

Prigogine's Thermodynamic Emergence and Continuous Topological Evolution

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Abstract: Irreversible processes in Open non-equilibrium thermodynamic systems, of topological dimension 4, can decay locally to Closed non-equilibrium thermodynamic states, of topological dimension 3, by means of continuous topological evolution. These topologically coherent, perhaps deformable, states of one or more components appear to "emerge" as compact 3D Contact submanifolds, defined as topological defects in the 4D Symplectic manifold. These emergent states are still far from equilibrium, as their topological (not geometrical) dimension is greater than 2. The 3D Contact submanifold admits a unique extremal Hamiltonian process (as well as fluctuation components). If the subsequent evolution is dominated by the Hamiltonian component, the emergent topological defects will maintain a relatively long-lived, topologically coherent, approximately non-dissipative structure. These defect structures yield an evolutionary behavior that can be associated with the idea of "stationary states" far from equilibrium. If the fluctuation (spinor) components are weak, but not zero, the emergent thermodynamic structure will ultimately decay, but only after a substantial "lifetime". Analytic solutions and examples of these processes of continuous topological evolution give credence, and a deeper understanding, to the general theory of self-organized states far from equilibrium, as conjectured by I. Prigogine. Moreover, in an applied sense, universal engineering design criteria can be developed to minimize irreversible dissipation and to improve system efficiency in general situations. As the methods are based on universal topological, not geometrical, ideas, the general thermodynamic results apply to all synergetic topological systems. It may come as a surprise, but ecological applications of thermodynamics need not be limited to the design specific hardware devices, but apply to all synergetic systems be they mechanical, biological, economical or political.

Prologue

THE POINT OF DEPARTURE

The point of departure starts with a topological (not statistical) formulation of Thermodynamics, which furnishes a universal foundation for the Partial Differential Equations of classical hydrodynamics and electrodynamics [41]. The topology that is of significance is defined in terms of Cartan's topological structure [35], which can be constructed from an exterior differential 1-form, A , defined on a pre-geometric domain of base variables. The topological method extends the classical geometrical approach to the study of non-equilibrium thermodynamic systems.

In particular, the methods permit non-statistical engineering design criteria to be developed for non-equilibrium systems. The theory of Topological Thermodynamics, based upon Continuous Topological Evolution [36] of Cartan's topological structure, can explain why topologically coherent, compact structures, far from equilibrium, will emerge as long-lived artifacts of thermodynamically irreversible, turbulent, continuous processes. I want to present the idea that:

Theorem 1 *The Pfaff Topological Dimension (PTD) of a Thermodynamic System can change dynamically and continuously via irreversible dissipative processes from a non-equilibrium turbulent state of $PTD = 4$ to an excited "topologically stationary, but excited," state of $PTD = 3$, which is still far from equilibrium! The $PTD=3$ state admits an extremal Hamiltonian evolutionary process which, if dominant, produces a relatively long lifetime.*

There exist C^2 smooth processes that can describe the topological evolution from an Open non-equilibrium turbulent domain of Pfaff Topological Dimension 4 to Closed, but non-equilibrium, domains of Pfaff Topological Dimension 3, and ultimately to equilibrium domains of Pfaff dimension 2 or less. The Topological domains of Pfaff Topological Dimension 3 emerge via thermodynamically irreversible, dissipative processes as topologically coherent, deformable defects, embedded in the turbulent environment of Pfaff Topological Dimension 4.

Now I am well aware of the fact that Thermodynamics (much less Topological Thermodynamics) is a topic often treated with apprehension. In addition, I must confess, that as undergraduates at MIT we used to call the required physics course in Thermodynamics, The Hour of Mystery! Let me present a few quotations (taken from Uffink, [40]) that describe the apprehensive views of several very famous scientists:

Any mathematician knows it is impossible to understand an elementary course in thermodynamics V. Arnold 1990.

It is always emphasized that thermodynamics is concerned with reversible processes and equilibrium states, and that it can have nothing to do with irreversible processes or systems out of equilibriumBridgman 1941

No one knows what entropy really is, so in a debate (if you use the term entropy) you will always have an advantage Von Neumann (1971)

On the other hand Uffink states:

Einstein, ..., remained convinced throughout his life that thermodynamics is the only universal physical theory that will never be overthrown.

I wish to demonstrate that from the point of view of Continuous Topological Evolution (which is based upon Cartan's theory of exterior differential forms) many of the mysteries of non-equilibrium thermodynamics, irreversible processes, and turbulent flows, can be resolved. In addition, the non-equilibrium methods can lead to many new processes and patentable devices and concepts.

Non-equilibrium thermodynamics can be constructed in terms of disconnected Cartan topology of Pfaff Topological Dimension of 3 or more. As irreversibility requires a change in topology, the point of departure for this article will be to use the thermodynamic theory of continuous topological evolution in 4D space-time. It can be demonstrated, by example, that the non-equilibrium component of the Cartan topology can support topological change, thermodynamic irreversible processes and turbulent solutions to the Navier-Stokes equations, while the equilibrium topological component cannot. In addition, it will be demonstrated that complex isotropic *macroscopic* Spinors are the source of topological fluctuations and irreversible processes in the topological dynamics of non-equilibrium systems. This, perhaps surprising, fact has been ignored by almost all researchers in classical hydrodynamics who use classic real vector analysis and symmetries to produce conservation laws, which do not require Spinor components. The flaw in such symmetrical based theories is that they describe evolutionary processes that are time reversible. Time irreversibility requires topological change.

EXTERIOR DIFFERENTIAL FORMS
overcomes the
LIMITATIONS of REAL VECTOR ANALYSIS

During the period 1965-1992 it became apparent that new theoretical foundations were needed to describe non-equilibrium systems and continuous irreversible processes - which

require topological (not geometrical) evolution. I selected Cartan's methods of exterior differential topology to encode Continuous Topological Evolution. The reason for this choice is that many years of teaching experience indicated that such methods were rapidly learned by all research scientists and engineers. In short:

1. Vector and Tensor analysis is not adequate to study the evolution of topology. The tensor constraint of diffeomorphic equivalences implies that the topology of the initial state must be equal to the topology of the final state. Turbulence is a thermodynamic, irreversible process which can not be described by tensor fields alone.
2. However, Cartan's methods of exterior differential systems and the topological perspective of Continuous Topological Evolution (not geometrical evolution) CAN be used to construct a theory of non-equilibrium thermodynamic systems and irreversible processes.
3. Bottom Line: Exterior differential forms carry topological information and can be used to describe topological change induced by processes. Real "Vector" directionfields alone cannot describe processes that cause topological change; but Spinor directionfields can.

A cornerstone of classic Vector (tensor) analysis is the constraint of functional equivalence with respect to diffeomorphisms. However, diffeomorphisms are a subset of a homeomorphisms, and homeomorphisms preserve topology. Hence to study topological change, Vector (tensor) analysis is not adequate. In topological thermodynamics, processes are defined in terms of directionfields which may or may not be tensors. The ubiquitous concepts of 1-1 diffeomorphic equivalence, and non-zero congruences, for the eigen directionfields of symmetric matrices do not apply to the eigen directionfields of antisymmetric matrices. The eigen direction fields of antisymmetric matrices (which are equivalent to Cartan's isotropic Spinors) may be used to define components of a thermodynamic process, but such Spinors have a null congruence (zero valued quadratic form), admit chirality, and are not 1-1. Where classic geometric evolution is described in terms of symmetries and conservation laws, topological evolution is described in terms of antisymmetries.

Cartan's theory of exterior differential forms is built over completely antisymmetric structures, and therefore is the method of choice for studying topological evolution. The exterior differential defines limit sets; the Lie differential defines continuous topological evolution. The concept of Spinors arise naturally in theories using Cartan's methods of exterior differential forms; i.e., Spinors are not added to the theory ad hoc. The Cartan theory of extended differential forms can be used to study topological change. The word *extended* is

used to emphasize the fact that differential forms are functionally well defined with respect a larger class of transformations than those used to define tensors. Extended differential forms behave as scalars with respect to C1 maps which do not have an inverse, much less an inverse Jacobian. Both the inverse map and the inverse Jacobian are required by a diffeomorphism. The exterior differential form on the final state of such C1 non-invertible maps permits the functional form of the differential form on the initial state to be functionally well defined in a retrodictive, pullback sense - not just at a point, but over a neighborhood.

Theorem 2 *Tensor fields can be neither retrodicted nor predicted in functional form by maps that are not diffeomorphisms [14].*

CONTINUOUS TOPOLOGICAL EVOLUTION

Objectives of CTE The objectives of the theory of Continuous Topological Evolution are to:

1. Establish the long sought for connection between irreversible thermodynamic processes and dynamical systems – without statistics!
2. Demonstrate the connection between thermodynamic irreversibility and Pfaff Topological Dimension equal to 4. The result suggests that “2-D Turbulence is a myth” for it is a thermodynamic system of Pfaff Topological Dimension equal to 3 [21].
3. Demonstrate that topological thermodynamics leads to universal topological equivalences between Electromagnetism, Hydrodynamics, Cosmology, and Topological Quantum Mechanics.
4. Demonstrate that Cartan’s methods of exterior differential forms permits important topological concepts to be displayed in a useful, engineering format.

New Concepts deduced from CTE The theory of Continuous Topological Evolution introduces several new important concepts that are not apparent in a geometric equilibrium analysis.

1. Continuous Topological Evolution is the dynamical equivalent of the FIRST LAW OF THERMODYNAMICS.
2. The Pfaff Topological Dimension, PTD, is a topological property associated with any Cartan exterior differential 1-form, A . The PTD can change via topologically continuous processes.

3. Topological Torsion is a 3-form (on any 4D geometrical domain) that can be used to describe irreversible processes which produce self-similar thermodynamic systems. As a 4D non-equilibrium direction field it is completely determined by the coefficient functions that encode the thermodynamic system. Other process directionfields are determined by the system topology based upon the 1-form of Action, A , and the refinement based on the topology of the 1-form of work, W .
4. Closed thermodynamic topological defects of Pfaff Topological Dimension 3 can emerge from Open thermodynamic systems of Pfaff Topological Dimension 4 by means of irreversible dissipative processes that represent topological evolution and change. When the topologically coherent defect structures emerge, their evolution can be dominated by a Hamiltonian component (modulo topological fluctuations), which maintains the topological deformation invariance, and yields hydrodynamic wakes [20] and other Soliton structures. These objects are of Pfaff Topological Dimension 3 and are far from equilibrium. They behave as if they were "stationary excited" states above the equilibrium ground state. Falaco Solitons are an easily reproduced hydrodynamic example that came to my attention in 1986 [42] [33]. A movie is available online [34].

