

THE QUEST FOR UNDERSTANDING IN RELATIVISTIC QUANTUM PHYSICS *

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ABSTRACT

We discuss the status and some perspectives of relativistic quantum physics.

1 Introduction

The end of the first half of the century coincided with a notable incision in the search for fundamental laws. The breakthrough in the handling of Quantum Electrodynamics had shown that old equations contained much more physically relevant information than one had dared to believe. It had restored faith in the power of quantum field theory. But side by side with the dominant feeling of great triumph there was a spectrum of mixed feelings ranging from bewilderment to severe criticism.

Dirac emphasized that there was no acceptable physical theory but only an ugly set of rules. Heisenberg felt that the success of renormalization had turned the minds

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away from the really important issues in shaping a new theory. Still there was the empirical fact that QED was capable of producing numbers agreeing with experiments to an unbelievable degree of accuracy without any radical changes in its foundations and that there lacked any indication that the general scheme of quantum field theory was at odds with experiments in high energy physics, though there were obviously great difficulties in eliminating conceptual and mathematical muddles abounding in the existing formulation. So it appeared that the time called for a period of consolidation, of patient work devoted to the separation of golden nuggets from the mud. What constitutes a quantum field theory? What is needed to extract the relevant physical information?

It is not our intention to present in this essay a retrospective of developments in the past fifty years. But it is important to recall some attitudes and prejudices prevailing at various periods, to recall the questions asked then and see to what extent they have been answered in a satisfactory way in order to have a basis for the assessment of open questions today, to recognize tasks and perspectives. Therefore we shall begin with a brief sketch of endeavors in the fifties and sixties. Since our article necessitated a severe restriction in the topics addressed and thus an unavoidable bias in the selection of references, it should not be used as a source for the “history of science”. We shall not be concerned with the disentangling of “who contributed what and when”. We shall also suppress technical details as much as possible and refer the reader to the easily accessible books, where detailed references may be found and the methods and techniques alluded to are fully described. For the first sections most of this is given in references [1] to [4].

For the most part, we shall use the language of an approach which is often, but inappropriately, called “Algebraic Quantum Field Theory”, because we feel that it provides the simplest and most natural formulation in which the relevant principles can be expressed and it also provides a powerful mathematical structure which can be precisely described and applies to a wide area. No quantum fields appear in this formulation. In fact, the relation to quantum fields is not as close as originally believed. In particular, it is important that it can also incorporate extended objects which generalize the field concept. Thus a better name is “Local Quantum Physics”. For details and references see [5].

Our main aim is to describe the questions that presented themselves at various times, follow the changes of perspective needed in answering them and indicate open questions to which we do not know the answer and which might suggest tasks to think about in the future.

2 Taking stock

In the construction of models in quantum field theory one usually starts from a classical field theory and tries to “quantize” it following as closely as possible the rules which had proven so successful in the transition from classical to quantum mechanics. The dynamical variables are now a set of fields which transform covariantly under some finite dimensional representation of the Lorentz group (e.g. spinors, vectors). The key element in characterizing the model is the Lagrangian from which the equations of motion and commutation relations can be guessed. One novel feature appeared: For some fields the commutator had to be replaced by the anticommutator in order to comply with the Pauli principle.

This scheme was immediately successful in the case of free fields. Such a field can be decomposed into a positive and a negative frequency part, yielding annihilation and creation operators for some type of particle. The theory then just describes an arbitrary number of identical, non-interacting particles. This feature was interpreted as a manifestation of the well known wave-particle dualism. There was, however, no easy way to extend this formalism to a theory of interacting fields. It became clear that the commutation relations could no longer be the canonical ones but must have stronger singularities; that the equations of motion were sick because the product of fields at the same point defied any simple definition; that one had to think more carefully about the relation between fields and particles.

What to keep and what to throw overboard? We shall divide the tentative answers given into two groups. The first, group *A*, concerns the general setting, the second, group *B*, the field concept. In the first group we have:

*A*₁) Keep the notion of space-time as a classical manifold with pre-given geometry (the Minkowski space \mathcal{M}) as the arena in which physics plays. Its symmetry group is the “Poincaré group”, generated by translations and Lorentz transformations.

*A*₂) Keep the standard formalism of quantum physics in which pure states are described as “rays” in a Hilbert space \mathcal{H} (unit vectors up to a phase factor) and observables as self adjoint operators acting in \mathcal{H} .

*A*₃) Incorporate the results of Wigner’s analysis: A symmetry is implemented by a “ray representation” of the symmetry group. In the case of the Poincaré group \mathcal{P} this is equivalent to a representation of the covering group $\tilde{\mathcal{P}}$ by unitary operators.

This provides already important physical information. For instance, the infinitesimal generators of the translations P^μ may be interpreted as observables corresponding to the total energy-momentum. It is a purely mathematical problem to determine

all irreducible representations and this problem has been solved. It turns out that an irreducible representation with positive energy describes the state space of a single stable particle. All other representations can be constructed by direct sums and tensor products from the irreducibles. Since the restriction to positive energies seems to be well motivated (for instance to ensure stability), one comes to the first basic postulate (axiom, principle):

S) The spectrum of the energy–momentum operators P^μ in \mathcal{H} is restricted to the closed forward cone $\overline{V}_+ = \{p : p_0 \geq |\mathbf{p}|\}$. One usually also assumes that there is a unique ground state Ω , the vacuum.

Since we are talking about field theory, we decide in group *B*:

*B*₁) Keep the idea that the basic dynamical variables, in terms of which all operators in \mathcal{H} should be expressed, are fields.

The naive idea that a field φ assigns to each space–time point x an operator $\varphi(x)$ in \mathcal{H} is not tenable. Therefore a considerable amount of mathematical care and sophistication is needed to avoid pitfalls. One may consider a field as an “operator valued distribution” on a suitably defined domain in \mathcal{H} or as a sesquilinear form on this domain. This being done, one may formulate the postulate *B*₂:

*B*₂) The theory is completely described by a finite number of covariant fields (each having a finite number of components).

Why fields? This question is asked at regular intervals. One strong argument was, of course, the success in QED. But more deeply, the notion of field allows us to encode the relativistic causal structure of space–time in the theory and this is implemented by the basic locality postulate.

L) Field quantities in regions which lie space–like to each other either commute or anticommute.

Experiments in high energy physics are concerned with particles and cross sections, not with fields. So one needs to know the connection between fields and particles. An important step in this direction were the asymptotic relations of Lehmann, Symanzik and Zimmermann [6] which provided an elegant algorithm relating correlation functions of fields to S–matrix elements.

The strategy of starting from precisely defined postulates, analyzing their consequences and focusing first on general structure instead of specific equations, created an enterprise with rather novel style (“Axiomatic Quantum Field Theory”). The emphasis on mathematical rigor, stating results in the form of theorems and lemmas, was instrumental in establishing a very fruitful discussion between mathematical

physicists and pure mathematicians, closing a deplorable gap. On the other hand, it was not to the taste of all parts of the physics community, as illustrated by a joke circulated in the early sixties: “The contribution of axiomatic quantum field theory to physics is smaller than any preassigned positive number ε ”. To balance this joke, we should also mention another one: “In the thirties, under the demoralizing influence of quantum theoretic perturbation theory, the mathematics required of a theoretical physicist was reduced to a rudimentary knowledge of the Latin and Greek alphabets” (Res Jost, as quoted in [1]).

Well, there is no point in arguing with good jokes. But they do contain messages which should be taken seriously. A belief that mathematics is the prime mover for progress in physics is not warranted and if it leads to an overemphasis on mathematical rigor, there is the danger of distracting attention from the essential points and contributing to a language problem so that different camps abandon the effort to understand each other’s vocabulary. On the other hand, a narrow view of what constitutes “real physics” fosters an ill founded snobbism. It takes many kinds of craftsmen to construct a building. The enterprise whose origins were sketched above did contribute to “real physics” in many ways. Not only by clarifying issues and proving or disproving conjectures but also by providing tools, essential for many subsequent developments. A prime example are the analyticity properties of n -point functions, derived as consequences of postulates S and L by Wightman [7], seminal for a variety of subsequent developments (dispersion relations, renormalization theory, Euclidean formulation etc.). And it raised new questions. One of them concerned a deeper understanding of the relation between fields and particles. We shall devote the next section to this.

3 Fields and particles

In the early years of quantum field theory, the prevailing picture was: There are a few types of “elementary” particles which serve as the building stones for more complex structures (from nuclei to crystals) and to each species of elementary particle there corresponds a fundamental quantum field. If this is a good picture then the relation between a basic field and the states of the corresponding particle (isolated before or after a collision) is adequately described by the LSZ-formalism and these relations may be regarded as an “asymptotic condition” needed for a reasonable interpretation of the theory. The problem of how to deal with composite particles (“bound states”) could be postponed as a later worry.

But evidence from a variety of sources eroded this simple picture. It was difficult

to decide whether newly discovered particles should be regarded as elementary and honored by associating a new basic field with them. It was recognized that no simple and clear distinction between elementary and composite was available, neither for fields nor for particles. In the process of developing a collision theory for composite particles, it was discovered that the LSZ-formalism could be rather easily extended to cover also this case. Furthermore, there was no need for any close connection between particle type and basic field. The asymptotic condition was not an extra condition but a consequence of the postulates S and L . All that was needed was the existence of a discrete part in the mass spectrum (single particle states, no matter whether elementary or composite) and the existence of some “quasilocal operators” connecting the vacuum with these states.¹

The term “quasilocal” brings us back to the original significance of the field concept, namely the establishment of a relation between space-time and the dynamical variables of the theory, to allow us to characterize those operations which (at least approximately) pertain to a specific region in space-time. In other words: there is no general field-particle duality. A particle is a stable quasilocal excitation. The determination of the types of particles appearing in the theory is a dynamical problem which bears some analogy with the determination of the ground state of an atom with one important difference. We cannot regard the particle as a composite of discrete, elementary objects.

But what about leptons, quarks and the parton picture? Is the success of the “Standard Model” not evidence to the contrary? Not really. The message of the Standard Model has much in common with the message received from QED. It does suggest that a field theory in which the property B_2 is concretized by a specific set of fermionic fields and gauge fields has more physical relevance than one dared to hope. But again this achievement is accompanied by a host of puzzling questions. From a formal point of view we have a successful field theory encompassing the postulates B_2 and L . But the dynamical variables do not operate in the space of physical states [12, 13]. The road to the Hilbert space to which the above mentioned items A_2 , A_3 and S refer is quite involved. This applies a fortiori to the description of particles,

¹For a long time (in many textbooks till present days) the formulation of collision theory was based on the comparison of a “free Hamiltonian” H_0 with the full Hamiltonian H . To adapt this formulation to many channel reactions with “bound states”, even in non relativistic quantum mechanics one had to find a different separation of free motion and interaction for each channel, a procedure which was not only difficult but highly nontransparent if due account of the Pauli principle was taken (the dispute about “post-prior antisymmetrization”). To our knowledge, the first natural approach to the problem was due to H. Ekstein [8] (compare also [9, 10]). The results in quantum field theory mentioned above were obtained in full clarity by D. Ruelle [11].

as illustrated by the fragmentation models used in the discussion of jets in high energy reactions. The important progress, the salient feature, is the discovery of a relevant set of charge quantum numbers (color, flavor, electric, weak). The finite number of elementary objects refers to these, not to particles. The basic fields are the vehicles to handle the creation and transport of such charges. But none of these fields is observable in the sense of A_2 .

