z = x \circ y\} \ . \tag{4.1}$$

As is well known, it is the second construction which is natural for the topological case, when we need to replace 'function' by 'continuous function with compact support' (or *locally compact support* for the QFT extended symmetry sectors), and in this case $R \cong \mathbb{C}$. The point we are making here is that to make the usual construction and end up with an algebra rather than an algebroid, is a procedure analogous to replacing a groupoid G by a semigroup $G' = G \cup \{0\}$ in which the compositions not defined in G are defined to be 0 in G'. We argue that this construction removes the main advantage of groupoids, namely the spatial component given by the set of objects.

However, at present the question of how one can use categorical duality in order to find the analogue of the diagonal of a Hopf algebra remains open. Such questions require further work and also future development of the theoretical framework proposed here for extended symmetries and the related fundamental aspects of quantum field theories. Another related problem that we see is to what extent the famous theory of C^*-algebras, and its uses in physics, can be naturally applied or extended to the novel situation as outlined in the text.

4.2. **The weak C*–Hopf algebroid.** Progressing to the next level of generality, let A denote an algebra with local identities in a commutative subalgebra $R \subset A$. We adopt the definition of a *Hopf algebroid structure on A over R* following Mrčun (2002). Relative to a ground field $\mathbb{F}$ (typically $\mathbb{F} = \mathbb{C}$ or $\mathbb{R}$), the definition commences by taking three $\mathbb{F}$–linear maps, the *comultiplication*

$$\Delta : A \longrightarrow A \otimes_R A \ , \tag{4.2}$$

the counit

$$\varepsilon : A \longrightarrow R \ , \tag{4.3}$$

and the *antipode*

$$S : A \longrightarrow A \ , \tag{4.4}$$

such that :

(i) Δ and ε are homomorphisms of left R–modules satisfying $(\mathrm{id} \otimes \varepsilon) \circ \Delta = \mathrm{id}$ and $(\varepsilon \otimes \mathrm{id}) \circ \Delta = \mathrm{id}$.

(ii) $\varepsilon|_R = \mathrm{id}$, $\Delta|_R$ is the canonical embedding $R \cong R \otimes_R R \subset A \otimes_R A$, and the two right R–actions on $A \otimes_R A$ coincide on ΔA .

(iii) $\Delta(ab) = \Delta(a)\Delta(b)$ for any $a, b \in A$.

(iv) $S|_R = \mathrm{id}$ and $S \circ S = \mathrm{id}$.

(v) $S(ab) = S(a)S(b)$ for any $a, b \in A$.

(vi) $\mu \circ (S \otimes \mathrm{id}) \circ \Delta = \varepsilon \circ S$, where $\mu : A \otimes_R A \longrightarrow A$ denotes the multiplication.

If R is a commutative subalgebra with local identities, then a *Hopf algebroid over* R is a quadruple $(A, \Delta, \varepsilon, S)$ where A is an algebra which has R for a subalgebra and has local identities in R, and where (Δ, ε, S) is a Hopf algebroid structure on A over R . Our interest lies in the fact that a Hopf–algebroid comprises a (universal) enveloping algebra for a quantum groupoid.

Definition 4.1. Let $(A, \Delta, \varepsilon, S)$ be a Hopf algebroid as above. We say that $(A, \Delta, \varepsilon, S)$ is a *weak C*–Hopf algebroid* when

(1) A is a unital C*–algebra (with $\mathbf{1}$) . We set $\mathbb{F} = \mathbb{C}$.

(2) The comultiplication $\Delta : A \longrightarrow A \otimes A$ is a coassociative $*$–homomorphism. The counit is a positive linear map $\varepsilon : A \longrightarrow R$ satisfying the above compatibility condition. The antipode S is a complex–linear anti–homomorphism and anti–cohomorphism $S : A \longrightarrow A$ (that is, it reverses the order of the multiplication and comultiplication), and is inverted under the *–structure: $S^{-1}(a) = S(a^*)^*$.

(3)

$$\begin{aligned} \Delta(\mathbf{1}) \equiv \mathbf{1}_{(1)} \otimes \mathbf{1}_{(2)} &= \text{projection} \\ \varepsilon(ap) &= \varepsilon(a\mathbf{1}_{(1)}) \cdot \varepsilon(\mathbf{1}_{(2)}p) \\ S(a_{(1)})a_{(2)} \otimes a_{(3)} &= (\mathbf{1} \otimes a) \cdot \Delta(\mathbf{1}) \ . \end{aligned} \tag{4.5}$$

Here $a_{(1)} \otimes a_{(2)}$ is shorthand notation for the expansion of $\Delta(a)$.

(4) The dual $\widehat{A}$ is defined by the linear maps $\hat{a} : A \longrightarrow \mathbb{C}$. The structure of $\widehat{A}$ is canonically dualized via the pairing and $\widehat{A}$ is endowed with a dual *–structure via $\langle \hat{a}^*, a \rangle_A = \overline{\langle \hat{a}, S(a)^* \rangle_A}$. Further, $(\widehat{A}, \widehat{\Delta}, \hat{\varepsilon}, \widehat{S})$ with $*$ and $\varepsilon = \widehat{\mathbf{1}}$, is a weak C*–Hopf algebroid.