PRESENTATION OUTLINE

The essay is constructed in several sections:

Section 1. Topological Thermodynamics In Section 1, the concepts of topological thermodynamics in a space-time variety are reviewed (briefly) in terms of Cartan's method of exterior differential forms. A thermodynamic system is encoded in terms of a 1-form of Action, A . Thermodynamic processes are encoded in terms of the Lie differential with respect to a directionfield, V , acting on the 1-form, A , to produce a 1-form, Q . The process directionfield can have Vector and Spinor components. The definition of the Lie differential is a statement of cohomology and defines Q as the composite of a 1-form, W , and a perfect differential, dU . The formula abstractly represents a dynamical version of the First Law of Thermodynamics. It is a statement about cohomology theory, where the difference between the inexact 1-form of Heat, Q , and the inexact 1-form of Work, W , is a perfect differential, $dU = Q - W$.

The existence of a 1-form on a 4D space-time variety generates a Cartan topology. If the Pfaff Topological (not geometrical) Dimension of the 1-form of Action, A , is 2 or less, then the thermodynamic system is an isolated or equilibrium system on the 4D variety. If the Pfaff Topological Dimension of A is greater than 3, then the system is a non-equilibrium system on the 4D variety. Examples of systems of Pfaff Topological Dimension 4 which admit

processes which are thermodynamically irreversible are given in the reference monographs (see footnote page 1).

Section 2. Applications In Section 2, the abstract formalism will be given a specific realization appropriate for fluids and plasmas in general. First, an electromagnetic format will be described because my teaching experience has demonstrated that the ideas of non-equilibrium phenomena are more readily recognized in terms of electromagnetic concepts. Then it will be demonstrated how the PDE's representing the Hamiltonian version of the hydrodynamic Lagrange-Euler equations arise from the constraint that the work 1-form, W , should vanish (Pfaff Topological Dimension of $W = 0$). Such processes are defined as extremal processes in the theory of the calculus of variations.

The Bernoulli flow can be obtained by constraining the thermodynamic Work 1-form to be exact, $W = d\Theta$ (Pfaff Topological Dimension 1), and the Helmholtz flow will follow from the constraint that the thermodynamic Work 1-form be closed, but not necessarily exact, $dW = 0$. Such reversible dynamical processes belong to the connected component of the Work 1-form.

Irreversible processes belong to the disconnected topological component of the Work 1-form. An important example is the process defined in terms of the Topological Torsion direction field on a Symplectic manifold. Such processes are self-similar relative to the 1-form of Action, A , and are thermodynamically irreversible.

Section 3. Closed States of Topological Coherence embedded as deformable defects in Turbulent Domains In Section 3, the problem of emergence of topologically coherent compact defects in an open thermodynamic system will be attacked from the point of continuous topological evolution. First, the properties of the different species of topological defects will be discussed. These defects are non-equilibrium closed domains (of $PTD = 3$) which can emerge by C2 smooth irreversible process in open domains (of $PTD = 4$), as excited states far from equilibrium, yet with long relative lifetimes. Falaco Solitons are an easily reproduced *experimental* example of such topological defects, and are discussed in detail in [42] [34].

The analytic properties of two different species of $PTD = 3$ defect domains will be given in detail. In addition, an analytic solution of a thermodynamically irreversible process that causes the defect domain to emerge will be demonstrated. An example of a process that creates the topological defect in finite time will be given.

Finally, an example will be given where by combinations of Spinor solutions (fluctuations) produce piecewise linear processes. These piecewise linear processes are thermodynamically reversible, while the Spinor solutions of which they are composed are not.

Section 4. Synergetic systems In Section 4, a few concluding remarks will be made about the ongoing research concerning topological fluctuations, as generated by Spinors. Such topological fluctuations can be associated with the concepts of pressure and temperature. The methods of fiber bundle theory can be used to extend the 4D thermodynamic domain.

However, a more striking implication of the thermodynamic topological method is that it can be applied to many different complex systems of synergetic components obeying logical rules that define a topological structure. Such systems need not be constrained to plasmas and fluids, but apply universally to all topological systems that can be encoded in terms of exterior differential forms. Such systems may be biological systems, political systems, economic systems, as well as mechanical systems. They all have common characteristics of emergence (conception and growth to maturity), by interacting irreversibly with their Open (Pfaff topological dimension 4) environment (by feeding and excreting waste). Maturity implies that the system has reached a slowly decaying, almost stationary, Closed (Pfaff topological dimension 3) state of relatively long lifetime, often followed by a rapid decay to Equilibrium ((Pfaff topological dimension 1) state, or death.

The bottom line is that Ecological studies need to include more complex systems, and these systems yield to the analysis of topological thermodynamics and continuous topological evolution.

1 Topological Thermodynamics

1.1 The Axioms of Topological Thermodynamics

The topological methods used herein are based upon Cartan's theory of exterior differential forms. The thermodynamic view assumes that the physical systems to be studied can be encoded in terms of a 1-form of Action Potentials (per unit source, or, per mole), A , on a four-dimensional variety of ordered independent variables, $\{\xi^1, \xi^2, \xi^3, \xi^4\}$. The variety supports a differential volume element $\Omega_4 = d\xi^1 \wedge d\xi^2 \wedge d\xi^3 \wedge d\xi^4$. This statement implies that the differentials of the $\mu = 4$ base variables are functionally independent. No metric, no connection, no constraint of gauge symmetry is imposed upon the four-dimensional pregeometric variety. Topological constraints can be expressed in terms of exterior differential systems placed upon this set of base variables [1].

In order to make the equations more suggestive to the reader, the symbolism for the variety of independent variables will be changed to the format $\{x, y, z, t\}$, but be aware that no constraints of metric or connection are imposed upon this variety, at this, thermodynamic, level. For instance, it is NOT assumed that the variety is spatially Euclidean.

With this notation, the Axioms of Topological Thermodynamics can be summarized as:

Axiom 1. *Thermodynamic physical systems can be encoded in terms of a 1-form of covariant Action Potentials, $A_\mu(x, y, z, t\dots)$, on a four-dimensional abstract variety of ordered independent variables, $\{x, y, z, t\}$. The variety supports differential volume element $\Omega_4 = dx \wedge dy \wedge dz \wedge dt$.*

Axiom 2. *Thermodynamic processes are assumed to be encoded, to within a factor, $\rho(x, y, z, t\dots)$, in terms of a contravariant Vector and/or complex Spinor directionfields, symbolized as $V_4(x, y, z, t)$.*

Axiom 3. *Continuous Topological Evolution of the thermodynamic system can be encoded in terms of Cartan's magic formula (see p. 122 in [10]). The Lie differential with respect to the process, ρV_4 , when applied to an exterior differential 1-form of Action, $A = A_\mu dx^\mu$, is equivalent, abstractly, to the first law of thermodynamics.*

$$\text{Cartan's Magic Formula } L_{(\rho \mathbf{V}_4)} A = i(\rho \mathbf{V}_4) dA + d(i(\rho \mathbf{V}_4) A), \quad (1)$$

$$\text{First Law} \quad : \quad W + dU = Q, \quad (2)$$

$$\text{Inexact Heat 1-form } Q = W + dU = L_{(\rho \mathbf{V}_4)} A, \quad (3)$$

$$\text{Inexact Work 1-form } W = i(\rho \mathbf{V}_4) dA, \quad (4)$$

$$\text{Internal Energy } U = i(\rho \mathbf{V}_4) A. \quad (5)$$

Axiom 4. *Equivalence classes of systems and continuous processes can be defined in terms of the Pfaff Topological Dimension and topological structure generated by of the 1-forms of Action, A , Work, W , and Heat, Q .*

Axiom 5. *If $Q \wedge dQ \neq 0$, then the thermodynamic process is irreversible.*

1.2 Cartan's Magic Formula \approx First Law of Thermodynamics

The Lie differential (not Lie derivative) is the fundamental generator of Continuous Topological Evolution. When acting on an exterior differential 1-form of Action, $A = A_\mu dx^\mu$, Cartan's magic (algebraic) formula is equivalent *abstractly* to the first law of thermodynamics:

$$L_{(\rho \mathbf{V}_4)} A = i(\rho \mathbf{V}_4) dA + d(i(\rho \mathbf{V}_4) A), \quad (6)$$

$$= W + dU = Q. \quad (7)$$

In effect, Cartan's magic formula leads to a topological basis of thermodynamics, where the thermodynamic Work, W , thermodynamic Heat, Q , and the thermodynamic internal energy, U , are defined *dynamically* in terms of Continuous Topological Evolution. In effect, the First Law is a statement of Continuous Topological Evolution in terms of deRham cohomology theory; the difference between two non-exact differential forms is equal to an exact differential, $Q - W = dU$.

My recognition (some 30 years ago) of this correspondence between the Lie *differential* and the First Law of thermodynamics has been the corner stone of my research efforts in applied topology.

It is important to realize that the Cartan formula is to be interpreted algebraically. Many textbook presentations of the Cartan-Lie differential formula presume a dynamic constraint, such that the vector field $\mathbf{V}_4(x, y, z, t)$ be the generator of a single parameter group. If true, then the topological constraint of Kinematic Perfection can be established as an exterior differential system of the format:

$$\mathbf{Kinematic\ Perfection} : \quad dx^k - \mathbf{V}^k dt \Rightarrow 0. \quad (8)$$

The topological constraint of Kinematic Perfection, in effect, defines (or presumes) a limit process. This constraint leads to the concept of the Lie *derivative*¹ of the 1-form A . The evolution then is represented by the infinitesimal propagation of the 1-form, A , down the

¹Professor Zbigniew Oziewicz told me that Sledobzinsky was the first to formulate the idea of the Lie derivative in his thesis (in Polish).

flow lines generated by the 1-parameter group. Cartan called this set of flow lines "the tube of trajectories".

However, such a topological, kinematic constraint is *not* imposed in the presentation found in this essay; the directionfield, \mathbf{V}_4 , may have multiple parameters. This observation leads to the important concept of topological fluctuations (about Kinematic Perfection), such as given by the expressions:

$$\text{Topological} \quad : \quad \text{Fluctuations}$$

$$(dx^k - \mathbf{V}^k dt) = \Delta \mathbf{x}^k \neq 0, \quad (\sim \text{Pressure}) \quad (9)$$

$$(dV^k - \mathbf{A}^k dt) = (\Delta \mathbf{V}^k) \neq 0, \quad (\sim \text{Temperature}) \quad (10)$$

$$d(\Delta \mathbf{x}^k) = -(d\mathbf{V}^k - \mathbf{A}^k dt) \wedge dt = -(\Delta \mathbf{V}^k) \wedge dt, \quad (11)$$

In this context it is interesting to note that in Felix Klein's discussions [4] of the development of calculus, he says

"The primary thing for him (Leibniz) was not the differential quotient (the derivative) thought of as a limit. The differential, dx , of the variable x had for him (Leibniz) actual existence..."