This accentuates an old question. If postulate L intends to express only the relativistic causal structure of space–time, then it should simply read

L_O) Observables relating to space–like separated regions commute.

The intrinsic information of the theory (as contrasted to the particular way in which the theory is described) ultimately concerns the relation between observables. This raises the question: How can we characterize the intrinsic structure and what is the role of quantum fields in it? This will be addressed in the next section.

4 Fields and algebras

If we want to adhere to the first group of assumptions, from A_1 to S in Section 2, and want to incorporate the causality principle in the form L_O , then we must classify the observables according to space–time regions, i.e. focus on a correspondence

$$\mathcal{O} \longrightarrow \mathcal{R}(\mathcal{O}) \tag{4.1}$$

between space–time regions \mathcal{O} and the algebras $\mathcal{R}(\mathcal{O})$ generated by the observables in the respective region. More precisely, \mathcal{O} shall denote an open, bounded region in \mathcal{M} .² As elements of the algebras we may take bounded operators acting in \mathcal{H} , thereby escaping the complications with domain problems. After all, the most elementary observables are projectors. This suggests that we specify $\mathcal{R}(\mathcal{O})$ to be von Neumann algebras (also called von Neumann rings).³ It appears to be the most natural choice since a von Neumann–algebra is the set of all bounded operators which commute with a given set of others. This fits well with the form L_O of the causality requirement.

In addition to the von Neumann algebras $\mathcal{R}(\mathcal{O})$, acting on \mathcal{H} , we have a representation of the Poincaré group by unitary operators, satisfying the spectrum condition S , and a distinguished Poincaré invariant pure ground state, the vacuum. An individual algebra $\mathcal{R}(\mathcal{O})$ carries hardly any physical information. It is the relation between

²To fix ideas, one may think of “double–cones”, the causal completions of open balls in space, e.g. $\mathcal{O}_r = \{x : |x_0| + |\mathbf{x}| < r\}$, and their Lorentz transforms.

³A brief account of some basic notions in the theory of operator algebras is given in the Appendix. For a thorough understanding of this area of mathematics see, for example, [14, 15].

algebras of different regions and the correspondence (4.1) which contains the physics. This correspondence requires, on the mathematical side, that the set $\{\mathcal{R}(\mathcal{O})\}$ of algebras satisfies certain structural requirements [5]. The essential ones are

- a) inclusion relations (“isotony”), i.e. $\mathcal{O}_1 \subset \mathcal{O}_2$ implies $\mathcal{R}(\mathcal{O}_1) \subset \mathcal{R}(\mathcal{O}_2)$,
- b) causality (the principle $L_{\mathcal{O}}$), and
- c) covariant action of the Poincaré group on the local algebras.

We shall call the set $\{\mathcal{R}(\mathcal{O})\}$ of algebras, labelled by the space–time regions \mathcal{O} and endowed with this structure, the “net” of local algebras.

It is clear that the correspondence (4.1) provides a starting point for the physical interpretation. It is, however, remarkable that nothing more is needed. In other words, the net of algebras defines the theory, including the full physical interpretation. Once the net is given, we can analyze its physical predictions in terms of particles, collision cross sections, etc. We shall not describe this here but just indicate the reasons. The net allows us to construct the mathematical counterparts of coincidence arrangements of detectors. The specification of what the individual detector detects need not be fed in. It suffices that we can extract information about the energy–momentum range it selects, using the action of the Poincaré group, and that we have information about the placement, using the net structure. Exploiting judiciously just the information from different geometric constellations one is able to disentangle the particles and their collision cross sections which are described by the net. In this analysis the spectrum condition S plays a significant role. The essential arguments are given in [16, 17]. For further details see [5].

Thus we may say that a net $\mathcal{O} \longrightarrow \mathcal{R}(\mathcal{O})$ gives an intrinsic description in which the physically relevant information is encoded. One might therefore conjecture that quantum fields should be regarded as a (more or less convenient) way to coordinatize the net. This point of view is supported by a result which Borchers obtained in the context of the LSZ formalism [18]. We get, for instance, the same physical information whether we consider a free field φ_0 or its Wick power $\varphi = : \varphi_0^3 :$ as the basic field, though there is an obvious difference in convenience. A less trivial example, where the identification of two at first sight very different looking field theories required much more work can be found in [19]. Other surprising examples have attracted much attention. So the equivalence of the Thirring model with the Sine–Gordon model [20] and recently explored equivalences in supersymmetric Yang–Mills theories [21].

As experienced in other areas of study, depending on the problem and on the taste of the investigator, there are advantages in the use of coordinates and there are advantages in using an intrinsic (coordinate free) formulation. So the clarification of

the relation between fields and algebras is an important issue.

From fields to algebras:

Heuristically one would like to define $\mathcal{R}(\mathcal{O})$ as the von Neumann algebra generated by the (smoothed out) observable fields in the region \mathcal{O} . Appealing to von Neumann’s double commutant theorem [14], this may be symbolically written as⁴

$$\mathcal{R}(\mathcal{O}) = \{\varphi(x) : x \in \mathcal{O}\}'' \tag{4.2}$$

where φ stands for the set of observable fields. Because of subtle questions concerning the commutativity of unbounded operators, it is, however, not clear from the outset whether this heuristic idea can be really implemented. The first steps in the analysis of this problem were taken by Borchers and Zimmermann [22]. They showed that if the vacuum Ω is an analytic vector for the fields, i.e. if the formal power series of the exponential function of a smeared field applied to Ω converges absolutely, then the passage from fields to local algebras via (4.2) can be accomplished. Further progress on this problem was made in [23], where it was shown that fields satisfying so-called linear energy bounds generate acceptable nets of local algebras. This result covers most of the interacting quantum field theories which have been rigorously constructed so far in the endeavor of “constructive quantum field theory”.

As for the general situation, the most comprehensive results are contained in [24] and the references quoted there. In that analysis certain specific positivity properties of Wightman functions were isolated as the crucial prerequisite for the passage from fields to algebras. Altogether the result of these investigations could be summarized by saying that, while the original form of the Wightman axioms is not sufficient to allow the transition from fields to algebras along the lines indicated by (4.2), this can be remedied by adding some rather unsuspecting requirements.

More serious is the fact that in (4.2) we were talking about “observable fields”. As already indicated, the development of field theory has led to a situation in which none of the basic fields is observable. The proper assessment of this problem becomes, however, clearer in following the opposite road.

From algebras to fields:

As already mentioned, the characterization of the theory by a net of local algebras is more general than the traditional field theoretic approach. It covers also the

⁴The commutant \mathcal{S}' of a set \mathcal{S} of operators consists of all bounded operators commuting with the elements of \mathcal{S} ; the double commutant \mathcal{S}'' is the commutant of \mathcal{S}' . It is assumed that the sets contain with every operator also its adjoint. This must be required correspondingly for the set of fields in (4.2).

case of observables which are not built from point-like objects but are localizable in extended (though finite) regions, such as Wilson loops or (finitely extended) Mandelstam strings. Nevertheless, the point field content is of great interest since we believe that it contains such distinguished observables as the components of the energy momentum tensor, certain currents etc. Heuristically, the point fields can be recovered from the algebra by a formula like

$$\{\varphi(x)\} = \bigcap_{\mathcal{O} \ni x} \overline{\mathcal{R}(\mathcal{O})}. \quad (4.3)$$

The bar on the right hand side indicates that one cannot take the intersection of the local algebras themselves, which is known to consist only of multiples of the identity. Therefore, one first has to complete the algebras in a suitable topology which allows the appearance of unbounded operators, respectively linear forms. This was carried through in [25], where the needed completion of the local algebras was defined with the help of “energy norms” which are sensitive to the energy–momentum transfer of the observables. Using this device, it was shown that, provided that the algebras are generated from sufficiently regular fields in the sense indicated by (4.2), one can recover the fields from the algebras via (4.3). From a general point of view it would be desirable to clarify the status of point fields without assuming their existence from the outset. An interesting proposal in this direction was recently made in [26]. We shall come back to it in Section 8.

Unobservable Fields:

One circle of nagging questions was known for a long time but mostly regarded as of minor importance. It begins with the original formulation L of the causality principle. Why the Bose–Fermi alternative? In the sequel of his discussion of the ray representations of the Poincaré group, Wigner noted that the relative phase between a state vector belonging to a double valued representation (spinorial wave function) and one belonging to integer spin could have no physical meaning. Then it was recognized that such limitations of the superposition principle occur also between states of different electric charge due to the principle of gauge invariance in QED and that they may be expected in still other circumstances [27]. This called for a modification of assumption A_2 in Section 2: The Hilbert space decomposes into a direct sum of mutually orthogonal subspaces, called the “coherent sectors”, and the relative phase between state vectors in different sectors is void of physical meaning. The unobservable fields can then be regarded as operators leading from one sector to another. One might be inclined to accept these so-called “superselection rules” as a fact of life, producing some slight complication such as the appearance of unobservable fields. But this is somewhat artificial and calls for a more natural explanation.

Could it be that the coherent sectors were just the modules (representation spaces) of inequivalent representations of one basic algebra? Let us remember that, at the birth of quantum mechanics, Dirac introduced the notion of “q–numbers” defining some abstract algebraic structure and that the equivalence of this with the wave mechanical formulation (i.e. ultimately with Hilbert space operators) depended on the uniqueness proof for the representation of the canonical commutation relations. Now it was known (in circles of mathematical physicists since the early fifties) that in the case of infinitely many degrees of freedom the uniqueness theorem failed. In fact, there was an innumerable host of inequivalent representations of canonical commutation relations. So it seemed that for the interpretation of the theory one needed more than an abstract algebra.

In mathematics the theory of a class of abstract algebras which allowed representations in Hilbert space, the so–called “C*–algebras”, had been developed. Therefore Irving Segal, one of the fathers of the mathematical theory, had advocated for several years to base the physical theory on an abstract C*–algebra. But quantum field theorists who were aware of the difficulty of an appropriate physical interpretation and of the problem of overabundance of inequivalent representations had no use for this advice.