5. Weak Hopf C*–symmetry

At this stage we note a schematic representation for our groupoid symmetries as follows :

Classical dynamical symmetry :

Lie groups $\Longrightarrow$ Lie algebras $\Longrightarrow$ Universal enveloping algebra $\Longrightarrow$ Quantization

Quantum symmetry :

Weak Hopf algebras $\Longleftarrow$ Representations $\Longleftarrow$ Quantum groups

The intention is to view the latter scheme in terms of *Weak Hopf C*–Symmetries* which we propose to do by incorporating the concepts of *rigged Hilbert spaces* and *sectional functions for a small category*.

As previously, let (G, τ) be a locally compact groupoid endowed with a (left) Haar system, and let $A = C^*(\mathsf{G}, \tau)$ be the convolution C*–algebra (we append A with $\mathbf{1}$ if necessary, so that A is unital). Now suppose we have consider a groupoid representation $\Lambda : (\mathsf{G}, \tau) \longrightarrow \{\mathcal{H}_x, \sigma_x\}_{x \in X}$ as was previously defined and respecting a compatible measure σ_x on $\mathcal{H}_x$. On taking a state ρ on A, we assume a parametrization

$$(\mathcal{H}_x, \sigma_x) := (\mathcal{H}_\rho, \sigma)_{x \in X} \ . \tag{5.1}$$

Further, each $\mathcal{H}_x$ is consider as a *rigged Hilbert space* Bohm and Gadella (1989), that is we have nested inclusions

$$\Phi_x \subset (\mathcal{H}_x, \sigma_x) \subset \Phi_x^{\times} \ , \tag{5.2}$$

in the usual way, where Φ_x is a dense subspace of $\mathcal{H}_x$ with the appropriate locally convex topology, and $\Phi_x^{\times}$ is the space of continuous antilinear functionals of Φ . For each $x \in X$, we require Φ_x to be invariant under Λ and Im $\Lambda | \Phi_x$ is a continuous representation of G on Φ_x . Representations of quantum groupoids derived for weak C*–Hopf algebras (or algebroids) modeled on rigged Hilbert spaces could be suitable generalizations in the framework of a Hamiltonian generated semigroup of time evolution of a physical system via integration of Schrödinger's equation $\iota\hbar \frac{\partial \psi}{\partial t} = H\psi$ as studied in the case of Lie groups (Wickramasekara and Bohm, 2006). The adoption of the rigged Hilbert spaces is also based on how the latter are recognized as reconciling the Dirac and von Neumann theories (Bohm and Gadella, 1989).

Next let G be a locally compact Hausdorff groupoid and X a locally compact Hausdorff space. In order to achieve a small C*–category we follow a suggestion of A. Seda (private communication) by using a general principle in the context of Banach bundles (Seda, 1976, 982)). Let $q = (q_1, q_2) : \mathsf{G} \longrightarrow X \times X$ be a continuous, open and surjective map. For each $z = (x, y) \in X \times X$, consider the fibre $\mathsf{G}_z = \mathsf{G}(x, y) = q^{-1}(z)$, and set $\mathcal{A}_z = C_0(\mathsf{G}_z) = C_0(\mathsf{G}(x, y))$ equipped with a uniform norm $\| \ \|_z$. Then we set $\mathcal{A} = \bigcup_z \mathcal{A}_z$. We form a Banach bundle $p : \mathcal{A} \longrightarrow X \times X$ as follows. Firstly, the projection is defined via the typical fibre $p^{-1}(z) = \mathcal{A}_z = \mathcal{A}_{(x,y)}$. Let $C_c(\mathsf{G})$ denote the continuous complex valued functions on G with compact support. We obtain a sectional function $\widetilde{\psi} : X \times X \longrightarrow \mathcal{A}$ defined via restriction as $\widetilde{\psi}(z) = \psi | \mathsf{G}_z = \psi | \mathsf{G}(x, y)$. Commencing from the vector space $\Gamma = \{\widetilde{\psi} : \psi \in C_c(\mathsf{G})\}$, the set $\{\widetilde{\psi}(z) : \widetilde{\psi} \in \Gamma\}$ is dense in $\mathcal{A}_z$. For each $\widetilde{\psi} \in \Gamma$, the function $\|\widetilde{\psi}(z)\|_z$ is continuous on X, and each $\widetilde{\psi}$ is a continuous section of $p : \mathcal{A} \longrightarrow X \times X$. These facts follow from Seda (1982, Theorem 1). Further, under the convolution product $f * g$, the space $C_c(\mathsf{G})$ forms an associative algebra over $\mathbb{C}$ (Seda, 1982, Theorem 3).

Definition 5.1. The data proposed for a *weak C*–Hopf symmetry* consists of:

(1) A weak C*–Hopf algebroid $(A, \Delta, \varepsilon, S)$, where as above, $A = C^*(\mathsf{G}, \tau)$ is constructed via sectional functions over a small category.