The Leibniz concept is followed throughout this presentation. It is important for the reader to remember that the concept of a differential form is different from the concept of a derivative, where a (topological) limit has been defined, thereby constraining the topological evolution.

The topological methods to be described below go beyond the notion of processes which are confined to equilibrium systems of kinematic perfection. Non-equilibrium systems and processes which are thermodynamically irreversible, as well as many other classical thermodynamic ideas, can be formulated in precise mathematical terms using the topological structure and refinements generated by the three thermodynamic 1-forms, A , W , and Q .

1.3 The Pfaff Sequence and the Pfaff Topological Dimension

1.3.1 The Pfaff Topological Dimension of the System 1-form, A

It is important to realize that the Pfaff Topological Dimension of the system 1-form of Action, A , determines whether the thermodynamic system is Open, Closed, Isolated or Equilibrium. Also, it is important to realize that the Pfaff Topological Dimension of the thermodynamic Work 1-form, W , determines a specific category of reversible and/or irreversible processes. It is therefore of some importance to understand the meaning of the Pfaff Topological Dimension of a 1-form. Given the functional format of a general 1-form, A , on a 4D variety

it is an easy step to compute the Pfaff Sequence, using one exterior differential operation, and several algebraic exterior products. For a differential 1-form, A , defined on a geometric domain of 4 base variables, the Pfaff Sequence is defined as:

$$\mathbf{Pfaff\ Sequence} \quad \{A, dA, A \wedge dA, dA \wedge dA \dots\} \quad (12)$$

It is possible that over some domains, as the elements of the sequence are computed, one of the elements (and subsequent elements) of the Pfaff Sequence will vanish. The number of non-zero elements in the Pfaff Sequence (PS) defines the Pfaff Topological Dimension (PTD) of the specified 1-form². Modulo singularities, the Pfaff Topological Dimension determines the minimum number M of N functions of base variables ($N \geq M$) required to define the topological properties of the connected component of the 1-form A .

The Pfaff Topological Dimension of the 1-form of Action, A , can be put into correspondence with the four classic topological structures of thermodynamics. Equilibrium, Isolated, Closed, and Open systems. The classic thermodynamic interpretation is that the first two structures do not exchange mass (mole numbers) or radiation with their environment. The Closed structure can exchange radiation with its environment but not mass (mole numbers). The Open structure can exchange both mass and radiation with its environment. The following table summarizes these properties. For reference purposes, I have given the various elements of the Pfaff sequence specific names:

Topological p-form name	PS element	Nulls	PTD	Thermodynamic system
Action	A	$dA = 0$	1	Equilibrium
Vorticity	dA	$A \wedge dA = 0$	2	Isolated
Torsion	$A \wedge dA$	$dA \wedge dA = 0$	3	Closed
Parity	$dA \wedge dA$	–	4	Open

Table 1 Applications of the Pfaff Topological Dimension.

The four thermodynamic systems can be placed into two disconnected topological categories. If the Pfaff Topological Dimension of A is 2 or less, the first category is determined by the closure (or differential ideal) of the 1-form of Action, $A \cup dA$. This Cartan topology is a connected topology. In the case that the Pfaff Topological Dimension is greater than 2, the Cartan topology is based on the union of two closures, $\{A \cup dA \cup A \wedge dA \cup dA \wedge dA\}$, and is a disconnected topology.

²The Pfaff Topological dimension has been called the "class" of a 1-form in the old literature. I prefer the more suggestive name.

It is a topological fact that there exists a (topologically) continuous C2 process from a disconnected topology to a connected topology, but there does not exist a C2 continuous process from a connected topology to a disconnected topology. This fact implies that topological change can occur continuously by a "pasting" processes representing the *decay* of turbulence by "condensations" from non-equilibrium to equilibrium systems. On the other hand, the *creation* of Turbulence involves a discontinuous (non C2) process of "cutting" into parts. This warning was given long ago [19] to prove that computer analyses that smoothly match value and slope will not replicate the *creation* of turbulence, but can faithfully replicate the *decay* of turbulence.

1.3.2 The Pfaff Topological Dimension of the Thermodynamic Work 1-form, W

The topological structure of the thermodynamic Work 1-form, W , can be used to refine the topology of the physical system; recall that the physical system is encoded by the Action 1-form, A .

Claim 3 *The PDE's that represent the system dynamics are determined by the Pfaff Topological Dimension of the 1-form of Work, W , and the 1-form of Action, A , that encodes the physical system.*

The Pfaff Topological Dimension of the thermodynamic Work 1-form depends upon both the physical system, A , and the process, \mathbf{V}_4 . In particular if the Pfaff Dimension of the thermodynamic Work 1-form is zero, ($W = 0$), then system dynamics is generated by an extremal vector field which admits a Hamiltonian realization. However, such extremal direction fields can occur only when the Pfaff Topological Dimension of the system encoded by A is odd, and equal or less than the geometric dimension of the base variables.

For example, if the geometric dimension is 3, and the Pfaff Topological Dimension of A is 3, then there exists a unique extremal field on the Contact manifold defined by dA . This unique directionfield is the unique eigen directionfield of the 3x3 antisymmetric matrix (created by the 2-form $F = dA$) with eigenvalue equal to zero.

If the geometric dimension is 4, and the Pfaff Topological Dimension of A is 3, then there exists a two extremal fields on the geometric manifold. These directionfields are those generated as the eigen directionfields of the 4x4 antisymmetric matrix (created by the 2-form $F = dA$) with eigenvalue equal to zero.

If the geometric dimension is 4, and the Pfaff Topological Dimension of A is 4, then there do not exist extremal fields on the Symplectic manifold defined by dA . All of the eigen directionfields of the 4x4 antisymmetric matrix (created by the 2-form $F = dA$) are complex isotropic spinors with pure imaginary eigenvalues not equal to zero.

1.4 Topological Torsion and other Continuous Processes.

1.4.1 Reversible Processes

Physical Processes are determined by directionfields³ with the symbol, \mathbf{V}_4 , to within an arbitrary function, ρ . There are several classes of direction fields that are defined as follows [5]:

$$\text{Associated Class} : i(\rho\mathbf{V}_4)A = 0, \quad (13)$$

$$\text{Extremal Class} : i(\rho\mathbf{V}_4)dA = 0, \quad (14)$$

$$\text{Characteristic Class} : i(\rho\mathbf{V}_4)A = 0, \quad (15)$$

$$\text{and} : i(\rho\mathbf{V}_4)dA = 0, \quad (16)$$

$$\text{Helmholtz Class} : d(i(\rho\mathbf{V}_4)dA) = 0, \quad (17)$$

Extremal Vectors (relative to the 1-form of Action, A) produce zero thermodynamic work, $W = i(\rho\mathbf{V}_4)dA = 0$, and admit a Hamiltonian representation. Associated Vectors (relative to the 1-form of Action, A) can be adiabatic if the process remains orthogonal to the 1-form, A . Helmholtz processes (which include Hamiltonian processes, Bernoulli processes and Stokes flow) conserve the 2-form of Topological vorticity, dA . All such processes are thermodynamically reversible. Many examples of these systems are detailed in the reference monographs (see footnote on page 1).

1.4.2 Irreversible Processes

There is one directionfield that is uniquely defined by the coefficient functions of the 1-form, A , that encodes the thermodynamic system on a 4D geometric variety. This vector exists only in non-equilibrium systems, for which the Pfaff Topological Dimension of A is 3 or 4. This 4 vector is defined herein as the topological Torsion vector, \mathbf{T}_4 . To within a factor, this directionfield⁴ has the four coefficients of the 3-form $A \wedge dA$, with the following properties:

³Which include both vector and spinor fields.

⁴A direction field is defined by the components of a vector field which establish the "line of action" of the vector in a projective sense. An arbitrary factor times the direction field defines the same projective line of action, just reparameterized. In metric based situations, the arbitrary factor can be interpreted as a renormalization factor.

Properties of : Topological Torsion \mathbf{T}_4 on Ω_4 (18)

$$i(\mathbf{T}_4)\Omega_4 = i(\mathbf{T}_4)dx\wedge dy\wedge dz\wedge dt = A\wedge dA, \quad (19)$$

$$W = i(\mathbf{T}_4)dA = \sigma A, \quad (20)$$

$$dW = d\sigma\wedge A + \sigma dA = dQ \quad (21)$$

$$U = i(\mathbf{T}_4)A = 0, \quad \mathbf{T}_4 \text{ is associative} \quad (22)$$

$$i(\mathbf{T}_4)dU = 0 \quad (23)$$

$$i(\mathbf{T}_4)Q = 0 \quad \mathbf{T}_4 \text{ is adiabatic} \quad (24)$$

$$L_{(\mathbf{T}_4)}A = \sigma A, \quad \mathbf{T}_4 \text{ is homogeneous} \quad (25)$$

$$L_{(\mathbf{T}_4)}dA = d\sigma\wedge A + \sigma dA = dQ, \quad (26)$$

$$Q\wedge dQ = L_{(\mathbf{T}_4)}A\wedge L_{(\mathbf{T}_4)}dA = \sigma^2 A\wedge dA \neq 0, \quad (27)$$

$$dA\wedge dA = d(A\wedge dA) = d\{i(\mathbf{T}_4)\Omega_4\} = (div_4\mathbf{T}_4)\Omega_4, \quad (28)$$

$$L_{(\mathbf{T}_4)}\Omega_4 = d\{i(\mathbf{T}_4)\Omega_4\} = (2\sigma)\Omega_4, \quad (29)$$

If the Pfaff Topological Dimension of A is 4 (an Open thermodynamic system), then \mathbf{T}_4 has a non-zero 4 divergence, (2σ) , representing an expansion or a contraction of the 4D volume element Ω_4 . The Heat 1-form, Q , generated by the process, \mathbf{T}_4 , is NOT integrable. Q is of Pfaff Topological Dimension greater than 2, when $\sigma \neq 0$. Furthermore the \mathbf{T}_4 process is locally adiabatic as the change of internal energy in the direction of the process path is zero. Therefore, in the Pfaff Topological Dimension 4 case, where $dA\wedge dA \neq 0$, the \mathbf{T}_4 direction field represents an *irreversible, adiabatic process*.

When σ is zero and $d\sigma = 0$, but $A\wedge dA \neq 0$, the Pfaff Topological Dimension of the system is 3 (a Closed thermodynamic system). In this case, the \mathbf{T}_4 direction field becomes a characteristic vector field which is both extremal and associative, and induces a Hamilton-Jacobi representation (the ground state of the system for which $dQ = 0$).