Two things were necessary before the idea of using abstract algebras could be implemented. On the one hand, strange as it may seem in retrospect, one had to recognize that we are not talking about a single algebra but about a net of algebras whose interpretation was hinged to space–time. Secondly, one had to realize that unitary inequivalence of representations was a much too fine distinction to be of any physical relevance because we can measure only with finite accuracy and consider only a finite number of observables at a time. Thus it was indeed possible and reasonable to consider the abstract algebraic structure as the primary definition of the theory and Hilbert space and representations as secondary.

To avoid confusion of concepts, we shall in the following use the Gothic letter \mathfrak{A} for a C*–algebra and \mathcal{R} for a von Neumann algebra, the symbol π to denote a representation³. Thus $\pi(\mathfrak{A})$ is a concrete algebra of operators in a Hilbert space. It leads us back to a von Neumann algebra, the double commutant

$$\mathcal{R} = \pi(\mathfrak{A})'' . \tag{4.4}$$

The reformulation of the theory so that the local algebras are considered as abstract C*–algebras $\mathfrak{A}(\mathcal{O})$ was done in [28].

So one had reached a point in relativistic quantum physics, reminiscent of the situation in quantum mechanics in 1926, where the primacy of algebraic relations was emphasized as the essence of the theory. It did suggest a natural way to understand

the appearance of different coherent sectors. But it raised new questions. On the one hand it was apparent that the selection of those representations usually discussed in quantum field theory resulted from some convenient idealizations, especially from simplifying assumptions concerning the physical situation at space-like infinity. A closer look at the “states of physical interest” showed, however, that this was not the whole truth. We have more information. This will be discussed in the next section.

5 States of physical interest

In the algebraic setting a “state” ω is considered as a positive, linear and normalized functional over the algebra \mathfrak{A} . It assigns to each $A \in \mathfrak{A}$ a complex number. It is real for self-adjoint elements and is then interpreted as an expectation value. Any such state gives rise to a representation π of the algebra on some Hilbert space \mathcal{H} , where it can be described by a unit vector (GNS-construction).³ Convex combinations of states give again states. This mathematical operation corresponds to the physical procedure of “mixing”. Conversely, pure states are the extremals in a convex decomposition (which, sometimes, may be physically meaningless).

In physics we consider primarily two classes of states, corresponding to different situations. In “particle physics” we are interested in states which are close to the vacuum, differing from it only by some more or less localizable disturbances. In statistical mechanics we are interested in states which are close to a thermal equilibrium state. These are idealized best by considering a medium with non-vanishing density, extending to infinity.⁵

One bonus of the formulation of the theory in terms of local algebras is that the common features of both classes become apparent and some powerful tools for a structure analysis in both areas emerge. Not only has the characterization of a thermal equilibrium state by the “thermal boundary condition”, arising from the work of Kubo and of Martin and Schwinger, a very simple form in the algebraic setting [29]. But it turned out surprisingly that this so-called KMS-condition has a very natural place in mathematics, the Tomita–Takesaki theory of modular automorphism groups [30], which plays a central role in the classification of von Neumann algebras.³

It was a remarkable experience that the ideas of Tomita and the paper [29] were presented at the same workshop, motivated by entirely different purposes and in complete ignorance of each other. This led to an intensive interaction between some

⁵Though one may focus on a system in a finite volume, this does not really change the above statement because one then has to specify the relation to the outside either by introducing a heat bath or by artificial boundary conditions.

groups of mathematicians and physicists from which both sides profited substantially. What is the crux of the matter? Suffice it here to say that Tomita and Takesaki studied von Neumann algebras for which there existed a vector which is both cyclic and separating for the algebra³. They found that such a vector (or rather the corresponding state) defines a distinguished one-parameter automorphism group for the algebra with some remarkable properties and a conjugation mapping of the algebra on its commutant. This group of modular automorphisms plays also an important role in physics. For instance, the extension of Gibbs' characterization of thermal equilibrium states to an infinitely extended medium is equivalent to the statement that equilibrium is described by any state whose modular group is some one-parameter subgroup of time translations and (global) gauge transformations. (In the non-relativistic limit, the latter corresponds to the conservation laws for independent species of particles instead of charges.)

In the case of zero temperature this formalism degenerates. In particular, the vacuum vector is not separating for the global algebra $\pi(\mathfrak{A})$ of all observables. However, a theorem of Reeh and Schlieder [31] tells us that it is cyclic and separating for $\mathcal{R}(\mathcal{O}) = \pi(\mathfrak{A}(\mathcal{O}))''$ whenever there is a non-void causal complement of the region \mathcal{O} . What can we say about the modular automorphism induced by the vacuum for such algebras? The first important discovery in this context was made by Bisognano and Wichmann [32] who determined these automorphisms for special regions, called “wedges”, such as

$$W = \{x : x_1 > |x_0|, x_2, x_3 \text{ arbitrary}\}. \quad (5.1)$$

They found that these automorphisms coincide with the Lorentz boosts, leaving the wedge invariant. The close connection of this fact to the Bekenstein–Hawking temperature of black holes was recognized somewhat later by Sewell [33] and more fully discussed in [34]. In the case of theories with conformal invariance, such a geometric significance of modular automorphisms could also be established for double cones [35].⁶

Besides such specific identifications, it was gradually realized that the von Neumann algebras of finitely extended regions are all of one universal type, irrespective of whether we consider thermal states or states in particle physics and that this is a consequence of important generic properties of the physical states. Again, this development originated from a bunch of quite different questions.

⁶For the full development of applications of modular theory to quantum field theory, see the contribution of H.–J. Borchers.

Phase space properties:

Since the connection between fields and particles is not very close and we even know field theoretical models which have no particle content whatsoever, one must ask for the conditions under which particles appear in the theory. We stated earlier that states of particles are quasilocal excitations. Naively, one would be inclined to define “localized states” by application of a local algebra $\mathcal{R}(\mathcal{O})$ to the vacuum. But the Reeh–Schlieder theorem tells us that this leads to a dense set in \mathcal{H} and all reminiscence of the region \mathcal{O} is lost. The reason for this paradox is that the vacuum state incorporates correlations between observables in far separated regions which cannot vanish exactly because of analytic properties of the correlation functions.

For a vector

$$\Psi = A\Omega \quad \text{with } A \in \mathcal{R}(\mathcal{O}) \quad (5.2)$$

there is the ratio between “cost and effect”,

$$c_A = \frac{\|A\|}{\|\Psi\|}, \quad (5.3)$$

which in general is larger than 1. If c_A is close to 1, then Ψ does describe an excitation which is approximately localized in \mathcal{O} , i.e. the expectation value of an observable in a region space-like to \mathcal{O} in this state is approximately equal to the vacuum expectation value. But as c_A gets larger, this significance of \mathcal{O} for the interpretation of Ψ is lost. In other words, denoting the unit ball of \mathcal{R} by \mathcal{R}_1 (the set of all $A \in \mathcal{R}$ with $\|A\| \leq 1$), the set of vectors $\mathcal{R}(\mathcal{O})_1\Omega$ characterizes a part of \mathcal{H} which, apart from vectors of very small length, describes approximate localization in \mathcal{O} .

If we choose for \mathcal{O} a bounded region, for instance the double cone² \mathcal{O}_r , and impose in addition a restriction of the total energy (and thereby also of the total linear momentum), we get a part of \mathcal{H} which we can attribute to a bounded region of phase space. The restriction of energy can be done, in a somewhat brutal fashion, by applying the projection operator P_E for energies below E to the set of vectors $\mathcal{R}(\mathcal{O}_r)_1\Omega$. A smooth cutoff function of the energy, such as $e^{-\beta H}$ for some sufficiently large positive β , is mathematically more convenient and leads to the subset of vectors in \mathcal{H}

$$\mathcal{N}_{\beta,r} = e^{-\beta H} \mathcal{R}(\mathcal{O}_r)_1 \Omega. \quad (5.4)$$

It was argued in [36] that a necessary condition for a physically reasonable theory is the “compactness criterion”

C) The set of vectors $\mathcal{N}_{\beta,r}$ is compact in the norm topology of Hilbert space.

In other words: for any choice of a positive number ε , the vectors in $\mathcal{N}_{\beta,r}$ with norm larger than ε are contained in the unit ball of some finite dimensional subspace of \mathcal{H} . As $\varepsilon \rightarrow 0$, the dimension $N_\varepsilon \rightarrow \infty$.

Twenty years later Buchholz and Wichmann [37] realized within a different context that the estimates in [36] could be considerably improved and that the criterion C should be replaced by

N) The set $\mathcal{N}_{\beta,r}$ is a nuclear set⁷ for sufficiently large β .

They argued that this requirement together with certain bounds on the nuclearity index in its dependence on r and β is necessary to ensure known thermodynamic properties, cf. also [38, 39] for further applications of this condition in the analysis of thermal states.

Several variants of the compactness and nuclearity criteria have been proposed. We shall not touch here the extensive work about their relation and consequences. References may be found in [5]. Rather, we shall focus in the following on an aspect which emerges from the foregoing discussion. Irrespective of whether we consider thermodynamics or particle physics, the von Neumann algebras of all bounded, contractible regions (such as double cones) are isomorphic.

The universal structure of local algebras:

In [40] Fredenhagen studied the following geometric constellation: the wedge W , defined in (5.1), and enclosed in it a sequence of double cones \mathcal{O}_{r_n} , tangent to the wedge at the origin, with decreasing radius $r_{n+1} = \lambda r_n$ for fixed $\lambda < 1$, so that they contract to the origin as $n \rightarrow \infty$. He found that a non-trivial “scaling limit” of the corresponding algebras could only exist if all the double cone algebras are of type III₁³. In [41] it was shown that the phase space properties imply that the local von Neumann algebras are hyperfinite. Moreover, according to our present knowledge, their center is trivial⁸, they are “factors”³. In [42] Haagerup had shown that all hyperfinite factors of type III₁ are isomorphic. Thus we conclude that all local algebras are isomorphic to a uniquely defined and well-studied mathematical object. This emphasizes once more that the physical information is not carried by a single algebra. We may compare this with the situation in non-relativistic quantum mechanics, where we encounter only type I algebras, irrespective of the system considered.

⁷It is contained in the image of the unit ball of \mathcal{H} under the mapping by a positive trace class operator. The trace of this operator, the “nuclearity index”, is a measure for the size of this set.

⁸The only physical reason for the appearance of a non-trivial center would be the possibility that superselection rules arising from the charge structure might be recognizable already within a bounded region. But there are good arguments against this. Still, a more careful consideration of this in the regime of local gauge theories might be warranted.