(2) A family of GNS representations

$$(\pi_\rho)_x : A \longrightarrow (\mathcal{H}_\rho)_x := \mathcal{H}_x \ , \tag{5.3}$$

where for each, $x \in X$, $\mathcal{H}_x$ is a rigged Hilbert space.

5.1. **Double groupoids.** We see the further advantage of the above procedure in moving to the double groupoid case (Brown and Spencer, 1976) in relationship to a C*–convolution algebroid. Here, the geometry of squares and their compositions leads to a common representation of a *double groupoid* in the following form:

$$\mathsf{D} = \begin{array}{ccc} S & \underset{t^1}{\overset{s^1}{\rightleftarrows}} & H \\ s_2 \downarrow\uparrow\downarrow t_2 & & s \downarrow\uparrow\downarrow t \\ V & \underset{t}{\overset{s}{\rightleftarrows}} & M \end{array} \tag{5.4}$$

where M is a set of 'points', H, V are 'horizontal' and 'vertical' groupoids, and S is a set of 'squares' with two compositions. The laws for a double groupoid make it also describable as a groupoid internal to the category of groupoids.

Given two groupoids H, V over a set M, there is a double groupoid $\square(H, V)$ with H, V as horizontal and vertical edge groupoids, and squares given by quadruples

$$\begin{pmatrix} & h & \\ v & & v' \\ & h' & \end{pmatrix} \tag{5.5}$$

for which we assume always that $h, h' \in H$, $v, v' \in V$ and that the initial and final points of these edges match in M as suggested by the notation, that is for example $sh = sv, th = sv', \ldots$, etc. The compositions are to be inherited from those of H, V, that is

(5.6)

$$\begin{pmatrix} & h & \\ v & & v' \\ & h' & \end{pmatrix} \circ_1 \begin{pmatrix} & h' & \\ w & & w' \\ & h'' & \end{pmatrix} = \begin{pmatrix} & h & \\ vw & & v'w' \\ & h'' & \end{pmatrix}, \begin{pmatrix} & h & \\ v & & v' \\ & h' & \end{pmatrix} \circ_2 \begin{pmatrix} & k & \\ v' & & v'' \\ & k' & \end{pmatrix} = \begin{pmatrix} & hk & \\ v & & v'' \\ & h'k' & \end{pmatrix}.$$

This construction is right adjoint to the forgetful functor which takes the double groupoid as in (5.4) to the pair of groupoids H, V over M. Now given a general double groupoid as above, we can define $S\begin{pmatrix} & h & \\ v & & v' \\ & h' & \end{pmatrix}$ to be the set of squares with these as horizontal and vertical edges.

This allows us to construct for at least a commutative C*–algebra A a double algebroid (i.e. a set with two algebroid structures)

$$A\mathsf{D} = \begin{array}{ccc} AS & \underset{t^1}{\overset{s^1}{\rightleftarrows}} & AH \\ s_2 \downarrow\uparrow\downarrow t_2 & & s \downarrow\uparrow\downarrow t \\ AV & \underset{t}{\overset{s}{\rightleftarrows}} & M \end{array} \tag{5.7}$$

for which

$$AS \begin{pmatrix} & h & \\ v & & v' \\ & h' & \end{pmatrix} \tag{5.8}$$

is the free A-module on the set of squares with the given boundary. The two compositions are then bilinear in the obvious sense. Alternatively, we can use the convolution construction $\bar{A}\mathsf{D}$ induced by the convolution C*–algebra over H and V. These ideas need further development in the light of the algebra of crossed modules of algebroids, developed in (Mosa, 1986, Brown and Mosa, 1986) as well as crossed cubes of (C*) algebras following Ellis (1988).

The next, natural extension of this *quantum algebroid* approach to QFT generalized symmetries can now be formulated in terms of *graded Lie algebroids* for a supersymmetry-based theory of *Quantum Gravity/ Supergravity*, as will be discussed in a later section.

6. Non–abelian algebroid representations of quantum state space geometry in quantum supergravity fields

Supergravity, in essence, is an extended supersymmetric theory of both matter and gravitation Weinberg (1995). A first approach to supersymmetry relies on a curved 'superspace' [44] and is analogous to supersymmetric gauge theories (see, for example, Sections 27.1 to 27.3 of Weinberg, 1995). Unfortunately, a complete non–linear supergravity theory might be forbiddingly complicated and furthermore, the constraints that need be made on the graviton superfield appear somewhat subjective Weinberg (1995). On the other hand, the second approach to supergravity is much more transparent than the first, albeit theoretically less elegant. The physical components of the gravitational superfield can be identified in this approach based on flat-space superfield methods (Chs. 26 and 27 of Weinberg, 1995). By implementing the weak-field approximation one obtains several of the most important consequences of supergravity theory, including masses for the hypothetical gravitino and gaugino 'particles' whose existence is expected from supergravity theories. Furthermore, by adding on the higher order terms in the gravitational constant to the supersymmetric transformation, the general coordinate transformations form a closed algebra and the Lagrangian that describes the interactions of the physical fields is invariant under such transformations. Quantization of such a flat-space superfield would obviously involve its 'deformation' as discussed in Section 2 above, and as a result its corresponding *supersymmetry algebra* would become *non–commutative*.