For any process and any system, equation (27) can be used as a test for irreversibility.

It seems a pity, that the concept of the Topological Torsion vector and its association with non-equilibrium systems, where it can be used to establish design criteria to minimize energy dissipation, has been ignored by the engineering community.

1.4.3 The Spinor class

It is rather remarkable (and only fully appreciated by me in February, 2005) that there is a large class of direction fields useful to the topological dynamics of thermodynamic systems (given herein the symbol $\rho\mathbf{S}_4$) that do not behave as vectors (with respect to rotations).

They are isotropic complex vectors of zero length, defined as Spinors by E. Cartan [2], but which are most easily recognized as the eigen directionfields relative to the antisymmetric matrix, $[F]$, generated by the component of the 2-form $F = dA$:

$$\text{The Spinor Class} \quad [F] \circ |\rho\mathbf{S}_4\rangle = \lambda |\rho\mathbf{S}_4\rangle \neq 0, \quad (30)$$

$$\langle \rho\mathbf{S}_4 | \circ |\rho\mathbf{S}_4\rangle = 0, \quad \lambda \neq 0 \quad (31)$$

In the language of exterior differential forms, if the Work 1-form is not zero, the process must contain Spinor components:

$$W = i(\rho\mathbf{S}_4)dA \neq 0 \quad (32)$$

As mentioned above, Spinors have metric properties, behave as vectors with respect to transitive maps, but do not behave as vectors with respect to rotations (see p. 3, [2]). Spinors generate harmonic forms and are related to conjugate pairs of minimal surfaces. The notation that a Spinor is a complex isotropic *directionfield* is preferred over the names "complex isotropic *vector*", or "null *vector*" that appear in the literature. As shown below, the familiar formats of Hamiltonian mechanical systems exclude the concept of Spinor process directionfields, for the processes permitted are restricted to be represented by direction fields of the extremal class, which have zero eigenvalues.

Remark 4 *Spinors are normal consequences of antisymmetric matrices, and, as topological artifacts, they are not restricted to physical microscopic or quantum constraints. According to the topological thermodynamic arguments, Spinors are implicitly involved in all processes for which the 1-form of thermodynamic Work is not zero. Spinors play a role in topological fluctuations and irreversible processes.*

The thermodynamic Work 1-form, W , is generated by a completely antisymmetric 2-form, F , and therefore, if not zero, must have Spinor components. In the odd dimensional Contact manifold case there is one eigen Vector, with eigenvalue zero, which generates the extremal processes that can be associated with a Hamiltonian representation. The other two eigendirection fields are Spinors. In the even dimensional Symplectic manifold case, any non-zero component of work requires that the evolutionary directionfields must contain Spinor components. All eigen directionfields on symplectic spaces are Spinors.

The fundamental problem of Spinor components is that there can be more than one Spinor direction field that generates the same geometric path. For example, there can be Spinors of left or right handed polarizations and Spinors of expansion or contraction that produce the same optical (null congruence) path. This result does not fit with the classic arguments

of mechanics, which require unique initial data to yield unique paths. Furthermore, the concept of Spinor processes can annihilate the concept of time reversal symmetry, inherent in classical hydrodynamics. The requirement of uniqueness is not a requirement of non-equilibrium thermodynamics, where Spinor "entanglement" has to be taken into account.

1.5 Emergent Topological Defects

Suppose an evolutionary process starts in a domain of Pfaff Topological Dimension 4, for which a process in the direction of the Topological Torsion vector, \mathbf{T}_4 , is known to represent an irreversible process. Examples can demonstrate that the irreversible process can proceed to a domain of the geometric variety for which the dissipation coefficient, σ , becomes zero. Physical examples [42] such as the skidding bowling ball proceed with irreversible dissipation ($PTD = 6$) until the "no-slip" condition is reached ($PTD = 5$). In fluid systems the topological defects can emerge as long lived states far from equilibrium. The process is most simply visualized as a "condensation" from a turbulent gas, such as the creation of a star in the model which presumes the universe is a very dilute, turbulent van der Waals gas near its critical point. The red spot of Jupiter, a hurricane, the ionized plasma ring in a nuclear explosion, Falaco Solitons, the wake behind an aircraft are all exhibitions of the emergence process to long lived topological structures far from equilibrium. It is most remarkable that the emergence of these experimental defect structures occurs in finite time.

The idea is that a subdomain of the original system of Pfaff Topological Dimension 4 can evolve continuously with a change of topology to a region of Pfaff Topological Dimension 3. The emergent subdomain of Pfaff Topological Dimension 3 is a topological defect, with topological coherence, and often with an extended lifetime (as a soliton structure with a dominant Hamiltonian evolutionary path), embedded in the Pfaff dimension 4 turbulent background.

The Topological Torsion vector in a region of Pfaff Topological Dimension 3 is an extremal vector direction field in systems of Pfaff Topological Dimension 3; it then has a zero 4D divergence, and leaves the volume element invariant. Moreover the existence of an extremal direction field implies that the 1-form of Action can be given a Hamiltonian representation, $P_k dq^k + H(P, q, t) dt$. In the domain of Pfaff dimension 3 for the Action, A , the subsequent continuous evolution of the system, A , relative to the process \mathbf{T}_4 , can proceed in an energy conserving, Hamiltonian manner, representing a "stationary" or "excited" state far from equilibrium (the ground state). This argument is based on the assumption that the Hamiltonian component of the direction field is dominant, and any Spinor components in the $PTD = 3$ domain, representing topological fluctuations, can be ignored. These excited states, far from equilibrium, can be interpreted as the evolutionary topological defects that

emerge and self-organize due to irreversible processes in the turbulent dissipative system of Pfaff dimension 4.

The descriptive words of self-organized states far from equilibrium have been abstracted from the intuition and conjectures of I. Prigogine [9]. The methods of Continuous Topological Evolution correct the Prigogine conjecture that "dissipative structures" can be caused by dissipative processes and fluctuations. The long-lived excited state structures created by irreversible processes are non-equilibrium, deformable topological defects almost void of irreversible dissipation. The topological theory presented herein presents for the first time a solid, formal, mathematical justification (with examples) for the Prigogine conjectures. Precise definitions of equilibrium and non-equilibrium systems, as well as reversible and irreversible processes can be made in terms of the topological features of Cartan's exterior calculus. Using Cartan's methods of exterior differential systems, thermodynamic irreversibility and the arrow of time can be well defined in a topological sense, a technique that goes beyond (and without) statistical analysis [23]. Thermodynamic irreversibility and the arrow of time requires that the evolutionary process produce topological change.

2 Applications

2.1 An Electromagnetic format

The thermodynamic identification of the terms in Cartan's magic formula are not whimsical. To establish an initial level of credence in the terminology, consider the 1-form of Action, A , where the component functions are the symbols representing the familiar vector and scalar potentials in electromagnetic theory. The coefficient functions have arguments over the four independent variables $\{x, y, z, t\}$,

$$A = A_\mu(x, y, z, t)dx^\mu = \mathbf{A} \circ d\mathbf{r} - \phi dt. \quad (33)$$

Construct the 2-form of field intensities as the exterior differential of the 1-form of Action,

$$F = dA = (\partial A_k / \partial x^j - \partial A_j / \partial x^k) dx^j \wedge dx^k \quad (34)$$

$$= F_{jk} dx^j \wedge dx^k = +\mathbf{B}_z dx \wedge dy \dots + \mathbf{E}_x dx \wedge dt \dots \quad (35)$$

The engineering variables are defined as electric and magnetic field intensities:

$$\mathbf{E} = -\partial \mathbf{A} / \partial t - \text{grad } \phi, \quad \mathbf{B} = \text{curl } \mathbf{A}. \quad (36)$$

Relative to the ordered set of base variables, $\{x, y, z, t\}$, define a process directionfield, $\rho \mathbf{V}_4$, as a 4-vector with components, $[\mathbf{V}, 1]$, with a scaling factor, ρ .

$$\rho[\mathbf{V}_4] = \rho[\mathbf{V}, 1]. \quad (37)$$

Note that this direction field can be used to construct a useful 3-form of (matter) current, C , in terms of the 4-volume element, $\Omega_4 = dx \wedge dy \wedge dz \wedge dt$:

$$C = i(\rho \mathbf{V}_4) dx \wedge dy \wedge dz \wedge dt = i(\mathbf{C}_4) \Omega_4. \quad (38)$$

The process 3-form, C , is not necessarily the same as electromagnetic charge current density 3-form of electromagnetic theory, J . The 4-divergence of C , need not be zero: $dC \neq 0$.

Using the above expressions, the evaluation of the thermodynamic work 1-form in terms of 3-vector engineering components becomes:

$$\text{The thermodynamic Work 1-form:} \quad W = i(\rho \mathbf{V}_4) dA = i(\rho \mathbf{V}_4) F, \quad (39)$$

$$\Rightarrow -\rho \{ \mathbf{E} + \mathbf{V} \times \mathbf{B} \} \circ d\mathbf{r} + \rho \{ \mathbf{V} \circ \mathbf{E} \} dt \quad (40)$$

$$= -\rho \{ \mathbf{f}_{Lorentz} \} \circ d\mathbf{r} + \rho \{ \mathbf{V} \circ \mathbf{E} \} dt. \quad (41)$$

$$\text{The Lorentz force} = -\{ \mathbf{f}_{Lorentz} \} \circ d\mathbf{r} \text{ (spatial component)} \quad (42)$$

$$\text{The dissipative power} = +\{ \mathbf{V} \circ \mathbf{E} \} dt \text{ (time component)}. \quad (43)$$

For those with experience in electromagnetism, note that the construction yields the format, automatically and naturally, for the "Lorentz force" as a *derivation consequence*, without further ad hoc assumptions. The dot product of a 3 component force, $\mathbf{f}_{Lorentz}$, and a differential spatial displacement, $d\mathbf{r}$, defines the elementary classic concept of "differential work". The 4-component thermodynamic Work 1-form, W , includes the spatial component and a differential time component, Pdt , with a coefficient which is recognized to be the "dissipative power", $P = \{\mathbf{V} \circ \mathbf{E}\}$. The thermodynamic Work 1-form, W , is not necessarily a perfect differential, and therefore can be path dependent. Closed cycles of Work need not be zero.

Next compute the Internal Energy term, relative to the process defined as $\rho\mathbf{V}_4$:

$$\textbf{Internal Energy: } U = i(\rho\mathbf{V}_4)A = \rho(\mathbf{V} \circ \mathbf{A} - \phi). \quad (44)$$

The result is to be recognized as the "interaction" energy density in electromagnetic plasma systems. It is apparent that the internal energy, U , corresponds to the interaction energy of the physical system; that is, U is the internal stress energy of system deformation. Therefore, the electromagnetic terminology can be used to demonstrate the premise that Cartan's magic formula is not just another way to state that the first law of thermodynamics makes practical sense. The topological methods permit the long sought for integration of mechanical and thermodynamic concepts, without the constraints of equilibrium systems, and/or statistical analysis.