Split property:

Of special interest are inclusion relations between algebras, see Section 9. We shall address here the case $\mathcal{R}(\mathcal{O}_1) \subset \mathcal{R}(\mathcal{O}_2)$, where the closure of the region \mathcal{O}_1 is contained in the interior of the bounded region \mathcal{O}_2 . The “split property” asserts that then there exist “intermediate” factors of type I such that³

$$\mathcal{R}(\mathcal{O}_1) \subset \mathcal{N} \subset \mathcal{R}(\mathcal{O}_2), \quad (5.5)$$

where \mathcal{N} denotes such a factor (to which we can, however, not assign a definite localization region). One of the consequences is the “statistical independence” in the situation where two regions \mathcal{O}_A and \mathcal{O}_B are space-like separated so that there is a region \mathcal{O} which properly contains \mathcal{O}_A and is disjoint from \mathcal{O}_B . In this case the von Neumann algebra generated by the two local algebras is isomorphic to their tensor product. In symbols

$$\mathcal{R}(\mathcal{O}_A) \vee \mathcal{R}(\mathcal{O}_B) \simeq \mathcal{R}(\mathcal{O}_A) \otimes \mathcal{R}(\mathcal{O}_B). \quad (5.6)$$

This may be regarded as a strengthened form of the locality postulate. It tells us that there are states which have no correlations between the two regions and that the Hilbert space in which the two algebras act can be factored into a tensor product $\mathcal{H}_A \otimes \mathcal{H}_B$, analogous to the notion of subsystems in non relativistic quantum mechanics. This, incidentally, implies that the discussion of entanglement and non-locality of EPR-correlations can be done in the same way as in quantum mechanics. But the distinction between causal effects, which are restricted by L_O , and EPR-correlations, which may persist over (large) spatial distances, is seen more clearly in the relativistic setting [43].

If two observers, nowadays called Alice and Bob, operate in two laboratories, there is nothing that Bob can do which changes the statistics of any experiment which Alice can make, as long as they operate at space-like separation. These statistics are governed by a well defined “partial state”, referring to the lab of Alice, which necessarily is impure because it ignores the situation outside her lab and it does not depend on Bob’s activities. However, if they look at the statistics of a coincidence experiment in a state representing a (common) ensemble, there will in general be correlations in the *joint* probability distribution.

Such correlations persist over large distances if they are related to some conservation law. This is not the surprising aspect of the EPR-correlations. It is encountered in “classical” situations, for instance a state of charge zero decomposing into two subsystems A and B , carrying opposite charge. Obviously, if Alice finds positive charge, she knows that Bob must observe negative charge in a coincidence experiment. The

quantum aspect comes from the possibility that Alice as well as Bob each can decide to choose among a set of incompatible measurements (which mutually are compatible). The correlations observed in the pairs of such choices are such that they cannot be explained if one tries to describe the total state as a probabilistic distribution over pairs of states of “subsystems” in any realistic sense. The separation of the observation labs does not correspond to a partition of the system into realistic subsystems. The notion of “state” should not be interpreted as the “mode of existence” of some “object” with ontological significance. It describes probability assignments for the occurrence of events (here the clicks of detectors in the labs of Alice and Bob). These are real and localized.

The split property has been considered for some time as an additional assumption and “standard split inclusions” have been studied in detail in [44]. The recognition in [37, 41] that it is a consequence of the phase space properties came as a gratifying surprise. If the level density in a theory increases too fast, then a finite distance between the boundaries of the regions \mathcal{O}_1 and \mathcal{O}_2 may be necessary for relation (5.5) to hold. If the regions have a common boundary point, then (5.5) does not hold.

6 Charges and statistics

The transition from operator algebras to abstract algebras in [28] was motivated by the desire for a natural understanding of the role of different coherent sectors whose existence had been pointed out in [27]. Related to this there was the challenge to understand the reasons for the Bose–Fermi alternative. The arguments in quantum mechanics were not adequate because their starting point, namely the description of a state of several indistinguishable particles by a wave function in configuration space, took for granted part of what had to be explained. Indeed, other possibilities called “parastatistics” had been suggested by H.S. Green [45].

In [28] it was argued that the choice of a particular representation was largely a matter of convenience and that superselection rules concerning charge and spin must result from idealizing the situation at space-like infinity. Since charges are of eminent physical importance, the necessary idealizations had to be understood clearly and related to the appearance of charge sectors and statistics. The extensive work devoted to this task in the seventies and eighties need not be reported here in detail. This is described in Chapter IV of [5] and in [46, 47], where the pertinent references may be found.

We shall take here, however, a closer look at the idealizations used and the ensuing differences in the properties of charges arising from them.

Sharply localizable charges:

The approach by Doplicher, Haag, Roberts (DHR picture, [48]) started from the idealization that we restrict attention to all those states ω which become indistinguishable from the vacuum ω_0 by any observation in the causal complement of a sufficiently large double cone² \mathcal{O}_r . In symbols,

$$\|\omega - \omega_0\|_{\mathfrak{A}(\mathcal{O}'_r)} \rightarrow 0 \quad \text{as } r \rightarrow \infty, \quad (6.1)$$

where \mathcal{O}'_r denotes the causal complement of \mathcal{O}_r .

It was clear from the outset that thereby one excludes electric charges from consideration because the flux of the field strength through a sphere of arbitrarily large radius measures the charge. But it was believed then that this feature was intimately related to the zero mass of the photon and could not happen in a theory with a mass gap. Though it is true that Gauss' law excludes the possibility of a mass gap [49], it turned out that condition (6.1) is too stringent a restriction even for a purely massive theory (see below).

From (6.1) it followed that there are also states in which the charge is localized sharply in a bounded region. This is surprising in view of the discussion in the last section where we saw that the notion of "localization of a state" is in general only a qualitative one. Indeed, this feature, following from the idealization (6.1), will have to be modified, most significantly in Section 7. It follows further in this picture that the charge structure arises from the existence of localized automorphisms or endomorphisms of the net, i.e. mappings which preserve the algebraic relations and act trivially on the algebras in the causal complement of some region \mathcal{O} . We call such mappings local morphisms. The main consequences of this picture, derived in [48, 50], can be quickly summarized:

- a) There is charge conjugation symmetry.
- b) There is a composition law of charges with permutation symmetry. It leads to an elaborate calculus of intertwiners between morphisms localized in different regions.
- c) Each charge is either of bosonic or fermionic type and this distinction is reflected in the intertwiner calculus. Note that this appears as a consequence of the intrinsic structure, outlined in Section 4, without ad hoc introduction of anticommuting elements.
- d) There is a compact group associated with the charge structure. It may be interpreted as the global gauge group. This is a deep result, obtained by Doplicher

and Roberts [50]. By establishing a novel duality theorem in group theory, they showed that the structure of local morphisms and intertwiners, alluded to above, defines precisely the dual object of a group.

- e) Elementary charges correspond to irreducible representations of this gauge group. The dimension of the representation corresponds to the order of the parastatistics associated with the charge.

The simplest case, where the gauge group is the Abelian group $U(1)$, gives the well-known situation where all integer values of the charge appear and we have standard Bose or Fermi statistics. For non-Abelian groups we get parastatistics.

These results are put in proper perspective if one carries through the same analysis for the case of two-dimensional space-time. There one essential step, namely the exchange of the position of two space-like separated charges by continuous motion, keeping them always space-like, is no longer possible and this leads to a much more complicated structure [51, 52]. The permutation group is replaced by the braid group, the Bose-Fermi alternative is changed by the appearance of anyons and plectons. In the case of less sharply localizable charges, discussed below, these features appear already in three-dimensional space-time. By now, there is an extensive literature covering these aspects. References and surveys are given in [47, 53].

Charges localized in space-like cones:

Borchers proposed a different selection criterion: Consider all representations satisfying the spectrum condition S . This also serves to exclude states where the matter density does not vanish at infinity, but it is weaker than the DHR-criterion. Buchholz and Fredenhagen used this criterion, restricting the analysis to massive theories [54]. They found that the charges were not necessarily localizable in bounded regions, there can occur representations where the optimal localization of charge needs a cone extending to space-like infinity.

The results (a) to (e) remain valid in this situation [54, 50], but one needs (at least) four-dimensional space-time to rule out the braid group. Even then, time honored arguments had to be reexamined if particles carrying such “BF-charges” were involved. But it turned out that no serious changes resulted and even the dispersion relations for S-matrix elements were not affected [55]. In a massive theory, the placement and direction of the charge carrying cone plays no role for the unitary equivalence class of the representation (the superselection sector). Only the topological property that the sphere at space-like infinity has to be punctured somewhere is relevant.

Absence of mass gap:

If there are excitations of arbitrarily small energy, then the description of the set of superselection rules becomes a formidable task due to “infrared clouds”. In the case of QED the discussion of this [56] led to some interesting consequences. The optimal localizability of charge in a space-like cone remains, but different placements of this cone correspond to different infrared clouds and thereby to unitarily inequivalent representations.

Since a charged particle is always accompanied by an infrared cloud which depends on its state of motion, its mass is not sharply defined and represents only a lower bound of the energy–momentum spectrum (infraparticle problem [57, 58, 59]). But it is possible to give a precise meaning to the notion of “improper state of a charged particle with sharp four–momentum” (a generalized Dirac ket) as a “weight” on the algebra \mathfrak{A} . In contrast to the case of neutral particles, a superposition of these improper states to form a wave packet with specified localization properties is not possible. Nevertheless, a clear formulation of collision theory for charged particles and hard photons is available [17].

Summary and questions:

The analysis described in this section started from the aim of understanding the superselection structure. In the massive case it led to the appearance of charge quantum numbers related to equivalence classes of local morphisms (charge creation). The laws of composition and conjugation of these morphisms miraculously turned out to correspond precisely to the dual object of a compact group (global gauge group). Locality leads to a permutation symmetry whose implementation demands (para) Bose or Fermi statistics. If, instead of Minkowski space, one considers theories in lower dimensions, then these statements must be modified.

But there is evidence that the charge structure has deeper roots and is not necessarily reflected by superselection rules. In the Standard Model it is associated with a “principle of local gauge invariance” which has not been incorporated in the scheme discussed so far. Superselection rules and observable charges appear only as the survivors in the “unbroken part” of a very large symmetry. The notion of “spontaneous symmetry breaking” relates to the possible existence of different “phases” with different behavior at space-like infinity, e.g. “long range order”. The observable charges may depend on the phase. We shall address the meaning of “symmetry” in general and of the “local gauge principle” in particular in the next section.

7 Symmetries, local gauge principle

The word “symmetry” has several connotations. In A_3 we used it in the “active” sense. An element of the symmetry group changes the physical situation, noticeable by an observer, to an equivalent situation. By “change of situation” we mean in standard quantum theory that the state and all observables are altered. “Equivalent” means that all laws of nature apply in unchanged form in the new situation. More often, however, symmetry is understood in the “passive” sense as providing alternative descriptions for the same physical situation, expressing the fact that the (known) laws of nature do not distinguish a preferred way of coordinatization within a class of equivalent ones. In either case, we have to consider reference frames (coordinate systems) for the symmetry group and for the objects on which it acts.