6.1. The Metric Superfield.

Because in supergravity both spinor and tensor fields are being considered, the gravitational fields are represented in terms of *tetrads*, $e^a_\mu(x)$, rather than in terms of the general relativistic metric $g_{\mu\nu}(x)$. The connections between these two distinct representations are as follows:

$$g_{\mu\nu}(x) = \eta_{ab}\; e^a_\mu(x) e^b_\gamma(x) \;, \tag{6.1}$$

with the general coordinates being indexed by μ, ν, etc., whereas local coordinates that are being defined in a locally inertial coordinate system are labeled with superscripts a, b, etc.;

η_{ab} is the diagonal matrix with elements +1, +1, +1 and -1. The tetrads are invariant to two distinct types of symmetry transformations–the local Lorentz transformations:

$$e^a_\mu(x) \longmapsto \Lambda^a_b(x) e^b_\mu(x) \ , \tag{6.2}$$

(where Λ^a_b is an arbitrary real matrix), and the general coordinate transformations:

$$x^\mu \longmapsto (x')^\mu(x) \ . \tag{6.3}$$

In a weak gravitational field the tetrad may be represented as:

$$e^a_\mu(x) = \delta^a_\mu(x) + 2\kappa \Phi^a_\mu(x) \ , \tag{6.4}$$

where $\Phi^a_\mu(x)$ is small compared with $\delta^a_\mu(x)$ for all x values, and $\kappa = \sqrt{8\pi G}$, where G is Newton's gravitational constant. As it will be discussed next, the supersymmetry algebra (SA) implies that the graviton has a fermionic superpartner, the hypothetical *gravitino*, with helicities ± 3/2. Such a self-charge-conjugate massless particle as the gravitiono with helicities ± 3/2 can only have *low-energy* interactions if it is represented by a Majorana field $\psi_\mu(x)$ which is invariant under the gauge transformations:

$$\psi_\mu(x) \longmapsto \psi_\mu(x) + \delta_\mu \psi(x) \ , \tag{6.5}$$

with $\psi(x)$ being an arbitrary Majorana field as defined by Grisaru and Pendleton (1977). The tetrad field $\Phi_{\mu\nu}(x)$ and the graviton field $\psi_\mu(x)$ are then incorporated into a term $H_\mu(x, \theta)$ defined as the *metric superfield*. The relationships between $\Phi_{\mu_\nu}(x)$ and $\psi_\mu(x)$, on the one hand, and the components of the metric superfield $H_\mu(x, \theta)$, on the other hand, can be derived from the transformations of the whole metric superfield:

$$H_\mu(x, \theta) \longmapsto H_\mu(x, \theta) + \Delta_\mu(x, \theta) \ , \tag{6.6}$$

by making the simplifying- and physically realistic assumption of a weak gravitational field. Further details can be found, for example, in Ch.31 of vol.3. of Weinberg, 1995). The interactions of the entire superfield $H_\mu(x)$ with matter would be then described by considering how a weak gravitational field, h_{μ_ν} interacts with an energy-momentum tensor $T^{\mu\nu}$ represented as a linear combination of components of a real vector superfield Θ^μ. Such interaction terms would, therefore, have the form:

$$I_{\mathcal{M}} = 2\kappa \int dx^4 [H_\mu \Theta^\mu]_D \ , \tag{6.7}$$

($\mathcal{M}$ denotes 'matter') integrated over a four-dimensional (Minkowski) space-time with the metric defined by the superfield $H_\mu(x, \theta)$. The term Θ^μ, as defined above, is physically a *supercurrent* and satisfies the conservation conditions:

$$\gamma^\mu \mathbf{D} \Theta_\mu = \mathbf{D} \ , \tag{6.8}$$

where $\mathbf{D}$ is the four-component super-derivative and X denotes a real chiral scalar superfield. This leads immediately to the calculation of the interactions of matter with a weak gravitational field as:

(6.9)
$$I_{\mathcal{M}} = \kappa \int d^4 x T^{\mu\nu}(x) h_{\mu\nu}(x) \ ,$$

It is interesting to note that the gravitational actions for the superfield that are invariant under the generalized gauge transformations $H_\mu \longmapsto H_\mu + \Delta_\mu$ lead to solutions of the Einstein field equations for a homogeneous, non-zero vacuum energy density ρ_V that correspond to either a de Sitter space for $\rho_V > 0$, or an anti-de Sitter space for $\rho_V < 0$. Such spaces can be represented in terms of the hypersurface equation

(6.10)
$$x_5^2 \pm \eta_{\mu,\nu} x^\mu x^\nu = R^2 \ ,$$

in a quasi-Euclidean five-dimensional space with the metric specified as:

(6.11)
$$ds^2 = \eta_{\mu,\nu} x^\mu x^\nu \pm dx_5^2 \ ,$$

with '+' for de Sitter space and '-' for anti-de Sitter space, respectively.