It is remarkable that although the symbols are different, the same basic constructions and conclusions apply to many classical physical systems. The correspondence so established between the Cartan magic formula acting on a 1-form of Action, and the first law of thermodynamics is taken both literally and seriously in this essay. The methods yield explicit constructions for testing when a process acting on a physical system is irreversible. The methods permit irreversible adiabatic processes to be distinguished from reversible adiabatic processes, analytically. Adiabatic processes need not be "slow" or quasi-static.

Given any 1-form, A , W , and/or Q , the concept of Pfaff Topological Dimension (for each of the three 1-forms, A , W , Q) permits separation of processes and systems into equivalence classes. For example, dynamical processes can be classified in terms of the topological Pfaff dimension of the thermodynamic Work 1-form, W . All extremal Hamiltonian systems have a thermodynamic Work 1-form, W , of topological Pfaff dimension of 1, ($dW = 0$). Hamiltonian systems can describe reversible processes in non-equilibrium systems for which the topological Pfaff dimension is 3. Such systems are topological defects whose topology is preserved by the Hamiltonian dynamics, but all processes which preserve topology are re-

versible. In non-equilibrium systems, topological fluctuations can be associated with Spinors of the 2-form, $F = dA$. Even if the dominant component of the process is Hamiltonian, Spinor fluctuations can cause the system (ultimately) to decay.

2.1.1 Topological 3-forms and 4-forms in EM format

Construct the elements of the Pfaff Sequence for the EM notation,

$$\{A, F = dA, A \wedge F, F \wedge F\}, \quad (45)$$

and note that the algebraic expressions of Topological Torsion, $A \wedge F$, can be evaluated in terms of 4-component engineering variables \mathbf{T}_4 as:

$$\textbf{Topological Torsion vector} \quad (46)$$

$$A \wedge F = i(\mathbf{T}_4)\Omega_4 = i(\mathbf{T}_4)dx \wedge dy \wedge dz \wedge dt \quad (47)$$

$$\mathbf{T}_4 = [\mathbf{T}, h] = -[\mathbf{E} \times \mathbf{A} + \mathbf{B}\phi, \mathbf{A} \circ \mathbf{B}]. \quad (48)$$

The exterior 3-form, $A \wedge F$, with physical units of $(\hbar/\text{unit_mole})^2$, is not found (usually) in classical discussions of electromagnetism⁵.

If \mathbf{T}_4 is used as to define the direction field of a process, then

$$L_{(\mathbf{T}_4)}A = \sigma A, \quad i(\mathbf{T}_4)A = 0. \quad (49)$$

$$\text{where } 2\sigma = \{div_4(\mathbf{T}_4)\} = 2(\mathbf{E} \circ \mathbf{B}). \quad (50)$$

The important (universal) result is that if the acceleration associated with the direction field, \mathbf{E} , is parallel to the vorticity associated with the direction field, \mathbf{B} , then according to the equations starting with eq. (18) et. seq. the process is dissipatively irreversible. This result establishes the design criteria for engineering applications to minimize dissipation from turbulent processes.

The Topological Torsion vector has had almost no utilization in applications of classical electromagnetic theory.

2.1.2 Topological Torsion quanta

The 4-form of Topological Parity, $F \wedge F$, can be evaluated in terms of 4-component engineering variables as:

$$\textbf{Topological Parity}$$

$$d(A \wedge F) = F \wedge F = \{div_4(\mathbf{T}_4)\}\Omega_4 = \{2\mathbf{E} \circ \mathbf{B}\} \Omega_4. \quad (51)$$

⁵The unit mole number is charge, e, in EM theory.

This 4-form is also known as the second Poincare Invariant of Electromagnetic Theory.

The fact that $F \wedge F$ need not be zero implies that the Pfaff Topological Dimension of the 1-form of Action, A , must be 4, and therefore A represents a non-equilibrium Open thermodynamic system. Similarly, if $F \wedge F = 0$, but $A \wedge F \neq 0$, then the Pfaff Topological Dimension of the 1-form of Action, A , must be 3, and the physical system is a non-equilibrium Closed thermodynamic system. When $F \wedge F = 0$, the corresponding three-dimensional integral of the closed 3-form, $A \wedge F$, when integrated over a closed 3D-cycle, becomes a deRham period integral, defined as the Torsion quantum. In other words, the closed integral of the (closed) 3-form of Topological Torsion becomes a deformation (Hopf) invariant with integral values proportional to the integers.

$$\mathbf{Torsion\ quantum} = \iiint_{3D_cycle} A \wedge F. \quad (52)$$

On the other hand, topological evolution and transitions between "quantized" states of Torsion require that the respective Parity 4-form is are not zero. As,

$$L_{(\mathbf{T}_4)}\Omega_4 = d\{i(\mathbf{T}_4)\Omega_4\} = (2\sigma) \Omega_4 = 2(\mathbf{E} \circ \mathbf{B}) \Omega_4 \neq 0, \quad (53)$$

it is apparent that the evolution of the differential volume element, Ω_4 , depends upon the existence and colinearity of both the electric field, \mathbf{E} , and the magnetic field, \mathbf{B} . It is here that contact is made with the phenomenological concept of "4D bulk" viscosity = 2σ . It is tempting to identify σ^2 with the concept of entropy production. Note that the Topological Torsion directionfield appears only in non-equilibrium systems. These results are universal and can be used in hydrodynamic systems discussed in that which follows.

2.2 A Hydrodynamic format

2.2.1 The Topological Continuum vs. the Geometrical Continuum

In many treatments of fluid mechanics the (geometrical) continuum hypothesis is invoked from the start. The idea is "matter" occupies all points of the space of interest, and that properties of the fluid can be represented by piecewise continuous functions of space and time, as long as length and time scales are not too small. The problem is that at very small scales, one has been led to believe the molecular or atomic structure of particles will become evident, and the "macroscopic" theory will breakdown. However, these problems of scale are geometric issues, important to many applications, but not pertinent to a topological perspective, where shape and size are unimportant. Suppose that the dynamics can be formulated in terms of topological concepts which are independent from sizes and shapes.

Then such a theory of a Topological Continuum would be valid at all scales. Such is the goal of this monograph.

Remark 5 *However, one instance where "scale" may have topological importance is associated with the example of a surface with a "teeny" hole. If the hole, no matter what its size, has a twisted ear (Moebius band) then the whole surface is non-orientable, no matter how "small" the hole. Could it be that the world of the quantum is, in effect, that of non-orientable defects embedded in an otherwise orientable manifold that originally had no such defects. Note further the strong correspondence with Fermions with non-oriented (half-integer) multiplet ribbons, and Bosons with oriented (integer) multiplet ribbons of both right and left twists [45].*

As will be developed below, the fundamental equations of exterior differential systems can lead to field equations in terms of systems of Partial Differential Equations (PDE's). The format of the fundamental theory will be in terms of objects (exterior differential forms) which, although composed of algebraic constructions of tensorial⁶ things, are in a sense scalars (or pseudo scalars) that are homogeneous with respect to concepts of scale. The theory then developed is applicable to hydrodynamics at all scales, from the microworld to the cosmological arena. The "breakdown" of the continuum model is not relevant. The topological system may consist of many disconnected parts when the system is not in thermodynamic equilibrium or isolation, and the parts can have topological obstructions or defects, some of which can be used to construct period integrals that are topologically "quantized". Hence the "quantization" of the micro-scaled geometric systems can have its genesis in the non-equilibrium theory of thermodynamics. However, from the topological perspective, the rational topological quantum values can also occur at all scales.

2.2.2 Topological Hydrodynamics

The axioms of Topological Thermodynamics are summarized in Section 1.1. For hydrodynamics (or electrodynamics) the axioms are essentially the same. Just exchange the word, hydrodynamics (or electrodynamics), for the word, thermodynamics, in the formats of Section 1.1

By 1969 it had become evident to me that electromagnetism (without geometric constraints), when written in terms of differential forms, was a topological theory, and that the concept of dissipation and irreversible processes required more than that offered by Hamiltonian mechanics. At that time I was interested in possible interactions of the gravitational field and the polarizations of an electromagnetic signal. One of the first ideas discovered

⁶Relative to diffeomorphisms.

about topological electrodynamics was that there existed an intrinsic transport theorem [11] that introduced the concept of what is now called Topological Spin, $A^{\wedge}G$, into electromagnetic theory [44]. As a transport theorem not recognized by classical electromagnetism, the first publication was as a letter to Physics of Fluids. That started my interest in a topological formulation of fluids.

It was not until 1974 that the Lie differential acting on exterior differential forms was established as the key to the problem of intrinsically describing dissipation and the production of topological defects in physical systems; but methods of visualization of such topological defects in classical electrodynamics were not known [12]. It was hoped that something in the more visible fluid mechanics arena would lend credence to the concepts of topological defects. The first formulations of the PDE's of fluid dynamics in terms of differential forms and Cartan's Magic formula followed quickly [13].

In 1976 it was argued that topological evolution was at the cause of turbulence in fluid dynamics, and the notion of what is now called Topological Torsion, $A^{\wedge}F$, became recognized as an important concept. It was apparent that streamline flow imposed the constraint that $A^{\wedge}F = 0$ on the equations of hydrodynamics. Turbulent flow, being the antithesis of streamline flow, must admit $A^{\wedge}F \neq 0$. In 1977 it was recognized that topological defect structures could become "quantized" in terms of deRham period integrals [15], forming a possible link between topology and both macroscopic and microscopic quantum physics. The research effort then turned back to a study of topological electrodynamics in terms of the dual polarized ring laser, where it was experimentally determined that the speed of an electromagnetic signal outbound could be different from the speed of an electromagnetic signal inbound: a topological result not within the realm of classical theory.

Then in 1986 the long sought for creation and visualization of topological defects in fluids [16] became evident. The creation of Falaco Solitons in a swimming pool was the experiment that established credence in the ideas of what had, by that time, become a theory of continuous topological evolution. It was at the Cambridge conference in 1989 [17] that the notions of topological evolution, hydrodynamics and thermodynamics were put together in a rudimentary form, but it was a year later at the Permb conference in 1990 [18] that the ideas were well established. The Permb presentation also suggested that the ambiguous (at that time) notion of coherent structures in fluids could be made precise in terms of topological coherence. A number of conference presentations followed in which the ideas of continuous thermodynamic irreversible topological evolution in hydrodynamics were described [19], but the idea that the topological methods of thermodynamics could be used to distinguish non-equilibrium processes and non-equilibrium systems and irreversible processes with out the use of statistics slowly came into being in the period 1985-2005 [22]. These efforts have been summarized in [41], and a collection of the old publications appears

in [46].