In the case of the Poincaré symmetries we have implicitly assumed that an observer can establish a global reference frame in Minkowski space (fixing a point as the origin and a Lorentzian tetrad). He does this with the help of some macroscopic bodies and clocks which are not included in the physical system considered in the theory. (They are part of the “observer side” of the Bohr–Heisenberg cut.) Then, keeping this reference frame fixed, he can interpret the Poincaré transformations in the active sense. We shall accept this idealization here and ignore its limitations (indicated by general relativity on the one hand and the quantum nature of bodies used in the establishment of the frame on the other hand).

The active interpretation is, however, not possible in many cases where the symmetry we speak about is more indirectly inferred (or assumed) and the macroscopic objects available do not define a reference frame which the observer could control and change. Thus the global gauge groups mentioned in the last section may be regarded as describing a symmetry of the theory. But the objects on which the group acts cannot be measurable quantities in the sense of standard quantum theory. They cannot be accommodated in the algebra of observables because there is no operational way of establishing a reference frame relating to this group and thus the observables cannot depend on it. They must be invariants. One can extend the net of observable algebras $\mathcal{R}(\mathcal{O})$ to a net of “field algebras” $\mathcal{F}(\mathcal{O})$ on which the global (compact) gauge group G acts [50]. But linear relations between elements of $\mathcal{F}(\mathcal{O})$ which transform according to inequivalent representations of G are void of physical meaning. This reflects the limitations of the superposition principle [27] and the related feature that the causality requirement, expressed by $L_{\mathcal{O}}$, does not apply to the net $\mathcal{F}(\mathcal{O})$.

Apart from the possibility of fixing a reference frame in an operational way, there is another problem whose recognition was one of the keys leading to the development of the theory of general relativity: The comparison of reference systems used by

observers in different regions is ambiguous because it requires some bridge connecting them (some transport of information [60]) and the choice of the bridge plays a role. If we regard *local* gauge transformations as an internal symmetry, then both aspects enter. There is no observable way to fix a reference frame and there is no unambiguous way of comparing frames in different locations. The observables must be independent of the choice of these frames. Thus, to incorporate the local gauge principle by specifying an internal symmetry group, we must again augment the algebraic scheme.

A symmetry is expressed by a mapping of the mathematical structure onto itself. In the case of the structure outlined in Sections 4 and 5, this means that it is described by an automorphism α of the algebra \mathfrak{A} which conserves the net structure. In other words, the image of the algebra $\mathcal{R}(\mathcal{O})$ must again be the algebra of some space–time region \mathcal{O}_α ,

$$\alpha\mathcal{R}(\mathcal{O}) = \mathcal{R}(\mathcal{O}_\alpha). \quad (7.1)$$

Since this should hold for arbitrarily small regions, the map $\mathcal{O} \rightarrow \mathcal{O}_\alpha$ must result from a point transformation g_α ,

$$x \rightarrow g_\alpha x, \quad (7.2)$$

which furthermore has to conserve the causal structure in Minkowski space. For a precise discussion see [61]. This limits g_α to the elements of the Poincaré group, possibly extended by dilations. Apart from these “geometrical symmetries”, which change the regions, there may be also internal symmetries, corresponding to automorphisms transforming each $\mathcal{R}(\mathcal{O})$ onto itself,

$$\alpha\mathcal{R}(\mathcal{O}) = \mathcal{R}(\mathcal{O}), \quad (7.3)$$

a prominent example being charge conjugation.

An automorphism of \mathfrak{A} is called “inner” if it is implemented by a unitary element U belonging to the global C^* -algebra \mathfrak{A} ,

$$\alpha A = UAU^{-1}, \quad A \in \mathfrak{A}. \quad (7.4)$$

None of the global symmetries can be inner since they act non–trivially on observables localized arbitrarily far, whereas \mathfrak{A} contains only quasilocal elements. However, it appears that a much more important notion is local implementability of (possibly only local) symmetries. This means that we focus attention on the action of α on some chosen algebra $\mathcal{R}(\mathcal{O})$. Then α may be called “locally inner” if there exists some finitely extended region $\widehat{\mathcal{O}}$ such that

$$\alpha A = UAU^{-1} \quad \text{for some } U \in \mathcal{R}(\widehat{\mathcal{O}}) \text{ and all } A \in \mathcal{R}(\mathcal{O}). \quad (7.5)$$

It is then important to characterize a lower bound for the choice of $\widehat{\mathcal{O}}$. If the symmetry shifts \mathcal{O} to \mathcal{O}_α , then the split property implies that one may choose for $\widehat{\mathcal{O}}$ in (7.5) any connected region which contains the closure of the causal completion of $\mathcal{O} \cup \mathcal{O}_\alpha$.

This entails the following analogue to Noether’s theorem in the algebraic setting: The infinitesimal generator of any continuous symmetry is locally implemented by a Hermitian operator which is affiliated with a slightly larger region [62]. (If the charge structure is adequately described by the considerations in Section 6, then this holds likewise for the global gauge symmetries acting on the field algebra.) One may expect that these local generators determine a pointlike field (density) ρ in the limit as \mathcal{O} shrinks to a point (cf. Section 4). Moreover, if the symmetry commutes with time translations, there should hold a continuity equation,

$$\dot{\rho} = \operatorname{div} \mathbf{j}, \tag{7.6}$$

where \mathbf{j} is again a Wightman field. As discussed in [62], there remain some unresolved ambiguities in carrying through this intuitive argument whose significance is not yet properly understood. But these remarks may indicate the role of pointlike fields, such as the components of the energy–momentum tensor and certain currents within the general scheme.

Let us finally discuss the case of internal symmetries relating to the local gauge principle. There the mathematical structure referring to the non–invariant elements is much more subtle. To fix ideas, let us think of an internal symmetry group like $SU(2)$ or $U(1)$. In the classical theory, the appropriate structure is described by fiber bundles in which the notion of a field configuration in some region \mathcal{O} is replaced by that of a section in some (associated) bundle. The section obtains physical relevance only in conjunction with a connection. This demands that we endow the region \mathcal{O} (assumed to be contractible) with a collection \mathcal{C} of paths linking a fixed reference point $x_0 \in \mathcal{O}$ uniquely to every other point $x \in \mathcal{O}$. Secondly, that we attach a “charge transporter” $\Gamma_{x_0 x}$ to each such path with the help of the connection form. Using this device, we can compare the elements in different fibers and thus obtain the analogue of an ordinary field, say $\varphi_{\mathcal{C}}(x)$, for which the algebraic operations of addition and multiplication at different points are meaningful and for which the transformation by elements of the symmetry group, referring now to all of \mathcal{O} , is defined.

In adapting this to the non–commutative situation, we meet several difficulties. The first concerns the proper assessment of the singular quantities associated with points and lines. Since the connection forms are no longer ordinary functions, there is the question of whether the $\varphi_{\mathcal{C}}(x)$ can still be regarded as “operator–valued distributions”, as assumed for ordinary fields in Section 2. This is presumably not the case. But let us for the moment ignore this problem and proceed as in Section 4. This

would lead us to local field algebras $\mathcal{F}_C(\mathcal{O})$. The fixed points within $\mathcal{F}_C(\mathcal{O})$ under the action of the group could be interpreted as elements of the observable algebras $\mathcal{R}(\mathcal{O})$ which have significance without reference to the choice of C . If furthermore we could find within $\mathcal{F}_C(\mathcal{O})$ subspaces of partial isometries transforming under a specific irreducible representation of the symmetry group, then we could define endomorphisms for $\mathcal{R}(\mathcal{O})$, in complete analogy to the case of global gauge symmetries. We would thus arrive in the area studied in [50] with one difference: instead of “localized endomorphisms” for all of \mathfrak{A} , we would have to consider now endomorphisms restricted to the algebra $\mathcal{R}(\mathcal{O})$ for some region. Also, we would follow the path in [50] in the opposite direction. Instead of starting with the endomorphisms and their intertwiners and ending with a group, we would start from the group from which the dual structure emanates.

The purpose of the excursion in the last paragraph was just to indicate some parallelism of the superstructure met in quantized gauge theories with that found in theories with a global gauge group. The main problem with this “as if” picture comes from the feature that we cannot remedy the singular nature of $\varphi_C(x)$ just by smearing out with a test function $f(x)$, keeping C fixed. On the other hand, the algebraic relations between objects referring to different choices of C are not meaningful. Work in lattice gauge theory [63] and perturbation theory [64] indicates, however, that “quantum charge transporters” Γ_{xy} may be definable as distributions in x and y . These objects, corresponding in the field theoretic setting to finite Mandelstam strings, would allow us to construct by algebraic operations special elements in the observable algebras $\mathcal{R}(\mathcal{O})$ for which we can distinguish two kinds of supports: “charge supports”, relating to the supports of the test functions in x , respectively y , used for the smearing of Γ_{xy} , and a “causal support”, involving in addition a bridge region between the charge supports. The intrinsic significance of the notion of “connection” should then be understood by studying the effect of cutting such objects between disjoint charge supports.

Let us add one comment. In general the group structure just distinguishes conjugacy classes within the group so that a reference frame is also needed to characterize individual group elements. In the case of an Abelian group this is not necessary. A conjugacy class consists only of a single element. This brings some simplifications since the symmetry may be locally implementable by an observable field ρ (the charge density in QED). The trivial action of the group on the observables is then expressed by Gauss’ law, i.e. the existence of a local observable field \mathbf{E} such that

$$\rho = \operatorname{div} \mathbf{E}. \tag{7.7}$$

As already mentioned, the consequences of this feature have been studied extensively.

There are many aspects of the local gauge principle we cannot touch here (mostly because they are not worked out in adequately clear form). The elaboration of this rich structure to a degree of conciseness comparable to that of the previous sections appears to us as a task worthy of the sweat of the noble. It is presumably an essential step towards the characterization of a specific theory within the general frame described in previous sections along the lines suggested by the progress of high energy physics in the past decades.

8 Short distance structure

The previous sections were concerned with the development of a conceptual frame and corresponding mathematical structure, which can be accepted as reasonably complete and simple and provides natural answers to a variety of questions coming mainly from quantum field theory. But it is, of course, of paramount importance to see how a specific theory can be characterized within this general frame. From the observation that the local gauge principle together with a postulate of “minimal coupling” and some knowledge about the relevant degrees of freedom fixes the choice of a Lagrangian in classical field theory almost uniquely, we may surmise that the information which is needed to define a theory, beyond that supplied already by general principles and specification of internal symmetries, is encoded in the short distance structure.