The space-time symmetry groups, or groupoids –as the case may be– are different from the 'classical' Poincaré symmetry group of translations and Lorentz transformations. Such space-time symmetry groups, in the simplest case, are therefore the $\mathrm{O}(4,1)$ group for the de Sitter space and the $\mathrm{O}(3,2)$ group for the anti–de Sitter space. A detailed calculation indicates that the transition from ordinary flat space to a bubble of anti-de Sitter space is *not* favored energetically and, therefore, the ordinary (de Sitter) flat space is stable (cf. Coleman and De Luccia, 1980), even though quantum fluctuations might occur to an anti–de Sitter bubble within the limits permitted by the Heisenberg uncertainty principle.

6.2. **Supersymmetry Algebras and Graded Lie Algebras.** It is well known that a *continuous symmetry transformations* can be represented in terms of a *Lie algebra* of linearly independent *symmetry generators* t_j that satisfy the commutation relations:

(6.12)
$$[t_j, t_k] = \iota \Sigma_l C_{jk} t_l \ ,$$

Supersymmetry is similarly expressed in terms of the symmetry generators t_j of a graded Lie algebra which satisfy relations of the general form:

(6.13)
$$t_j t_k - (-1)^{\eta_j \eta_k} t_k t_j = \iota \Sigma_l C^l_{jk} t_l \ .$$

The generators for which $\eta_j = 1$ are fermionic whereas those for which $\eta_j = 0$ are bosonic. The coefficients C^l_{jk} are structure constants satisfying the following conditions:

(6.14)
$$C^l_{jk} = -(-1)^{\eta_j \eta_k} C^l_{jk} \ .$$

If the generators $_j$ are quantum Hermitian operators, then the structure constants satisfy the reality conditions $C^*_{jk} = -C_{jk}$.

The standard computational approach in QM utilizes the S-matrix approach, and therefore, one needs to consider the general, *graded* Lie algebra of *supersymmetry generators* that commute with the S-matrix. If one denotes the fermionic generators by Q, then $U^{-1}(\Lambda)QU(\Lambda)$ will also be of the same type when $U(\Lambda)$ is the quantum operator corresponding to arbitrary, homogeneous Lorentz transformations Λ^{μ_ν} . Such a group of generators provide therefore a representation of the homogeneous Lorentz group of transformations $\mathbb{L}$. The irreducible representation of the homogeneous Lorentz group of transformations provides therefore a classification of such individual generators.

6.3. **Graded Lie Algebras.** A set of quantum operators Q_{jk}^{AB} form an $\mathbf{A}, \mathbf{B}$ representation of the group $\mathbf{L}$ defined above which satisfy the commutation relations:

$$[\mathbf{A}, Q_{jk}^{AB}] = -[\Sigma_j' J_{jj'}^A Q_{j'k}^{AB}] \ , \tag{6.15}$$

and

$$[\mathbf{B}, Q_{jk}^{AB}] = -[\Sigma_{j'} J_{kk'}^A Q_{jk'}^{AB}] \ , \tag{6.16}$$

with the generators $\mathbf{A}$ and $\mathbf{B}$ defined by $\mathbf{A} \equiv (1/2)(\mathbf{J} \pm i\mathbf{K})$ and $\mathbf{B} \equiv (1/2)(\mathbf{J} - i\mathbf{K})$, with $\mathbf{J}$ and $\mathbf{K}$ being the Hermitian generators of rotations and 'boosts', respectively.

In the case of the two-component Weyl-spinors Q_{jr} the Haag–Lopuszanski–Sohnius (HLS) theorem applies, and thus the fermions form a *supersymmetry algebra* defined by the anti-commutation relations:

$$\begin{aligned} [Q_{jr}, Q_{ks}^*] &= 2\delta_{rs}\sigma_{jk}^{\mu} P_\mu \ , \\ [Q_{jr}, Q_{ks}] &= e_{jk} Z_{rs} \ , \end{aligned} \tag{6.17}$$

where P_μ is the 4–momentum operator, $Z_{rs} = -Z_{sr}$ are the bosonic symmetry generators, and σ_μ and $\mathbf{e}$ are the usual 2×2 Pauli matrices. Furthermore, the fermionic generators commute with both energy and momentum operators:

$$[P_\mu, Q_{jr}] = [P_\mu, Q_{jr}^*] = 0 \ . \tag{6.18}$$

The bosonic symmetry generators Z_{ks} and Z_{ks}^* represent the set of *central charges* of the supersymmetric algebra:

$$[Z_{rs}, Z_{tn}^*] = [Z_{rs}^*, Q_{jt}] = [Z_{rs}^*, Q_{jt}^*] = [Z_{rs}^*, Z_{tn}^*] = 0 \ . \tag{6.19}$$

From another direction, the Poincaré symmetry mechanism of special relativity can be extended to new algebraic systems (Tanasă, 2006). In Moultaka et al. (2005) in view of such extensions, consider invariant-free Lagrangians and bosonic multiplets constituting a symmetry that interplays with (abelian) U(1)–gauge symmetry that may possibly be described in categorical terms, in particular, within the notion of a *cubical site* (Grandis and Mauri, 2003).