3 Closed States of Topological Coherence embedded as deformable defects in Turbulent Domains

In this section 3, the problem of C2 smoothness will be attacked from the point of view of topological thermodynamics. First, two distinct examples will be given demonstrating two different emergent $PTD = 3$ states, that emerge from different 4D rotations (see p. 108, [39]). Then, an example demonstrating the decay of a $PTD = 4$ state into a $PTD = 3$ state will be given in detail. The electromagnetic notation will be used, but the results can be converted into hydrodynamic format using the techniques found in section 3.

3.1 Examples of $PTD = 3$ domains and their Emergence

In section 3, it was demonstrated that there are solutions (thermodynamic processes) to the Navier Stokes equations in non-equilibrium thermodynamic domains. The properties of those $PTD = 3$ domains which emerge by C2 irreversible solutions from domains of $PTD = 4$ are of particular interest. From section 2, it is apparent that the key feature of $PTD = 3$ domains is that the electric \mathbf{E} field (acceleration field \mathbf{a} in hydrodynamics) must be orthogonal to the magnetic \mathbf{B} field (vorticity field $\boldsymbol{\omega}$ in hydrodynamics). There are 8 cases to consider (including chirality),

Pfaff Topological Dimension 3

$$\mathbf{E} = \mathbf{0}, \quad \pm \mathbf{B} \neq \mathbf{0}, \quad (54)$$

$$\mathbf{B} = \mathbf{0}, \quad \pm \mathbf{E} \neq \mathbf{0}, \quad (55)$$

$$\mathbf{E} \circ \mathbf{B} = \mathbf{0}, \quad \text{with chirality choices, } \pm \mathbf{E} = \pm \mathbf{B} \neq \mathbf{0}, \quad (56)$$

of which two will be discussed in detail.

The Finite Helicity case (both \mathbf{E} and \mathbf{B} finite) $PTD = 3$ Start with the 4D thermodynamic domain, and first consider the 1-form of Action, A , with the format⁷:

$$A = A_x(z)dx + A_y(z)dy - \phi(z)dt, \quad (57)$$

⁷The +E, +B chirality has been selected.

and its induced 2-form, $F = dA$,

$$F = dA = (\partial A_x(z)/\partial z)dz \wedge dx + (\partial A_y(z)/\partial z)dz \wedge dy - (\partial \phi(z)/\partial z)dz \wedge dt, \quad (58)$$

$$= B_x(z)dz \wedge dx - B_y(z)dz \wedge dy + E_z(z)dz \wedge dt. \quad (59)$$

The 3-form of Topological Torsion 3-form becomes

$$i(\mathbf{T}_4)\Omega_4 = A \wedge F \text{ where} \quad (60)$$

$$\mathbf{T}_4(z) = [E_z A_y + \phi B_x, -E_z A_x + \phi B_y, 0, A_x B_x + A_y B_y] \quad (61)$$

$$\text{with } \text{div}_4(\mathbf{T}_4(z)) = 2(\mathbf{E} \circ \mathbf{B}) = 0, \quad \mathbf{A} \circ \mathbf{B} \neq 0. \quad (62)$$

The Zero Helicity case (both \mathbf{E} and \mathbf{B} finite) PTD = 3 Start with the 4D thermodynamic domain, and consider the 1-form of Action, A , with the format:

$$A = A_x(x, y)dx + A_y(x, y)dy - \phi(x, y)dt, \quad (63)$$

and its induced 2-form, $F = dA$,

$$F = dA = \{(\partial A_y(x, y)/\partial x) - (\partial A_x(x, y)/\partial x)dx \wedge dy\} \\ - (\partial \phi(x, y)/\partial x)dx \wedge dt - (\partial \phi(x, y)/\partial y)dy \wedge dt, \quad (64)$$

$$= B_z(x, y)dx \wedge dy + E_x(x, y)dx \wedge dt + E_y(x, y)dy \wedge dt. \quad (65)$$

The 3-form of Topological Torsion 3-form becomes,

$$i(\mathbf{T}_4)\Omega_4 = A \wedge F \quad \text{where} \quad (66)$$

$$\mathbf{T}_4(x, y) = [0, 0, (E_x A_y - E_y A_x) + \phi B_z, 0] \quad (67)$$

$$\text{with } \text{div}_4(\mathbf{T}_4(x, y)) = 2(\mathbf{E} \circ \mathbf{B}) = 0, \quad \mathbf{A} \circ \mathbf{B} = 0. \quad (68)$$

This case of zero helicity ($\mathbf{A} \circ \mathbf{B} = 0$), has the Topological Torsion vector, $\mathbf{T}_4(x, y)$, colinear with the \mathbf{B} field.

Zero Helicity case: PTD = 4 decays to PTD = 3 The two distinct cases, modulo chirality, are suggestive of the idea (see p. 108 [39]) that the rotation group of a 4D domain is not simple. The example, immediately above, is particularly useful because the algebra of the decay from Pfaff dimension 4 to 3 is transparent.

Start with the 4D thermodynamic domain, and consider the 1-form of Action, A , with the format:

$$A = A_x(x, y)dx + A_y(x, y)dy - \phi(x, y, z, t)dt, \quad (69)$$

and its induced 2-form, $F = dA$,

$$F = dA = \{(\partial A_y(x, y)/\partial x) - (\partial A_x(x, y)/\partial x)dx \wedge dy\} \\ -(\partial\phi(x, y, z)/\partial x)dx \wedge dt - (\partial\phi(x, y, z)/\partial y)dy \wedge dt - (\partial\phi(x, y, z)/\partial z)dz \wedge dt, \quad (70)$$

$$= B_z(x, y)dx \wedge dy \quad (71)$$

$$+E_x(x, y, z, t)dx \wedge dt + E_y(x, y, z, t)dy \wedge dt + E_z(x, y, z, t)dy \wedge dt. \quad (72)$$

The 3-form of Topological Torsion 3-form becomes,

$$i(\mathbf{T}_4)\Omega_4 = A \wedge F \text{ with } \text{PTD}(A) = 4 \quad (73)$$

$$\mathbf{T}_4(x, y, z, t) = [-E_z A_y, +E_z A_x, (E_x A_y - E_y A_x) + \phi B_z, 0] \quad (74)$$

$$\text{with } \text{div}_4(\mathbf{T}_4(x, y, z, t)) = 2\{E_z(x, y, z, t)B_z(x, y)\} \neq 0. \quad (75)$$

In this case, the helicity ($\mathbf{A} \circ \mathbf{B} = 0$) is still zero, but now the Topological Torsion vector, $\mathbf{T}_4(x, y, z, t)$, has three spatial components. Moreover, the Process generated by $\mathbf{T}_4(x, y, z)$ is thermodynamically irreversible, as $(\mathbf{E} \circ \mathbf{B}) \neq 0$. The example 1-form is of $\text{PTD} = 4$.

To demonstrate the emergence of the $\text{PTD} = 3$ state, suppose the potential function in this example has the format,

$$\phi = \psi(x, y) + \varphi(z)e^{-\alpha t} \quad (76)$$

$$E_z(z, t) = -(\partial\varphi(z)/\partial z)e^{-\alpha t} = E_z(z)e^{-\alpha t}. \quad (77)$$

Then the irreversible dissipation function decays as $\{E_z(z)B_z\}e^{-\alpha t}$. By addition of Spinor fluctuation terms to represent the very small components of irreversible dissipation at late times, the $\text{PTD} = 3$ solution,

$$\mathbf{T}_4(x, y) = [0, 0, (E_x A_y - E_y A_x) + \phi B_z, 0] \quad (78)$$

becomes dominant, and represents a long lived "stationary" state far from equilibrium, modulo the small Spinor decay terms⁸.

To demonstrate the emergence of the $\text{PTD} = 3$ state in finite time, suppose the potential function in this example has the format,

$$\phi = \psi(x, y) \pm \varphi(z)\sqrt{(-t - t_c)^3} \quad (79)$$

$$E_z(z, t) = \mp(\partial\varphi(z)/\partial z)\sqrt{(-t - t_c)^3} = \pm E_z(z)\sqrt{(-t - t_c)^3}. \quad (80)$$

⁸The experimental fact that the defect structures emerge in finite time is still an open *topological* problem, although some *geometric* success has been achieved through Ricci flows.

Then the irreversible dissipation function decays in a cuspidal way (typical of the approach to an edge of regression of an envelope function) according to the formula, $\{E_z(z)B_z\}\sqrt{(-t - t_c)^3}$. The *PTD* of the system is 4 for $t < t_c$, and becomes equal to 3 for $t = t_c$.

3.2 Piecewise Linear Vector Processes vs. C2 Spinor processes

It will be demonstrated on thermodynamic spaces of Pfaff Topological Dimension 3, that there exist piecewise continuous processes (solutions to the Navier-Stokes equations) which are thermodynamically reversible. These Vector processes can be fabricated by combinations of Spinor processes, each of which is irreversible. This topological result demonstrates, by example, the difference between piecewise linear 3-manifolds and smooth complex manifolds. It appears that the key feature of the irreversible processes is that they have a fixed point of "rotation or expansion".

Consider those abstract physical systems that are represented by 1-forms, A , of Pfaff Topological Dimension 3. The concept implies that the topological features can be described in terms of 3 functions (of perhaps many geometrical coordinates and parameters) and their differentials. For example, if one presumes the fundamental independent base variables are the set $\{x, y, z\}$, with an exterior differential oriented volume element consisting of a product⁹ of exact 1-forms $\Omega_3 = +dx \wedge dy \wedge dz$, (then a local) Darboux representation for a physical system could have the appearance,

$$A = xdy + dz. \quad (81)$$

The objective is to use the features of Cartan's magic formula to compute the possible evolutionary features of such a system. The evolutionary dynamics is essentially the first law of thermodynamics:

$$L_{\rho\mathbf{V}}A = i(\rho\mathbf{V})dA + di(\rho\mathbf{V})A = W + dU = Q. \quad (82)$$

The elements of the Pfaff sequence for this Action become,

$$A = xdy + dz., \quad (83)$$

$$dA = dx \wedge dy, \quad (84)$$

$$A \wedge dA = dx \wedge dy \wedge dz, \quad (85)$$

$$dA \wedge dA = 0. \quad (86)$$

Note that for this example the coefficient of the 3-form of Topological Torsion is not zero, and depends upon the Enstrophy (square of the Vorticity) of the fluid flow.

⁹More abstract systems could be constructed from differential forms which are not exact.