Scaling algebras:

Renormalization group methods have proven to be a powerful tool for the classification of this structure in quantum field theory. So it is gratifying that they have a very simple counterpart in the algebraic approach [65]. The essential idea is to consider functions \underline{A} of a scaling parameter $\lambda \in \mathbb{R}^+$ with values in the algebra of observables. In other words, the “value” $\underline{A}(\lambda)$ is an element of \mathfrak{A} . These functions form, under the obvious pointwise defined algebraic operations, a normed algebra $\underline{\mathfrak{A}}$ on which the Poincaré transformations (x, Λ) act by automorphisms $\underline{\alpha}_{x, \Lambda}$ related to those in \mathfrak{A} by

$$(\underline{\alpha}_x \underline{A})(\lambda) = \alpha_{\lambda x} \underline{A}(\lambda), \quad (\underline{\alpha}_\Lambda \underline{A})(\lambda) = \alpha_\Lambda \underline{A}(\lambda). \quad (8.1)$$

The norm is defined by

$$\| \underline{A} \| = \sup_{\lambda} \| \underline{A}(\lambda) \|. \quad (8.2)$$

The local structure of the original net is lifted to $\underline{\mathfrak{A}}$ by setting

$$\underline{\mathfrak{A}}(\mathcal{O}) = \{ \underline{A} : \underline{A}(\lambda) \in \mathcal{R}(\lambda \mathcal{O}), \lambda \in \mathbb{R}^+ \}, \quad (8.3)$$

and the momentum space properties of the elements \underline{A} of the scaling algebra are controlled by the requirement that $\underline{\alpha}_x \underline{A}$ and $\underline{\alpha}_\Lambda \underline{A}$ depend norm-continuously on x and Λ , respectively. The latter condition entails that the values $\underline{A}(\lambda)$ of \underline{A} , being localized in $\lambda\mathcal{O}$, have a momentum transfer of order λ^{-1} , in accord with the uncertainty principle. In this way one obtains a local, Poincaré covariant net which is canonically associated with the original theory.

One may regard the values $\underline{A}(\lambda)$ as observables in the theory “at scale λ ” corresponding to a change of the original unit of length and thereby of the metric tensor by the factor λ . The graph of a function \underline{A} establishes a relation between observables at different scales, in analogy to renormalization group transformations. However, in contrast to the field theoretic setting, there is no need to identify individual observables at different scales. All functions satisfying the constraints indicated above are admitted. It may seem strange at first sight that with such loose constraints the net $\underline{\mathfrak{A}}$ could provide any interesting information. But this may be understood by recalling that the relevant physical information is contained in the net structure and only the identification of the sets of operators associated to regions is necessary. For that reason one has much more freedom in choosing the relation between observables at different scales.

The next step is to describe the states in the theory at given scale λ with the help of the scaling algebra. This can be done by lifting the states ω of the underlying theory at scale $\lambda = 1$ to $\underline{\mathfrak{A}}$, setting

$$\underline{\omega}_\lambda(\underline{A} \underline{B} \cdots \underline{C}) = \omega(A(\lambda) B(\lambda) \cdots C(\lambda)). \quad (8.4)$$

The short distance properties of the theory can then be analyzed by proceeding to

$$\underline{\omega}_0(\underline{A} \underline{B} \cdots \underline{C}) = \lim \omega(A(\lambda) B(\lambda) \cdots C(\lambda)), \quad (8.5)$$

where we understand the symbol \lim as denoting any limit point of the sequence on the right hand side for $\lambda \rightarrow 0$. The existence of such limit points is guaranteed by general mathematical theorems⁹. Any such limit point is a pure vacuum state on $\underline{\mathfrak{A}}$, irrespective of the state ω from which one starts.

By the GNS construction one obtains from $\underline{\omega}_0$ a representation of a local net of von Neumann algebras, acting in a Hilbert space, which we call the scaling limit of the theory. Three distinct possibilities can arise. The limit may yield a classical theory (commutative algebras). This arises when all functions \underline{A} become multiples of the unit element as $\lambda \rightarrow 0$. Secondly, there may exist many different limit theories (indicating

⁹To use a favorite expression of R.V. Kadison: “Highly efficient abstract nonsense”. Specifically, it is the weak compactness of the unit ball in state space which is used.

the presence of an “unstable ultraviolet fixed point”). The third alternative that the limit points (8.5) define a unique theory and that this is not classical is, of course, the most interesting one. It may be regarded as a distinctive mark characterizing renormalizable theories with a stable ultraviolet fixed point in an intrinsic way, i.e. without reference to perturbation expansions or other approximation methods. One may expect that the scaling limit theory is simpler than the original one; in the extreme case it may turn out to be a theory of free fields (asymptotic freedom) [67].

This suggests that it is a reasonably well defined mathematical problem to investigate in the algebraic setting the existence and uniqueness of theories which have prescribed symmetries in the sense of the preceding section and are asymptotically free. The aims of such an enterprise would be similar to those pursued in “constructive quantum field theory”, but the methods may complement previous efforts [68].

Germ of states:

There is another way to look at the short distance structure [26] which yields a somewhat different type of information, relating to point fields and operator product expansions [66]. Any state ω has (as an expectation functional) a restriction to each subalgebra $\mathcal{R}(\mathcal{O})$, called the partial state in \mathcal{O} . For given $\mathcal{R}(\mathcal{O})$ we may consider the set of the corresponding partial states, or rather its complex hull $\Sigma(\mathcal{O})$, which is a Banach space with distinguished positive cone. The maps $\Sigma(\mathcal{O}_2) \rightarrow \Sigma(\mathcal{O}_1)$, which are obtained by restricting the functionals in $\Sigma(\mathcal{O}_2)$ to $\mathcal{R}(\mathcal{O}_1)$ if $\mathcal{O}_1 \subset \mathcal{O}_2$, induce the structure of a presheaf on the collection of $\Sigma(\mathcal{O})$: A partial state on \mathcal{O}_1 corresponds to an equivalence class of partial states in \mathcal{O}_2 . One may define an equivalence relation with respect to a point x ,

$$\psi \underset{x}{\simeq} \psi', \tag{8.6}$$

meaning that there exists some neighborhood of x in which the restrictions of ψ and ψ' coincide. We shall call such an equivalence class $\{\psi\}_x$ a “germ at the point x ”.

The nuclearity property, discussed in Section 5, suggests that one may obtain a tractable description of such germs in the following way: Focus attention on functionals ψ_E with total energy below E and restrict them to the algebras² $\mathcal{R}(\mathcal{O}_r)$. The resulting spaces are denoted by $\Sigma_E(\mathcal{O}_r)$. Disregarding functionals of small norm, each of them is finite dimensional. A measure for the accuracy of these finite approximations is the distance $\delta_n(E, r)$ between the unit balls in $\Sigma_E(\mathcal{O}_r)$ and in the closest n -dimensional subspace of functionals. This distance decreases as $r \rightarrow 0$ and increases as $E \rightarrow \infty$; moreover, for fixed E, r it decreases with growing n .

Under reasonable assumptions, the distance functions $\delta_n(E, r)$ vanish with increasing n of increasingly high order as $r \rightarrow 0$. In the example of the theory of a free

scalar field φ_0 one has for $Er < 1$

$$\delta_n(E, r) \simeq (Er)^{d_n}, \quad \text{where } d_1 = 1, \quad d_2 = 2, \quad d_3 = \dots = d_7 = 3 \text{ etc.} \quad (8.7)$$

Proceeding to the dual picture (the co-germs, which are associated with the algebra), this gives an increasing number of pointlike fields which are needed to distinguish the functionals in $\Sigma_E(\mathcal{O}_r)$ with increasing accuracy $\delta_n(E, r)$. In the free scalar theory, the unit operator corresponds to $n = 1$. For $n = 2$, the field φ_0 is needed also, for $n \leq 7$ the 4 derivatives $\partial_\mu \varphi_0$ enter and the Wick power $:\varphi_0^2:$. Ultimately, all elements of the Borchers class appear. The field equations give a reduction in the number of new independent elements for increasing n . Thus one has an ordering of the elements of the Borchers class according to their significance in the regime of small Er and one may consider approximation schemes corresponding to operator product expansions.

9 Inclusions

There are two distinct types of questions in which the study of inclusions of algebras plays a role. The obvious one in our context comes from the inclusions of regions in space-time. The less obvious one from endomorphisms of the algebra \mathfrak{A} . We shall begin with the latter because it provides another surprising example for “pre-stabilized harmony” between physics and mathematics.

The analysis in Section 6 related charge creation to the existence of endomorphisms of \mathfrak{A} . This led to the recognition that, associated with each type of charge, there is a “statistics parameter” λ which, in the case of four-dimensional space-time, could only take the values $\pm n^{-1}$, where n is an integer, the statistics dimension, and the sign distinguishes the Bose and Fermi case [48]. In two-dimensional space-time there is a much wider range of possibilities. The “statistics dimension” $|\lambda|^{-1}$ can take non-integer values and, instead of a sign, complex phase factors can appear. Instead of the Bose-Fermi alternative one has “braid group statistics” [51].

Motivated by a quite different circle of questions in mathematics, Vaughan Jones discovered that for certain inclusions of type II factors there exists an “index” which can take only a restricted set of values and that there is a relation of this structure with representations of the braid group. This initiated a mathematical development leading for example to substantial generalizations and applications to the theory of knots. It took several years till the close connection of these mathematical developments with the composition laws of charge quantum numbers was recognized by Longo [69], who showed that the statistical dimension is a (generalized) Jones index. While in four-dimensional space-time the latter is restricted to integers and this

feature is connected with the permutation group, the full complexity appears in the analysis of possible charge structures in two-dimensional space-time.

Coming now to the inclusions in the net of algebras, we may restrict ourselves here to a few remarks since this is extensively discussed in the contribution of Borchers to this issue. The central result is that a few algebras suffice to determine the whole net as well as the operators representing the Poincaré group and the TCP-operator. This amazing fact, recognized by Wiesbrock [70, 71] using basic results of Borchers [72], can be made plausible intuitively by starting from the discovery of Bisognano and Wichmann [32] which indicated that the modular automorphism group for the vacuum state and the wedge region (5.1) gives the Lorentz transformations in the x_0 - x_1 -plane. If one takes a second wedge, included in the first one, one obtains “half-sided modular inclusions” for which the modular operators and conjugations generate a whole family of algebras and the translation operator in a light-like direction in the x_0 - x_1 -plane. Repeating this construction, changing x_1 to x_2 and x_3 , the whole net is obtained by intersections and the full Poincaré group is obtained.

One comment should be added. Each of the six algebras used in the construction is isomorphic to the unique hyperfinite type III₁-factor. Their association with specific regions in Minkowski space could be regarded as secondary. It only serves to fix a general relation between these algebras. So one may replace the assumption $L_{\mathcal{O}}$, implying that the label \mathcal{O} in the basic correspondence (4.1) should be interpreted as a region in Minkowski space, by a weaker one [73]. The main structural relations needed refer to inclusions and complements and these are directly encoded in the algebraic relations.

10 Summary, comments, conclusions, perspectives

Looking back at the understanding of relativistic quantum physics fifty years ago, it may be fair to say that the second half of the century brought no revolution comparable in impact to the radical changes of basic concepts which shook the first three decades of this century. It was a period of steady evolution, but it led to significant changes of perspective.