We shall proceed to introduce in the next section generalizations of the concepts of Lie algebras and graded Lie algebras to the corresponding Lie *algebroids* that may also be regarded as C*–convolution representations of *quantum gravity groupoids* and superfield (or supergravity) supersymmetries. This is therefore a novel approach to the proper representation of the *non-commutative geometry of quantum space-times*–that are *curved* (or 'deformed') by the presence of *intense* gravitational fields–in the framework of *non-Abelian, graded Lie algebroids.* Their correspondingly *deformed quantum gravity groupoids* (QGG) should, therefore, adequately represent supersymmetries modified by the presence of such intense gravitational fields on the Planck scale. Quantum fluctuations that give rise to quantum 'foams' at the Planck scale may be then represented by *quantum homomorphisms* of such QGGs. If the corresponding graded Lie algebroids are also *integrable*, then one can reasonably expect to recover in the limit of $\hbar \to 0$ the Riemannian geometry of General Relativity and the *globally hyperbolic space-time* of Einstein's classical gravitation theory (GR), as a result of such an integration to the *quantum gravity fundamental groupoid* (QGFG). The following subsection will define the precise mathematical concepts underlying our novel quantum supergravity and extended supersymmetry notions.

7. The relationship with quantum algebras : algebras of Observables– Lie bialgebroids

7.1. **Lie algebroids and Lie bialgebroids.** One can think of a Lie algebroid as generalizing the idea of a tangent bundle where the tangent space at a point is effectively the equivalence class of curves meeting at that point (thus suggesting a groupoid approach), as well as serving as a site on which to study infinitesimal geometry (see e.g. Mackenzie, 2005). Specifically, let M be a manifold and let $\mathfrak{X}(M)$ denote the set of vector fields on M . Recall that a Lie algebroid over M consists of a vector bundle $E \longrightarrow M$, equipped with a Lie bracket $[\ ,\ ]$ on the space of sections $\Gamma(E)$, and a bundle map $\Upsilon : E \longrightarrow TM$, usually called the *anchor*. Further, there is an induced map $\Upsilon : \Gamma(E) \longrightarrow \mathfrak{X}(M)$, which is required to be a map of Lie algebras, such that given sections $\alpha, \beta \in \Gamma(E)$ and a differentiable function f, the following Leibniz rule is satisfied :

$$[\alpha, f\beta] = f[\alpha, \beta] + (\Upsilon(\alpha))\beta \ . \tag{7.1}$$

A typical example of a Lie algebroid is when M is a Poisson manifold and $E = T^*M$ (the cotangent bundle of M).

Now suppose we have a Lie groupoid G:

$$r, s \ : \ \mathsf{G} \underset{s}{\overset{r}{\rightrightarrows}} \mathsf{G}^{(0)} \ = M \ . \tag{7.2}$$

There is an associated Lie algebroid $\mathcal{A} = \mathcal{A}(\mathsf{G})$, which in the guise of a vector bundle, it is the restriction to M of the bundle of tangent vectors along the fibers of s (ie. the s–vertical vector fields). Also, the space of sections $\Gamma(\mathcal{A})$ can be identified with the space of s–vertical, right–invariant vector fields $\mathfrak{X}^s_{inv}(\mathsf{G})$ which can be seen to be closed under $[\ ,\ ]$, and the latter induces a bracket operation on $\Gamma(A)$ thus turning $\mathcal{A}$ into a Lie algebroid. Subsequently, a Lie algebroid $\mathcal{A}$ is integrable if there exists a Lie groupoid G inducing $\mathcal{A}$.

7.2. **Graded Lie algebroids.** A grading on a Lie algebroid follows by endowing a graded Jacobi bracket on the smooth functions $C^\infty(M)$ (see Grabowski and Marmo, 2001). A Graded Jacobi bracket of degree k on a $\mathbb{Z}$–graded associative commutative algebra $\mathcal{A} = \bigoplus_{i\in\mathbb{Z}} \mathcal{A}$ consists of a graded bilinear map

$$\{\cdot\ ,\ \cdot\} : \mathcal{A} \times \mathcal{A} \longrightarrow \mathcal{A}\ , \tag{7.3}$$

of degree k (that is, $|\{a,b\}| = |a| + |b| + k$) satisfying :