3.3 The Vector Processes

Relative to the position vector $\mathbf{R} = [x, y, z]$ of ordered topological coordinates $\{x, y, z\}$, consider the 3 abstract, linearly independent, orthogonal (supposedly) vector direction fields:

$$\mathbf{V}_x = \begin{vmatrix} 1 \\ 0 \\ 0 \end{vmatrix} \rangle, \quad (87)$$

$$\mathbf{V}_y = \begin{vmatrix} 0 \\ 1 \\ 0 \end{vmatrix} \rangle, \quad (88)$$

$$\mathbf{E} = \begin{vmatrix} 0 \\ 0 \\ 1 \end{vmatrix} \rangle. \quad (89)$$

These direction fields can be used to define a class of (real) Vector processes, but these real vectors do not exhibit the complex Spinor class of eigendirection fields for the 2-form, dA . The Spinor eigendirection fields are missing from this basis frame. The important fact is that thermodynamic processes defined in terms of a real basis frame (and its connection) are incomplete, as such processes ignore the complex spinor direction fields.

For each of the real direction fields, deform the (assumed) process by an arbitrary function, ρ . Then construct the terms that make up the First Law of topological thermodynamics. First construct the contractions to form the internal energy for each process,

$$U_{\mathbf{V}_x} = i(\rho\mathbf{V}_x)A = 0, \quad dU_{\mathbf{V}_x} = 0, \quad (90)$$

$$U_{\mathbf{V}_y} = i(\rho\mathbf{V}_y)A = \rho x, \quad dU_{\mathbf{V}_y} = d(\rho x), \quad (91)$$

$$U_{\mathbf{E}} = i(\rho\mathbf{E})A = \rho, \quad dU_{\mathbf{E}} = d\rho. \quad (92)$$

The *extremal* vector \mathbf{E} is the unique eigenvector with eigenvalue zero relative to the maximal rank antisymmetric matrix generated by the 2-form, dA . The *associated* vector \mathbf{V}_x (relative to the 1-form of Action, A , is orthogonal to the y, z plane. Recall that any associated vector represents a local adiabatic process, as the Heat flow is transverse to the process. The linearly independent thermodynamic Work 1-forms for evolution in the direction of the 3 basis vectors are determined to be,

$$W_{\mathbf{V}_x} = i(\rho\mathbf{V}_x)dA = +\rho dy, \quad (93)$$

$$W_{\mathbf{V}_y} = i(\rho\mathbf{V}_y)dA = -\rho dx, \quad (94)$$

$$W_{\mathbf{E}} = i(\rho\mathbf{E})dA = 0. \quad (95)$$

From Cartan's Magic Formula representing the First Law as a description of topological evolution,

$$L_{(\mathbf{V})}A = i(\rho\mathbf{V})dA + d(i(\rho\mathbf{V})A) \equiv Q, \quad (96)$$

it becomes apparent that,

$$Q_{\mathbf{V}_x} = +\rho dy, \quad dQ_{\mathbf{V}_x} = +d\rho \hat{d}y, \quad (97)$$

$$Q_{\mathbf{V}_y} = +x d\rho, \quad dQ_{\mathbf{V}_y} = -d\rho \hat{d}x, \quad (98)$$

$$Q_{\mathbf{E}} = d\rho \quad dQ_{\mathbf{E}} = 0, \quad (99)$$

All processes in the extremal direction satisfy the conditions that $Q_{\mathbf{E}} \hat{d}Q_{\mathbf{E}} = 0$. Hence, all extremal processes are reversible. It is also true that evolutionary processes in the direction of the other basis vectors, separately, are reversible, as the 3-form $Q \hat{d}Q$ vanishes for \mathbf{V}_x , \mathbf{V}_y , or \mathbf{E} . Hence all such *piecewise* continuous, *transitive*, processes are thermodynamically reversible.

Note further that the "rotation" induced by the antisymmetric matrix $[dA]$ acting on \mathbf{V}_x yields \mathbf{V}_y and the 4th power of the matrix yields the identity rotation,

$$[dA] \circ |\mathbf{V}_x\rangle = |\mathbf{V}_y\rangle, \quad (100)$$

$$[dA]^2 \circ |\mathbf{V}_x\rangle = -|\mathbf{V}_x\rangle, \quad (101)$$

$$[dA]^4 \circ |\mathbf{V}_x\rangle = +|\mathbf{V}_x\rangle. \quad (102)$$

This concept is a signature of Spinor phenomena.

3.4 The Spinor Processes

Now consider processes defined in terms of the Spinors. The eigendirection fields of the antisymmetric matrix representation of $F = dA$,

$$[F] = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad (103)$$

are given by the equations:

$$\text{EigenSpinor1 } |Sp1\rangle = \begin{pmatrix} 1 \\ \sqrt{-1} \\ 0 \end{pmatrix} \quad \text{Eigenvalue} = +\sqrt{-1}, \quad (104)$$

$$\text{EigenSpinor2 } |Sp2\rangle = \begin{pmatrix} 1 \\ -\sqrt{-1} \\ 0 \end{pmatrix} \quad \text{Eigenvalue} = -\sqrt{-1} \quad (105)$$

$$\text{EigenVector1 } |E\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad \text{Eigenvalue} = 0 \quad (106)$$

Now consider the processes defined by ρ times the Spinor eigendirection fields. Compute the change in internal energy, dU , the Work, W and the Heat, Q , for each Spinor eigendirection field:

$$U_{\rho\mathbf{SP}_1} = i(\rho\mathbf{SP}_1)A = \sqrt{-1}\rho x \quad d(U_{\rho\mathbf{SP}_1}) = \sqrt{-1}d(\rho x), \quad (107)$$

$$U_{\rho\mathbf{SP}_2} = i(\rho\mathbf{SP}_2)A = -\sqrt{-1}\rho x \quad d(U_{\rho\mathbf{SP}_2}) = -\sqrt{-1}d(\rho x), \quad (108)$$

$$U_{\rho\mathbf{E}} = i(\rho\mathbf{E})A = \rho, \quad d(U_{\rho\mathbf{E}}) = d\rho, . \quad (109)$$

$$W_{\rho\mathbf{SP}_1} = i(\rho\mathbf{SP}_1)dA = \rho(dy - \sqrt{-1}dx), \quad (110)$$

$$W_{\rho\mathbf{SP}_2} = i(\rho\mathbf{SP}_2)dA = +\rho(dy + \sqrt{-1}dx) \quad (111)$$

$$W_{\rho\mathbf{E}} = i(\rho\mathbf{V}_1)dA = 0, . \quad (112)$$

$$Q_{\rho\mathbf{SP}_1} = L_{i(\rho\mathbf{SP}_1)}A = \rho(dy - \sqrt{-1}dx) + \sqrt{-1}d(\rho x), \quad (113)$$

$$Q_{\rho\mathbf{SP}_2} = L_{i(\rho\mathbf{SP}_2)}A = \rho(dy + \sqrt{-1}dx) - \sqrt{-1}d(\rho x), \quad (114)$$

$$Q_{\rho\mathbf{E}} = L_{i(\rho\mathbf{E})}A = d\rho. \quad (115)$$

3.5 Irreversible Spinor processes

Next compute the 3-forms of $Q \hat{d}Q$ for each direction field, including the spinors:

$$Q_{\rho\mathbf{E}} \hat{d}Q_{\rho\mathbf{E}} = 0, \quad (116)$$

$$Q_{\rho\mathbf{SP}_1} \hat{d}Q_{\rho\mathbf{SP}_1} = -\sqrt{-1}\rho d\rho \hat{d}x \hat{d}y, \quad (117)$$

$$Q_{\rho\mathbf{SP}_2} \hat{d}Q_{\rho\mathbf{SP}_2} = +\sqrt{-1}\rho d\rho \hat{d}x \hat{d}y. \quad (118)$$

It is apparent that evolution in the direction of the Spinor fields can be irreversible in a thermodynamic sense, if $d\rho \hat{d}x \hat{d}y$ is not zero. This is not true for the "piecewise linear" combinations of the complex Spinors that produce the real vectors, \mathbf{V}_x and \mathbf{V}_y .

Evolution in the direction of "smooth" combinations of the base vectors may not satisfy the reversibility conditions, $Q \hat{d}Q = 0$, when the combination involves a fixed point in the x, y plane. For example, it is possible to consider smooth rotations (polarization chirality) in the x, y plane:

$$V_{\text{rotation right}} = \mathbf{V}_x + \sqrt{-1}\mathbf{V}_y = Sp1, \quad (119)$$

$$Q \hat{d}Q = -\sqrt{-1}\rho d\rho \hat{d}x \hat{d}y. \quad (120)$$

$$V_{\text{rotation left}} = \mathbf{V}_x - \sqrt{-1}\mathbf{V}_y = Sp2, \quad (121)$$

$$Q \hat{d}Q = +\sqrt{-1}\rho d\rho \hat{d}x \hat{d}y. \quad (122)$$

The non-zero value of $Q \hat{d}Q$ for the continuous rotations are related to the non-zero Godbillon-Vey class [7]. A key feature of the rotations is that they have a fixed point in the plane; the motions are not transitive. If the physical system admits an equation of state of the form, $\theta = \theta(x, y, \rho) = 0$, then the rotation or expansion processes are not irreversible.

Note that the (supposedly) Vector processes of the preceding subsection are combinations of the Spinor processes,

$$\mathbf{V}_x = (a \cdot Sp1 + b \cdot Sp2)/2 \quad (123)$$

$$\mathbf{V}_y = -\sqrt{-1}(a \cdot Sp1 - b \cdot Sp2)/2. \quad (124)$$

Almost always, a process defined in terms a linear combinations of the Spinor direction fields will generate a Heat 1-form, Q , that does not satisfy the Frobenius integrability theorem, and therefore all such processes are thermodynamically irreversible: $Q \hat{d}Q \neq 0$. However, with the requirement that a^2 is precisely the same as b^2 , then either piecewise linear process is reversible, for $Q \hat{d}Q = 0$.

If the coefficients, and therefore the Spinor contributions, have slight fluctuations, the cancellation of the complex terms is not precise. Then either of the (now approximately) piecewise continuous process will NOT be reversible due to Spinor fluctuations.

Remark 6 *The facts that piecewise (sequential) C1 transitive evolution along a set of direction fields in odd (3) dimensions can be thermodynamically reversible, $Q \hat{d}Q = 0$, while (smooth) C2 evolution processes composed from complex Spinors can be thermodynamically irreversible, $Q \hat{d}Q \neq 0$, is a remarkable result which appears to have a relationship to Nash's theorem on C1 embedding. Physically, the results are related to tangential discontinuities such as hydrodynamic wakes.*

For systems of Pfaff dimension 4, all of the eigendirection fields are Spinors. The Spinors occur as two conjugate pairs. If the conjugate variables are taken to be x,y and z,t then the z,t spinor pair can be interpreted in terms of a chirality of expansion or contraction, where the x,y pair can be interpreted as a chirality of polarization. In this sense it may be said that thermodynamic time irreversibility is an artifact of dimension 4.