We discussed here the synthesis of quantum theory and special relativity, incorporating some information from other sources and striving to bring out the essentials of a coherent conceptual and mathematical structure. On the side of quantum theory, we started from the orthodox position, distinguishing the observer and his instruments from the “physical system” which here, in principle, could be the whole universe minus the observer. We also adopted the point of view of Heisenberg and Dirac that

observables and manipulations of the system by the observer are mathematically described by elements of a non commutative algebra and that the (abstract) algebraic relations constitute the essence of the theory. Keeping in mind Niels Bohr’s message that we must be able to “tell our friends what we have done and what we have learned” and his conclusion that this forces us to describe the side of the observer in the “language of classical physics”, we note that indeed we retain one classical anchor, namely classical space–time in which we describe the placement of instruments and the Poincaré symmetry, used in the active sense of pushing around instruments.

The bridge between the mathematical formalism and the communication with our friends is provided by the correspondence (4.1) between space–time regions and algebras and by the realization of the Poincaré group by automorphisms (7.1), (7.2) of the net of local algebras. Its combination allows us to describe geometric constellations of instruments and to analyze the energy transfer between the instruments and the “system”, which suffices for a full physical interpretation of the information contained in the mathematical scheme.

How much do we know about the net of algebras? The two central pillars come from the relativistic causal structure of space–time, expressed by the postulate L_O of Section 3, and from the stability requirement, expressed as postulate S in Section 2. But the closer study in Section 5 leads to important refinements. On the one hand it shows that if we focus attention only on regions of finite extension, there is a faithful representation of the abstract algebraic elements by operators acting on a Hilbert space \mathcal{H} . Each individual subalgebra $\mathcal{R}(\mathcal{O})$ is isomorphic to a universal, known and well–studied object: the unique hyperfinite factor of type III_1 .

On the other hand, the consideration of different classes of states brings out clearly that the abstract algebras are the basic objects whereas their representations in terms of operators in Hilbert space is a matter of convenience which may be adapted to the situation under consideration. Inequivalent representations of one and the same net of abstract algebras describe different idealizations which are useful in different regimes. Prime examples are thermal equilibrium states and states carrying some (global) charge quantum number (Section 6).

Postulate S is strengthened to the nuclearity postulate N , implying roughly that finite phase space volumes correspond to finite dimensional subspaces of \mathcal{H} . The locality principle L_O is strengthened to the “split property” (5.5) which allows a factorization of \mathcal{H} for disjoint, space–like separated regions in analogy to the notion of subsystems in non relativistic quantum mechanics. This, incidentally, implies that the discussion of entanglement and non–locality of the EPR–correlations can be done in the same way as in quantum mechanics. But in this setting the distinction between

causal effects whose propagation is limited by light cones, as demanded by L_O , and EPR–correlations, which may persist over (large) spatial distances, is seen clearly.

Comment: If instead of Minkowski space one considers curved space–time, then the algebraic part of the theory carries over smoothly since the net structure refers only to inclusion relations and causal complementation and both remain well defined if the metric structure is classically given in terms of a gravitational background field [74]. The loss of Poincaré symmetry demands, however, that the stability requirement S must be replaced. Suggestions of how this may be done have been proposed. See e.g. [75, 76, 5, 73]. The most interesting physical consequences which can be treated in this setting are the Bekenstein temperature and Hawking radiation associated with black holes. One should note, however, that our present understanding of the stability requirement is not fully satisfactory.

We must now face up to the essential task of defining one specific theory within the still rather general frame. The most significant progress in high energy theory in the past decades has been the development of the Standard Model. It combines the choice of specific internal symmetry groups with the sharpened locality principle which, more than eighty years ago, had led to the general theory of relativity: there is no preferred global reference frame; the relation between frames in different locations depends on the choice of a path connecting them.

But here and now we talk about reference frames for the degrees of freedom associated with internal symmetries, not about the frame for space–time coordinates. The incorporation of internal symmetries subject to this “local gauge principle”, which demands that there is no preferred global reference system for them, is addressed in Section 7. It is not a straightforward task to transfer the notions of sections and connections, familiar from the classical formulation with fiber bundles, to the quantum level. We briefly sketched an approach which could lead to an intrinsic understanding of the meaning of quantum connection. One essential aspect appears to be that in addition to the causal support \mathcal{O} , used in the correspondence (4.1), one must introduce finer distinctions in $\mathcal{R}(\mathcal{O})$ by so–called “charge supports”. Specifically, one needs special elements in $\mathcal{R}(\mathcal{O})$ with disjoint, complementary charge supports related to representations of the gauge group.

Much remains to be done in the development of such ideas till a concise and complete structure is reached. But it is an effort well worth–while since it can open the gate to a very wide field. We mentioned already the need for a good definition of a specific theory along the lines suggested by the Standard Model. Combined with the short distance analysis (Section 8), the question of existence and uniqueness could be approached in precise mathematical terms. But beyond that let us mention some

old dreams: the supersymmetric unification of internal and geometric symmetries, treated as local gauge symmetries. This may, in fact, even suggest a natural approach to the synthesis of general relativity and quantum physics since, once we sacrifice the global nature of translations, we may also treat the Lorentz group as an internal $SL(2,C)$ -symmetry.

None of these perspectives is of a truly revolutionary nature. They constitute a natural development of existing ideas. But it seems that this development has not yet reached its end, its essential limits. The road from QED to QCD exemplifies that old equations and principles, properly understood and adapted, contain a lot of relevant new physics. The problems mentioned above indicate the wide range of efforts still needed to clarify and round off this era.

At the same time one cannot ignore the signs indicating the approach of some radical change in basic concepts. What will be the role of space-time in the paradigm of a future theory and how will the orthodox position of quantum theory be affected? It is already evident that the classical anchor, provided by the operational interpretation of space-time as the bridge between the mathematical formalism of the theory and the simple language needed for “telling what we have learned” cannot be pushed to extremes. We do not place and control instruments in regions of 10^{-16} cm extension. If we look for a synthesis of quantum physics with general relativity, for instance along the lines indicated above, then this means that we introduce on the side of the mathematical structure of the theory substantially more detailed ontological extrapolations than can be directly related to observations. The needed bridge on which Bohr insisted (for good reasons) must be established on an intermediate level, such as the definition of some (classical) background which must first be derived from the theory as an approximation under suitable circumstances.

There arises the question of how we can (within the scope of physics) divide the universe into distinct, individual parts to which we can give a name. This is indeed the main message brought home by the EPR-type experiments, because we see that the notions of “system” and “state” are approximate or relative unless we consider the whole universe as the system. As long as we regard space-time as a pregiven continuum, we may use this for the purpose of subdivision. If we give up this anchor, then what remains?

If we believe in a fundamental indeterminism of the theory, then we must distinguish between the realm of facts and the realm of possibilities, represented by probability assignments. The former is, at present, reduced to “observation results”, the latter to the notion of “state”. Strictly speaking, an observation result is a macroscopic change which enters the consciousness of several human beings. This

is certainly necessary for testing a theory. But hardly as a basic concept. Can it be generalized by the notion of an “event” which does not depend on the senses and consciousness of humans? Is the role of space–time ultimately just the set of relations within a pattern of events? Does the distinction between potentialities and facts imply a fundamental significance of the arrow of time? Facts belong to the past, possibilities to the future. For some tentative steps in such directions compare [77].

Let us conclude this essay with the acknowledgment that there remain many questions and we are very far from a “theory of everything”.

Appendix

For the convenience of the reader, we collect here some facts and notions from the theory of operator algebras which are used in the main text.

In the algebraic approach to quantum theory, the basic mathematical objects are C^* –algebras. A C^* –algebra \mathfrak{A} is a complex linear space, equipped with an associative product, a $*$ –operation (defining the adjoint) and a distinguished norm $\| \cdot \|$. With respect to the corresponding norm topology, \mathfrak{A} is complete, i.e. a Banach space. We assume that \mathfrak{A} contains a unit element 1.

A state ω on \mathfrak{A} is a complex linear functional which attains non–negative values on all elements of the positive cone $\mathfrak{A}_+ = \{A^*A : A \in \mathfrak{A}\} \subset \mathfrak{A}$ and which is normalized, $\omega(1) = 1$. It is a basic fact, established by Gelfand, Naimark and Segal (GNS construction), that any state ω determines (a) some Hilbert space \mathcal{H} , (b) a mapping π from \mathfrak{A} into the algebra of bounded linear operators on \mathcal{H} which preserves the algebraic relations (i.e. a homomorphism) and (c) some normalized vector $\Phi \in \mathcal{H}$ such that

$$\omega(A) = (\Phi, \pi(A)\Phi) \quad \text{for } A \in \mathfrak{A}. \quad (\text{A.1})$$

In this way, \mathfrak{A} is mapped to a concrete C^* –algebra $\pi(\mathfrak{A})$ of Hilbert space operators and the state ω is interpreted as an expectation functional on $\pi(\mathfrak{A})$.

On a Hilbert space one can introduce the notion of weak convergence of sequences of bounded operators (all matrix elements of the sequence converge). The resulting limits are again bounded linear operators. It is therefore meaningful to proceed from $\pi(\mathfrak{A})$ to its “weak closure” $\mathcal{R} = \pi(\mathfrak{A})^-$, i.e. the set of operators consisting of $\pi(\mathfrak{A})$ and all weak limit points. The set \mathcal{R} is again a C^* –algebra but, in contrast to $\pi(\mathfrak{A})$, it is also closed with respect to weak limits. Such weakly closed algebras are called von Neumann algebras.¹⁰

¹⁰Similarly to the case of C^* –algebras, there exists also an abstract version of von Neumann algebras, the W^* –algebras. In the present context, we do not need to distinguish between those.

In the analysis of von Neumann algebras \mathcal{R} one uses various notions which enter also in the present discussion. The “center” \mathcal{Z} of \mathcal{R} is the subalgebra of operators commuting with all operators in \mathcal{R} . If this center consists only of multiples of the unit operator, \mathcal{R} is called a “factor”. The decomposition of an algebra into factors is unique and corresponds to the simultaneous spectral resolution of all operators in the center. Another important notion is “hyperfiniteness”, which means that the algebra can be approximated (in the sense of weak limits) by its finite dimensional subalgebras.

In their seminal investigation, entitled “rings of operators”, von Neumann and Murray found that there were several types of factors. The ones with which physicists were familiar (“type I”) correspond to the algebra of all bounded operators on some Hilbert space. Different factors of type I can thus be distinguished by the dimension of the underlying space. Then there was a continuous generalization, called type II, in which a trace could still be defined for a class of elements. Everything else was lumped together as “type III”.