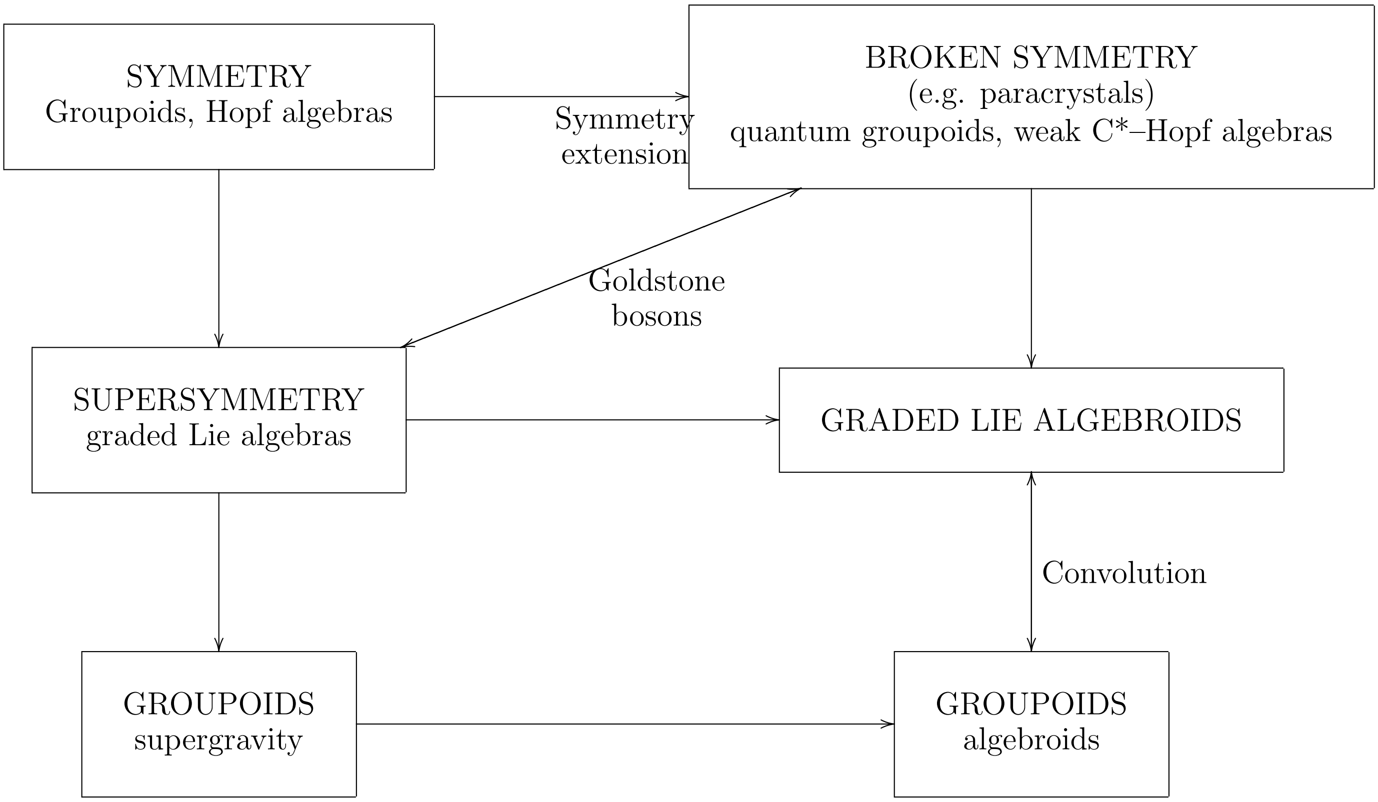
1. $\{a,b\} = -(-1)^{\langle a+k,b+k\rangle}\{b,a\}$ (graded anticommutativity)

2. $\{a,bc\} = \{a,b\}c + (-1)^{\langle a+k,b\rangle}\ b\{a,c\} - \{a,\mathbf{1}\}bc$ (graded generalized Leibniz rule)

3. $\{\{a,b\},c\} = \{a,\{b,c\}\} - (-1)^{\langle a+k,b+k\rangle}\{b,\{a,c\}\}$ (graded Jacobi identity)

where $\langle\ \cdot,\cdot\rangle$ denotes the usual pairing in $\mathbb{Z}^n$. Item 2. says that $\{\ ,\ \}$ corresponds to a first–order bidifferential operator on $\mathcal{A}$, and an odd Jacobi structure corresponds to a generalized Lie bialgebroid.

7.3. **Graded Lie bialgebroids for symmetry breaking.** A *Lie bialgebroid* is a Lie algebroid E such that $E^* \longrightarrow M$ also has a Lie algebroid structure. Lie bialgebroids are often thought of as the infinitesimal variations of Poisson groupoids. Specifically, with regards to a Poisson structure Λ, if $(\mathsf{G} \rightrightarrows M, \Lambda)$ is a Poisson groupoid and if $E\mathsf{G}$ denotes the Lie algebroid of G, then $(E\mathsf{G}, E^*\mathsf{G})$ is a Lie bialgebroid. Conversely, a Lie bialgebroid structure on the Lie algebroid of a Lie groupoid can be integrated to a Poisson groupoid structure. Examples are Lie bialgebras which correspond bijectively with simply connected Poisson Lie groups.



8. Appendix

8.1. **von Neumann Algebras.** Let $\mathcal{H}$ denote a complex (separable) Hilbert space. A *von Neumann algebra* $\mathcal{A}$ acting on $\mathcal{H}$ is a subset of the algebra of all bounded operators $\mathcal{L}(\mathcal{H})$ such that:

(1) $\mathcal{A}$ is closed under the adjoint operation (with the adjoint of an element T denoted by T^*).

(2) $\mathcal{A}$ equals its bicommutant, namely:
$$\mathcal{A} = \{A \in \mathcal{L}(\mathcal{H}) : \forall B \in \mathcal{L}(\mathcal{H}), \forall C \in \mathcal{A},\ (BC = CB) \Rightarrow (AB = BA)\}\ .$$

If one calls a *commutant* of a set $\mathcal{A}$ the special set of bounded operators on $\mathcal{L}(\mathcal{H})$ which commute with all elements in $\mathcal{A}$, then this second condition implies that the commutant of the commutant of $\mathcal{A}$ is again the set $\mathcal{A}$.