It is remarkable that a rotation and an expansion can be combined (eliminating the fixed point) to produce a thermodynamically *reversible* process.

Ian Stewart points out that there are three types of manifold structure: piecewise linear, smooth, topological. Theorems on piecewise-linear manifolds may not be true on smooth manifolds. The work above seems to describe such an effect. Piecewise continuous processes are reversible, where smooth continuous processes are not (see page 106, [39])!

4 Epilogue: Topological Universality

This Section 4 comments on the universality of the methods of continuous topological evolution and topological thermodynamics and the extension of thermodynamic concepts to the theory of fiber bundles. Recall that complex topological systems can have domains that are defined in terms of exterior differential forms. Such complex systems can consist of many disconnected topological sub-domains, all cooperating in a synergetic way in terms of the many *simultaneous* topologies that can be imposed upon a geometric set. For example in the previous discussion it became apparent that the simultaneous, but different topologies, of A , W , and Q , imposed upon the 4D geometric variety, influenced the thermodynamic interpretations. Such ideas can be extended to fiber bundles.

All such complex systems admit the possibility of non-equilibrium emergent "birth" of primitive odd dimensional "stationary" states far from equilibrium, with gradual pubescence accumulation of matter resulting in a stage of "maturity". The maturity stage can be (relatively speaking) an almost stationary state with slow decay. Then at later stages the mature system dies.

Such descriptions experimentally appear to describe almost every observable synergetic system, including the classical thermodynamic models, but also such things as biological systems, economic systems and political systems. The hand of thermodynamics touches each of these interacting complexes. I conjecture that methods which can treat the ecological effects of synergetic political and economic systems can be developed in terms of continuous topological evolution and topological thermodynamics.

4.1 The Cartan-Hilbert Action 1-form

To demonstrate by example how the topological methods can be extended to higher dimensional systems and fiber bundles, consider those physical systems that can be described by a function $L(\mathbf{q}, \mathbf{v}, t)$ and a 1-form of Action (per unit mole) given by Cartan-Hilbert format,

$$A = L(\mathbf{q}^k, \mathbf{v}^k, t)dt + \mathbf{p}_k \cdot (d\mathbf{q}^k - \mathbf{v}^k dt). \quad (125)$$

The classic Lagrange function, $L(\mathbf{q}^k, \mathbf{v}^k, t)dt$, is extended to include fluctuations in the kinematic variables, $(d\mathbf{q}^k - \mathbf{v}^k dt) \neq 0$. It is no longer assumed that the equation of Kinematic Perfection is satisfied. Fluctuations of the topological constraint of Kinematic Perfection are permitted;

$$\textbf{Topological Fluctuations in position: } \Delta\mathbf{q} = (d\mathbf{q}^k - \mathbf{v}^k dt) \neq 0. \quad (126)$$

As the fluctuations are 1-forms, it is some interest to compute their Pfaff Topological Dimension. The first step in the construction of the Pfaff Sequence is to compute the exterior differential of the fluctuation 1-form:

$$\textbf{Fluctuation 2-form: } \quad d(\Delta\mathbf{q}) = -(d\mathbf{v}^k - \mathbf{a}^k dt) \wedge dt \quad (127)$$

$$= -\Delta\mathbf{v} \wedge dt, \quad (128)$$

$$\textbf{Topological Fluctuations in velocity} \quad : \quad \Delta\mathbf{v} = (d\mathbf{v}^k - \mathbf{a}^k dt) \neq 0. \quad (129)$$

It is apparent that the Pfaff Topological Dimension of the fluctuations is at most 3, as $\Delta\mathbf{q} \wedge \Delta\mathbf{v} \wedge dt \neq 0$, and has a Heisenberg component,

When dealing with fluctuations in this prologue, the geometric dimension of independent base variables will not be constrained to the 4 independent base variables of the Thermodynamic model. At first glance it appears that the domain of definition is a $(3n+1)$ -dimensional variety of independent base variables, $\{\mathbf{p}_k, \mathbf{q}^k, \mathbf{v}^k, t\}$. Do not make the assumption that the \mathbf{p}_k are constrained to be canonically defined. Instead, consider \mathbf{p}_k to be a (set of) Lagrange multiplier(s) to be determined later. Also, do not assume at this stage that \mathbf{v} is a kinematic velocity function, such that $(d\mathbf{q}^k - \mathbf{v}^k dt) \Rightarrow 0$. The classical idea is to assert that topological fluctuations in position are related to pressure, and topological fluctuations in velocity are related to temperature.

For the given Action, construct the Pfaff Sequence (12) in order to determine the Pfaff dimension or class [8] of the Cartan-Hilbert 1-form of Action. The top Pfaffian is defined as the non-zero p-form of largest degree p in the sequence. The top Pfaffian for the Cartan-

Hilbert Action is given by the formula,

Top Pfaffian is 2n+2

$$(dA)^{n+1} = (n+1)! \{ \sum_{k=1}^n (\partial L / \partial v^k - p_k) dv^k \} \wedge \Omega_{2n+1}, \quad (130)$$

$$\Omega_{2n+1} = dp_1 \wedge \dots \wedge dp_n \wedge dq^1 \wedge \dots \wedge dq^n \wedge dt. \quad (131)$$

The formula is a bit surprising in that it indicates that the Pfaff Topological Dimension of the Cartan-Hilbert 1-form is $2n+2$, and not the geometrical dimension $3n+1$. For $n=3$ "degrees of freedom", the top Pfaffian indicates that the Pfaff Topological Dimension of the 2-form, dA is $2n+2=8$. The value $3n+1=10$ might be expected as the 1-form was defined initially on a space of $3n+1$ "independent" base variables. The implication is that there exists an irreducible number of independent variables equal to $2n+2=8$ which completely characterize the differential topology of the first order system described by the Cartan-Hilbert Action. It follows that the exact 2-form, dA , satisfies the equations

$$(dA)^{n+1} \neq 0, \text{ but } A \wedge (dA)^{n+1} = 0. \quad (132)$$

Remark 7 *The idea that the 2-form, dA , is a symplectic generator of even maximal rank, $2n+2$, implies that ALL eigendirection fields of the 2-form, $F=dA$, are complex isotropic Spinors, and all processes on such domains have Spinor components.*

The format of the top Pfaffian requires that the bracketed factor in the expression above, $\{ \sum_{k=1}^n (\partial L / \partial v^k - p_k) dv^k \}$, can be represented (to within a factor) by a perfect differential, dS .

$$dS = (n+1)! \{ \sum_{k=1}^n (\partial L / \partial v^k - p_k) dv^k \}. \quad (133)$$

The result is also true for any closed addition γ added to A ; e.g., the result is "gauge invariant". Addition of a closed 1-form does not change the Pfaff dimension from even to odd. On the other hand the result is not renormalizable, for multiplication of the Action 1-form by a function can change the algebraic Pfaff dimension from even to odd.

On the $2n+2$ domain, the components of $(2n+1)$ -form $T = A \wedge (dA)^n$ generate what has been defined herein as the Topological Torsion vector, to within a factor equal to the Torsion Current. The coefficients of the $(2n+1)$ -form are components of a contravariant vector density \mathbf{T}^m defined as the Topological Torsion vector, the same concept as defined previously on a 4D thermodynamic domain, but now extended to $(2n+2)$ -dimensions. This vector is orthogonal (transversal) to the $2n+2$ components of the covector, \mathbf{A}_m . In other words,

$$A \wedge T = A \wedge (A \wedge (dA)^n) = 0 \Rightarrow i(\mathbf{T})(A) = \sum \mathbf{T}^m \mathbf{A}_m = 0. \quad (134)$$

This result demonstrates that the extended Topological Torsion vector represents an adiabatic process. This topological result does not depend upon geometric ideas such as metric. It was demonstrated above that, on a space of 4 independent variables, evolution in the direction of the Topological Torsion vector is irreversible in a thermodynamic sense, subject to the symplectic condition of non-zero divergence, $d(A \wedge dA) \neq 0$. The same concept holds on dimension $2n+2$.

The $2n+2$ symplectic domain so constructed can not be compact without boundary for it has a volume element which is exact. By Stokes theorem, if the boundary is empty, then the surface integral is zero, which would require that the volume element vanishes; but that is in contradiction to the assumption that the volume element is finite. For the $2n+2$ domain to be symplectic, the top Pfaffian can never vanish. The domain is therefore orientable, but has two components, of opposite orientation. Examination of the constraint that the symplectic space be of dimension $2n+2$ implies that the Lagrange multipliers, \mathbf{p}_k , cannot be used to define momenta in the classical "conjugate or canonical" manner.

Define the non-canonical components of the momentum, $\hbar k_j$, as,

$$\text{non-canonical momentum: } \hbar k_j = (p_j - \partial L / \partial v^j), \quad (135)$$

such that the top Pfaffian can be written as,

$$(dA)^{n+1} = (n+1)! \{ \Sigma_{j=1}^n \hbar k_j dv^j \} \wedge \Omega_{2n+1}, \quad (136)$$

$$\Omega_{2n+1} = dp_1 \wedge \dots \wedge dp_n \wedge dq^1 \wedge \dots \wedge dq^n \wedge dt. \quad (137)$$

For the Cartan-Hilbert Action to be of Pfaff Topological Dimension $2n+2$, the factor $\{ \Sigma_{j=1}^n \hbar k_j dv^j \} \neq 0$. It is important to note, however, that as $(dA)^{n+1}$ is a volume element of geometric dimension $2n+2$, the 1-form $\Sigma_{j=1}^n \hbar k_j dv^j$ is exact (to within a factor, say $T(\mathbf{q}^k, t, \mathbf{p}_k, S_{\mathbf{v}})$); hence,

$$\Sigma_{j=1}^n \hbar k_j dv^j = T dS_{\mathbf{v}}. \quad (138)$$

Tentatively, this 1-form, $dS_{\mathbf{v}}$, will be defined as the Topological Entropy production relative to topological fluctuations of momentum, kinematic differential position and velocity. If $\hbar k_j$ is defined as the deviation about the canonical definition of momentum, $\hbar k_j = \Delta p_j$, and noting the the expression for the top Pfaffian can be written as $(n+1)! \{ \Sigma_{j=1}^n \hbar k_j \Delta v^j \} \wedge \Omega_{2n+1}$, leads to an expression for the entropy production rate in the suggestive "Heisenberg" format:

$$T dS_{\mathbf{v}} = \Delta p_j \Delta v^j. \quad (139)$$

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