The Tomita–Takesaki theory provided tools for a finer subdivision. In this theory, the basic ingredients are, besides a von Neumann algebra \mathcal{R} , cyclic and separating vectors Ψ , i.e. vectors for which $\mathcal{R}\Psi$ is dense in the Hilbert space and which are annihilated by none of the operators in \mathcal{R} , apart from 0. Given such a pair (\mathcal{R}, Ψ) , one can consistently define an antilinear operator S , the Tomita conjugation, setting

$$S A \Psi = A^* \Psi \quad \text{for } A \in \mathcal{R}. \tag{A.2}$$

The conjugation S can be decomposed in a unique way (polar decomposition) into the product $S = J \Delta^{1/2}$ of an anti-unitary operator J and a positive self-adjoint operator $\Delta^{1/2}$ whose square is called the modular operator affiliated with (\mathcal{R}, Ψ) . It is a central result in this theory that the corresponding unitary operators Δ^{it} , $t \in \mathbb{R}$, map by their adjoint action the algebra \mathcal{R} onto itself. These maps are the modular automorphisms mentioned at various points in the main text.

Based on these notions, an essentially complete classification of factors \mathcal{R} was achieved by Alain Connes [78], who showed that the “spectral invariant”, obtained from the intersection of the spectra of modular operators affiliated with \mathcal{R} , is (disregarding the value 0) always a closed subgroup of the multiplicative group of positive real numbers. All such groups occur in this classification, but we mention only the ones which will concern us here. In the case of the types I and II the group consists only of the unit element. The opposite situation is that the group consists of all positive real numbers. This was called “type III₁”. As was shown later by Haagerup [42], the hyperfinite factor of type III₁ is unique. It is this factor which generically appears in quantum field theory.

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References

- [1] R.F. Streater and A.S. Wightman, *PCT, Spin and Statistics, and all that*, Benjamin 1964
- [2] R. Jost, *The General Theory of Quantized Fields*, American Math. Soc. 1965
- [3] N.N. Bogolubov and D.V. Shirkov, *Introduction to the theory of quantized fields*, Interscience 1958
- [4] N.N. Bogolubov, A.A. Logunov and I.T. Todorov, *Introduction to Axiomatic Quantum Field Theory*, Benjamin 1975
- [5] R. Haag, *Local Quantum Physics: Fields, Particles, Algebras*, 2nd revised ed., Springer 1996
- [6] H. Lehmann, K. Symanzik and W. Zimmermann, *Nuovo Cimento* **1** (1955) 425
- [7] A.S. Wightman, *Phys. Rev.* **101** (1956) 860
- [8] H. Ekstein, *Phys. Rev.* **101** (1956) 880 and *Nuovo Cimento* **4** (1956) 1017
- [9] R. Haag, *Phys. Rev.* **112** (1958) 669
- [10] W. Brenig and R. Haag, *Fortschr. Phys.* **7** (1959) 183
- [11] D. Ruelle, *Helv. Phys. Acta* **35** (1962) 147
- [12] F. Strocchi, p. 551 in: *Mathematical Methods in Theoretical Physics*, Proceedings Boulder Col. 1971, W.E.Brittin ed., Colorado Assoc. Univ. Press 1973
- [13] T. Kugo and I. Ojima, *Prog. Theor. Phys. Suppl.* **66** (1979) 1
- [14] R.V. Kadison and J.R. Ringrose, *Fundamentals of the Theory of Operator Algebras, Vol. 1 and Vol. 2*, Academic Press 1986
- [15] O. Bratteli and D.W. Robinson, *Operator Algebras and Quantum Statistical Mechanics, Vol. 1*, Springer 1979
- [16] H. Araki and R. Haag, *Commun. Math. Phys.* **4** (1967) 77
- [17] D. Buchholz, M. Poppmann and U. Stein, *Phys. Lett.* **B267** (1991) 377

- [18] H.-J. Borchers, *Nuovo Cimento* **15** (1960) 784
- [19] K.-H. Rehren, *Commun. Math. Phys.* **178** (1996) 453
- [20] S. Coleman, *Phys. Rev.* **D11** (1975) 2088
- [21] K. Intriligator and N. Seiberg, *Nucl. Phys. Proc. Suppl.* **45BC** (1996) 1
- [22] H.-J. Borchers and W. Zimmermann, *Nuovo Cimento* **31** (1963) 1047
- [23] W. Driessler and J. Fröhlich, *Ann. Inst. H. Poincaré* **27** (1977) 221
- [24] H.-J. Borchers and J. Yngvason, *Rev. Math. Phys.* **Special Issue** (1992) 15
- [25] K. Fredenhagen and J. Hertel, *Commun. Math. Phys.* **80** (1981) 555
- [26] R. Haag and I. Ojima, *Ann. Inst. H. Poincaré* **64** (1996) 385
- [27] J.C. Wick, A.S. Wightman and E.P. Wigner, *Phys. Rev.* **88** (1952) 101
- [28] R. Haag and D. Kastler, *J. Math. Phys.* **5** (1964) 848
- [29] R. Haag, N.M. Hugenholtz and M. Winnink, *Comm. Math. Phys.* **5** (1967) 215
- [30] M. Takesaki, *Tomita's Theory of Modular Hilbert Algebras and its Applications*, Lecture Notes in Mathematics, Springer 1970
- [31] H. Reeh and S. Schlieder, *Nuovo Cimento* **22** (1961) 1051
- [32] J. Bisognano and E.H. Wichmann, *J. Math. Phys.* **16** (1975) 985 and *J. Math. Phys.* **17** (1976) 303
- [33] G. Sewell, *Phys. Rev. Lett.* **79A** (1980) 23
- [34] R. Haag, H. Narnhofer and U. Stein, *Commun. Math. Phys.* **94** (1984) 219
- [35] P. Hislop and R. Longo, *Commun. Math. Phys.* **84** (1982) 71
- [36] R. Haag and J.A. Swieca, *Commun. Math. Phys.* **1** (1965) 308
- [37] D. Buchholz and E.H. Wichmann, *Commun. Math. Phys.* **106** (1986) 321
- [38] D. Buchholz and P. Junglas, *Commun. Math. Phys.* **121** (1989) 255
- [39] J. Bros and D. Buchholz, *Nucl. Phys.* **B429** (1994) 291, *Ann. Inst. H. Poincaré* **64** (1996) 495, and *Z. Phys.* **C55** (1992) 509
- [40] K. Fredenhagen, *Commun. Math. Phys.* **97** (1985) 79

- [41] D. Buchholz, C. D'Antoni and K. Fredenhagen, *Commun. Math. Phys.* **111** (1987) 123
- [42] U. Haagerup, *Acta Math.* **158** (1987) 95
- [43] S.J. Summers and R. Werner, *Commun. Math. Phys.* **110** (1987) 1004 and *Lett. Math. Phys.* **33** (1995) 321
- [44] S. Doplicher and R. Longo, *Invent. Math.* **73** (1984) 493
- [45] H.S. Green, *Phys. Rev.* **90** (1953) 270
- [46] H. Baumgärtel and M. Wollenberg, *Causal Nets of Operator Algebras*, Akademie Verlag 1992
- [47] D. Kastler (ed.), *The Algebraic Theory of Superselection Sectors and Field Theory: Introduction and Recent Results*, Proceedings Palermo 1989, World Scientific 1990
- [48] S. Doplicher, R. Haag and J.E. Roberts, *Commun. Math. Phys.* **23** (1971) 199 and *Commun. Math. Phys.* **35** (1974) 49
- [49] J.A. Swieca, *Phys. Rev.* **D13** (1976) 312
- [50] S. Doplicher and J.E. Roberts, *Commun. Math. Phys.* **131** (1990) 51
- [51] K. Fredenhagen, K.-H. Rehren and B. Schroer, *Commun. Math. Phys.* **125** (1989) 201 and *Rev. Math. Phys.* **Special Issue** (1992) 111
- [52] J. Fröhlich and F. Gabbiani, *Rev. Math. Phys.* **2** (1991) 251
- [53] B. Schroer, *Modular Localization and Nonperturbative Local Quantum Physics*, Lecture Notes, Rio de Janeiro: CBPF, 1998, preprint hep-th/9805093
- [54] D. Buchholz and K. Fredenhagen, *Commun. Math. Phys.* **84** (1982) 1
- [55] J. Bros and H. Epstein, p. 330 in: *XIth International Congress of Mathematical Physics. Paris 1994*, D. Iagolnitzer ed., Internat. Press 1995
- [56] D. Buchholz, *Commun. Math. Phys.* **85** (1982) 85
- [57] B. Schroer, *Fortschr. Phys.* **11** (1963) 1
- [58] J. Fröhlich, G. Morchio and F. Strocchi, *Annals Phys.* **119** (1979) 241
- [59] D. Buchholz, *Phys. Lett.* **B174** (1986) 331
- [60] G. Mack, *Fortsch. Phys.* **29** (1981) 135

- [61] H. Araki, Rev. Math. Phys. **Special Issue** (1992) 1
- [62] D. Buchholz, S. Doplicher and R. Longo, Ann. Phys. **170** (1986) 1
- [63] E. Seiler, *Gauge Theories as a Problem of Constructive Quantum Field Theory and Statistical Mechanics*, Springer 1982
- [64] O. Steinmann, Ann. Phys. **157** (1984) 232
- [65] D. Buchholz and R. Verch, Rev. Math. Phys. **7** (1995) 1195 and Rev. Math. Phys. **10** (1998) 775
- [66] K. Wilson and W. Zimmermann, Commun. Math. Phys. **24** (1971) 87
- [67] D. Buchholz, Nucl. Phys. **B469** (1996) 333
- [68] J. Glimm and A. Jaffe, *Quantum Physics: A Functional Integral Point of View*, Springer 1987
- [69] R. Longo, Commun. Math. Phys. **126** (1989) 217 and Commun. Math. Phys. **130** (1990) 285
- [70] H.W. Wiesbrock, Commun. Math. Phys. **157** (1993) 83, Erratum Commun. Math. Phys. **184** (1997) 683, and Commun. Math. Phys. **193** (1998) 269
- [71] R. Kähler and H.W. Wiesbrock, “Modular theory and the reconstruction of 4-dimensional quantum field theories”, preprint
- [72] H.-J. Borchers, Commun. Math. Phys. **143** (1992) 315
- [73] D. Buchholz, O. Dreyer, M. Florig and S.J. Summers, “Geometric modular action and spacetime symmetry groups”, preprint math-ph/9805026, to appear in Rev. Math. Phys.
- [74] J. Dimock, Commun. Math. Phys. **77** (1980) 219
- [75] R. Wald, *Quantum Field Theory in Curved Spacetime and Black Hole Thermodynamics*, Univ. Chicago Pr. 1994
- [76] R. Brunetti, K. Fredenhagen and M. Köhler, Commun. Math. Phys. **180** (1996) 633
- [77] R. Haag, Commun. Math. Phys. **123** (1990) 245, Commun. Math. Phys. **180** (1996) 733, Z. Naturforschung **54a** (1999) 2, and “Objects, Events, Localization”, ESI-preprint 541 (1998)
- [78] A. Connes, Ann. Sci. Ecole Norm. Sup. **6** (1973) 133