On the other hand, a von Neumann algebra $\mathcal{A}$ inherits a *unital* subalgebra from $\mathcal{L}(\mathcal{H})$, and according to the first condition in its definition $\mathcal{A}$ does indeed inherit a **-subalgebra* structure, as further explained in the next section on C*-algebras. Furthermore, we have notable *Bicommutant Theorem* which states that $\mathcal{A}$ *is a von Neumann algebra if and only if* $\mathcal{A}$ *is a *-subalgebra of* $\mathcal{L}(\mathcal{H})$, *closed for the smallest topology defined by continuous maps* $(\xi, \eta) \longmapsto (A\xi, \eta)$ *for all* $< A\xi, \eta) >$ *where* $< ., . >$ *denotes the inner product defined on* $\mathcal{H}$. For further instruction on this subject, see e.g. Aflsen and Schultz (2003), Connes (1994).

8.2. **Groupoids.** Recall that a groupoid G is, loosely speaking, a small category with inverses over its set of objects $X = \mathrm{Ob}(\mathsf{G})$. One often writes G_x^y for the set of morphisms in G from x to y . *A topological groupoid* consists of a space G, a distinguished subspace $\mathsf{G}^{(0)} = \mathrm{Ob}(\mathsf{G}) \subset \mathsf{G}$, called *the space of objects* of G, together with maps

$$r, s \; : \; \mathsf{G} \underset{s}{\overset{r}{\rightrightarrows}} \mathsf{G}^{(0)} \tag{8.1}$$

called the *range* and *source maps* respectively, together with a law of composition

$$\circ \; : \; \mathsf{G}^{(2)} := \mathsf{G} \times_{\mathsf{G}^{(0)}} \mathsf{G} = \{ \; (\gamma_1, \gamma_2) \in \mathsf{G} \times \mathsf{G} \; : \; s(\gamma_1) = r(\gamma_2) \; \} \longrightarrow \mathsf{G} \; , \tag{8.2}$$

such that the following hold :

(1) $s(\gamma_1 \circ \gamma_2) = r(\gamma_2)$, $r(\gamma_1 \circ \gamma_2) = r(\gamma_1)$, for all $(\gamma_1, \gamma_2) \in \mathsf{G}^{(2)}$.

(2) $s(x) = r(x) = x$, for all $x \in \mathsf{G}^{(0)}$.

(3) $\gamma \circ s(\gamma) = \gamma$, $r(\gamma) \circ \gamma = \gamma$, for all $\gamma \in \mathsf{G}$.

(4) $(\gamma_1 \circ \gamma_2) \circ \gamma_3 = \gamma_1 \circ (\gamma_2 \circ \gamma_3)$.

(5) Each γ has a two–sided inverse γ^{-1} with $\gamma\gamma^{-1} = r(\gamma)$, $\gamma^{-1}\gamma = s(\gamma)$.

It is usual to call $\mathsf{G}^{(0)} = \mathrm{Ob}(\mathsf{G})$ *the set of objects* of G . For $u \in \mathrm{Ob}(\mathsf{G})$, the set of arrows $u \longrightarrow u$ forms a group G_u, called the *isotropy group of* G *at* u. For a further study of groupoids we refer to Brown (2006), Connes (1994) .

8.3. **Haar systems for locally compact topological groupoids.** Let

$$\mathsf{G} \underset{s}{\overset{r}{\rightrightarrows}} \mathsf{G}^{(0)} \; = X \tag{8.3}$$

be a locally compact, locally trivial topological groupoid with its transposition into transitive (connected) components. Recall that for $x \in X$, the *costar of* x denoted $\mathrm{CO}^*(\mathrm{x})$ is defined as the closed set $\bigcup\{\mathsf{G}(y, x) : y \in \mathsf{G}\}$, whereby

$$\mathsf{G}(x_0, y_0) \hookrightarrow \mathrm{CO}^*(\mathrm{x}) \longrightarrow \mathrm{X} \; ,$$

is a principal $\mathsf{G}(x_0, y_0)$–bundle relative to fixed base points (x_0, y_0) . Assuming all relevant sets are locally compact, then following Seda (1976), a *(left) Haar system on* G denoted (G, τ) (for later purposes), is defined to comprise of i) a measure κ on G, ii) a measure μ on X and iii) a measure μ_x on $\mathrm{CO}^*(\mathrm{x})$ such that for every Baire set E of G, the following hold on setting $E_x = E \cap \mathrm{CO}^*(\mathrm{x})$:

(1) $x \mapsto \mu_x(E_x)$ is measurable.

(2) $\kappa(E) = \int_x \mu_x(E_x) \; d\mu_x$.

(3) $\mu_z(tE_x) = \mu_x(E_x)$, for all $t \in \mathsf{G}(x, z)$ and $x, z \in \mathsf{G}$.

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